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Nonconforming finite element approximations of the Steklov eigenvalue problem ☆

Yidu Yang *, Qin Li, Sirui Li

School of Mathematics and Computer Science, Guizhou Normal University, GuiYang, 550001, China

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ABSTRACT

This paper deals with nonconforming finite element approximations of the Steklov eigenvalue problem. For a class of nonconforming finite elements, it is shown that the j-th approximate eigenpair converges to the j-th exact eigenpair and error estimates for eigenvalues and eigenfunctions are derived. Furthermore, it is proved that the j-th eigenvalue derived by the EQ_1^{rot} element gives lower bound of the j-th exact eigenvalue, whereas the nonconforming Crouzeix–Raviart element and the Q_1^{rot} element provide lower bounds of the large eigenvalues. Numerical results are presented to confirm the considered theory.

1. Introduction

Steklov eigenvalue problems, in which the eigenvalue parameter appears in the boundary condition, arise in a number of applications. For instance, they are found in the study of surface waves (see [8]), in the analysis of stability of mechanical oscillators immersed in a viscous fluid (see [14] and the references therein), in the study of the vibration modes of a structure in contact with an incompressible fluid (see, for example [9]) and in the analysis of the antiplane shearing on a system of collinear faults under slip-dependent friction law (see [11]). There are many interesting problems for the one-dimensional case, like those of vibrations of a pendulum (see [1] and the references therein), those of eigen oscillations of mechanical systems with boundary conditions containing the frequency (see [19]), and many others (see [12], [22] et al.).

Bramble and Osborn [10] studied the Galerkin method for the approximation of Steklov eigenvalues of non-selfadjoint second order elliptic operators. Andreev and Todorov [3] studied the isoparametric variant of finite element method for an approximation of Steklov eigenvalue problems for second-order selfadjoint elliptic differential operators. Armentano and Padra [6] proposed and analyzed an a posteriori error estimator, of the residual type, for the linear finite element approximation of the Steklov eigenvalue problem. Han and Guan [17], Han, Guan and He [18], Huang and Lü [20], and Tang, Guan and Han [24] studied the boundary element method for Steklov eigenvalue problems. However, a few papers deal with nonconforming finite elements.

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* Corresponding author.
E-mail addresses: ydyang@gznu.edu.cn (Y. Yang), liqin362429@163.com (Q. Li), lisirui1983@gznu.edu.cn (S. Li).

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It is well known that nonconforming finite element methods play an important role in the numerical approximation of elliptic partial differential equations when conforming methods and others seem too costly or unstable. So it is natural and meaningful for us to study nonconforming finite element approximations of Steklov eigenvalue problems.

In this paper, we consider the following model problem

$$-\Delta u + u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = \lambda u \quad \text{on } \partial\Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain and $\frac{\partial}{\partial \nu}$ is the outward normal derivative on $\partial\Omega$.

The variational problem associated with (1) is given by: find $\lambda \in \mathbb{R}$ and $u \in H^1(\Omega)$ with $\|u\|_b = 1$ satisfying

$$a(u, v) = \lambda b(u, v), \quad \forall v \in H^1(\Omega), \tag{2}$$

where

$$a(u, v) = \int_{\Omega} \nabla u \nabla v + uv\, dx,$$

$$b(u, v) = \int_{\partial\Omega} uv\, ds, \quad \|u\|_b = b(u, u)^{\frac{1}{2}}.$$

Evidently, the bilinear form $a(\cdot,\cdot)$ is symmetric, continuous and coercive over the product space $H^1(\Omega) \times H^1(\Omega)$.

From [10], the problem (2) has a countable infinite set of eigenvalues, all having finite multiplicity and being strictly positive, without finite accumulation point. We arrange them as $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots \to +\infty$ (here each eigenvalue occurs as many times as given by its multiplicity). The associated eigenfunctions $u_j \in H^2(\Omega)$.

Let π_h be a regular mesh of Ω (see [13], pp. 131), and $K \in \pi_h$ be affine-equivalent to a reference element $\hat{K}$. Let F and $\hat{F}$ be edges of K and $\hat{K}$ respectively.

We introduce three nonconforming finite elements: the nonconforming Crouzeix–Raviart element, the Q_1^{rot} element and the EQ_1^{rot} element. Their corresponding spaces are defined as follows.

The nonconforming Crouzeix–Raviart element space, proposed by Crouzeix and Raviart [15], is defined by $S^h = \{v \in L_2(\Omega)\colon\ v|_K \in \operatorname{span}\{1, x, y\}$ is continuous at the midpoints of the edges of elements$\}$.

The Q_1^{rot} element space, proposed by Rannacher and Turek [23] and Arbogast and Chen [4], is defined by

$$S^h = \left\{ v \in L_2(\Omega)\colon\ v|_K \in \operatorname{span}\{1, x, y, x^2 - y^2\}, \int_F v|_{K_1}\, ds = \int_F v|_{K_2}\, ds \text{ if } K_1 \cap K_2 = F \right\}.$$

The EQ_1^{rot} element space, provided by Lin, Tobiska and Zhou (see [21]), is defined by

$$S^h = \left\{ v \in L_2(\Omega)\colon\ v|_K \in \operatorname{span}\{1, x, y, x^2, y^2\}, \int_F v|_{K_1}\, ds = \int_F v|_{K_2}\, ds \text{ if } K_1 \cap K_2 = F \right\}.$$

Note that the Q_1^{rot} element and EQ_1^{rot} element in the paper are defined on rectangular meshes.

All the above nonconforming elements have the following common characters:

1) The space of shape functions contains the complete polynomials of degree 1;
2) $v \in S^h$ is integrally continuous at the common edge F between the neighboring elements K_1 and K_2, i.e.,

$$\int_F v|_{K_1}\, ds = \int_F v|_{K_2}\, ds \text{ if } K_1 \cap K_2 = F;$$

3) $S^h \not\subset H^1(\Omega)$, $S^h \subset L^2(\Omega)$, and $\delta S^h \subset L^2(\partial\Omega)$, where δS^h denotes the space of restrictions to $\partial\Omega$ of functions in S^h.

Note that the results of Sections 2 and 3 hold true for the nonconforming elements with the above three characters.

The nonconforming finite element approximation of (2) is the following:

Find $\lambda_h \in R$ and $u_h \in S^h$ with $\|u_h\|_b = 1$ such that

$$a_h(u_h, v) = \lambda_h b(u_h, v), \quad \forall v \in S^h, \tag{3}$$

where

$$a_h(u_h, v) = \sum_{K \in \pi_h} \int_K \nabla u_h \nabla v + u_h v\, dx.$$

Define $\|\cdot\|_h = (\sum_{K \in \pi_h} \|\cdot\|_{1,K}^2)^{\frac{1}{2}}$. Evidently, $\|\cdot\|_h$ is the norm on S^h and $a_h(\cdot,\cdot)$ is uniformly S^h-elliptic. In fact,

$$a_h(v, v) = \|v\|_h^2, \quad \forall v \in S^h.$$

We arrange the eigenvalues of problem (3) as $0 < \lambda_{1,h} \leqslant \lambda_{2,h} \leqslant \cdots \leqslant \lambda_{j,h} \leqslant \cdots \leqslant \lambda_{N,h}$, where $N = \dim S^h$ and each eigenvalue occurs as many times as given by its multiplicity.

The rest of the paper is organized as follows. In Section 2 we analyze the nonconforming element approximations of the corresponding source problem. The error estimates are derived. In Section 3 we study nonconforming element approximations of the Steklov eigenvalue problem. We prove that the j-th approximate eigenpair converges to the j-th exact eigenpair and error estimates for eigenvalues and eigenfunctions are obtained. In Section 4 we discuss lower bounds of the exact eigenvalue of the Steklov eigenvalue problem using the Crouzeix–Raviart element, the Q_1^{rot} element and the EQ_1^{rot} element. We prove that the j-th eigenvalue derived by the EQ_1^{rot} element gives lower bound of the j-th exact eigenvalue, whereas the nonconforming Crouzeix–Raviart element and the Q_1^{rot} element provide lower bounds of the large eigenvalues. Finally, numerical experiments in Section 5 are carried out to verify our theoretical analysis.

Throughout this paper, C denotes a generic positive constant independent of h, which may not be the same at each occurrence.

2. Nonconforming element approximations of the corresponding source problem

Consider the source problem (4) associated with (1), and the approximate source problem (5) associated with (3):

$$u \in H^1(\Omega),$$
$$a(u,v) = b(f,v), \quad \forall v \in H^1(\Omega). \tag{4}$$
$$u_h \in S^h,$$
$$a_h(u_h,v) = b(f,v), \quad \forall v \in S^h. \tag{5}$$

Having in mind that $H^s(\Omega)$ denotes the Sobolev space with real order s on Ω, $\|\cdot\|_s$ is the norm on $H^s(\Omega)$ and $H^0(\Omega) = L_2(\Omega)$.

Lemma 2.1. *Assume $f \in L_2(\partial\Omega)$, then $u \in H^{\frac{3}{2}}(\Omega)$ and*

$$\|u\|_{\frac{3}{2}} \leqslant C_p \|f\|_b. \tag{6}$$

Further, assume that $f \in H^{\frac{1}{2}}(\partial\Omega_j)$ $(j = 1, 2, \ldots, J)$ and $\partial\Omega = \bigcup_{j=1}^{J} \partial\Omega_j$, where $\partial\Omega_j$ are straight segments. Then $u \in H^2(\Omega)$ and

$$\|u\|_2 \leqslant C_p \sum_{j=1}^{J} \|f\|_{\frac{1}{2},\partial\Omega_j}. \tag{7}$$

Proof. See (4.10) in [10] and Proposition 4.4 in [9] for details. □

From Lemma 7.1.1 in [25], the following lemma likewise holds.

Lemma 2.2. *For any $w \in H^r(K)$,*

$$\int_{\partial K} |w|^2\, ds \leqslant C\left\{h_K^{-1}\|w\|_{0,K}^2 + h_K^{2r-1}|w|_{r,K}^2\right\} \quad \left(\frac{1}{2} \leqslant r \leqslant 1\right),$$

where the positive constant C is independent of w and the diameter h_K of K.

Proof. Using the trace theorem over the reference element $\hat{K}$, we have

$$\int_{\partial K} |w|^2\, ds = \int_{\partial\hat{K}} |\hat{w}|^2 \frac{|\partial K|}{|\partial\hat{K}|}\, d\hat{s} \leqslant C h_K \|\hat{w}\|_{0,\partial\hat{K}}^2 \leqslant C h_K \|\hat{w}\|_{r,\hat{K}}^2$$
$$= C h_K\left\{\|\hat{w}\|_{0,\hat{K}}^2 + |\hat{w}|_{r,\hat{K}}^2\right\} \leqslant C\left\{h_K^{-1}\|w\|_{0,K}^2 + h_K^{2r-1}|w|_{r,K}^2\right\}, \quad \forall w \in H^r(K). \qquad \square$$

Let F be an edge of K, define:

$$P_0^F f = \frac{1}{|F|}\int_F f\, ds, \qquad R_0^F f = f - P_0^F f, \tag{8}$$

$$P_0^K f = \frac{1}{|K|}\int_K f\, dx, \qquad R_0^K f = f - P_0^K f. \tag{9}$$

Lemma 2.3. *Let $w \in H^s(K)$. Then*

$$\left\| R_0^K w \right\|_{0,K} \leqslant Ch^s |w|_{s,K} \quad (0 \leqslant s \leqslant 1). \tag{10}$$

Proof. From the interpolation theory, we easily see that (10) is valid. □

Lemma 2.4. *For any $f \in L_2(F)$,*

$$\begin{aligned} &\left\| P_0^F f \right\|_{0,F} \leqslant \|f\|_{0,F}, \\ &\left\| R_0^F f \right\|_{0,F} \leqslant \|f - v\|_{0,F}, \quad \forall v \in P_0(K). \end{aligned} \tag{11}$$

Proof. It is not difficult to prove that $P_0^F : L_2(F) \to P_0(F)$ is an orthogonal projection operator. Thus the above inequalities hold. □

Let $E_h(u, v) = a_h(u, v) - b(f, v)$ be the consistency error term of nonconforming finite element.

Theorem 2.1. *Let u is a solution of (4) and $u \in H^2(\Omega)$, then $E_h(u, v)$ can be estimated by*

$$E_h(u, v) \leqslant Ch|u|_2 \left(\sum_K |v|_{1,K}^2 \right)^{\frac{1}{2}}, \quad \forall v \in H^1(\Omega) + S^h. \tag{12}$$

Let u be the solution of (4) and $u \in H^{\frac{3}{2}}(\Omega)$, then the following estimate holds.

$$E_h(u, v) \leqslant Ch^{\frac{1}{2}} |u|_{\frac{3}{2}} \left(\sum_K |v|_{1,K}^2 \right)^{\frac{1}{2}}, \quad \forall v \in H^1(\Omega) + S^h, \tag{13}$$

where the positive constant C is independent of u and h.

Proof. Here we use the standard method (see, for example [25], §7.2.1).

By Green's formula, we have

$$\begin{aligned} E_h(u, v) = a_h(u, v) - b(f, v) &= \sum_{K \in \pi_h} \int_K \nabla u \nabla v + uv \, dx - \int_{\partial\Omega} f v \, ds \\ &= \int_\Omega (-\Delta u + u) v \, dx + \sum_{K \in \pi_h} \int_{\partial K} \frac{\partial u}{\partial \nu} v \, ds - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, ds. \end{aligned}$$

Since u is a solution of (4), we have $\int_\Omega (-\Delta u + u) v \, dx = 0$. Thus,

$$\begin{aligned} E_h(u, v) &= \sum_{k \in \pi_h} \int_{\partial K} \frac{\partial u}{\partial \nu} v \, ds - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, ds \\ &= \sum_{K \in \pi_h} \int_{\partial K} \sum_{i=1}^2 \partial_i u v \nu_i \, ds - \int_{\partial\Omega} \sum_{i=1}^2 \partial_i u v \nu_i \, ds \\ &= \sum_{i=1}^2 \sum_{F \not\subset \partial\Omega} \int_F (\partial_i u v|_{K^+} - \partial_i u v|_{K^-}) \nu_i^+ \, ds \\ &= \sum_{i=1}^2 \sum_{F \not\subset \partial\Omega} \int_F \partial_i u [v] \nu_i^+ \, ds, \end{aligned} \tag{14}$$

where $[v]$ denotes the jump of v on F and $[v] = (v|_{K^+} - v|_{K^-})|_F$.

Since $v \in H^1(\Omega) + S^h$ and functions in S^h is integrally continuous at F, $\int_F [v] \, ds = 0$.

Now we estimate $\int_F \partial_i u [v] \nu_i^+ \, ds$, $F \not\subset \partial\Omega$ on the right-hand side of (14).

From (9), we have

$$\begin{aligned}\int_F \partial_i u[v]\nu_i^+\,ds &= \nu_i^+\int_F \partial_i u[v]\,ds = \nu_i^+\int_F R_0^K(\partial_i u)[v]\,ds + \nu_i^+\int_F P_0^K(\partial_i u)[v]\,ds\\ &= \nu_i^+\int_F R_0^K(\partial_i u)[v]\,ds + \nu_i^+ P_0^K(\partial_i u)\int_F [v]\,ds = \nu_i^+\int_F R_0^K(\partial_i u)[v]\,ds\\ &= \nu_i^+\int_F R_0^K(\partial_i u)R_0^F\big([v]\big)\,ds + \nu_i^+\int_F R_0^K(\partial_i u)P_0^F\big([v]\big)\,ds. \end{aligned} \tag{15}$$

Note that

$$P_0^F\big([v]\big) = \frac{1}{|F|}\int_F [v]\,ds = 0.$$

After we take it in (15) and make use of Schwarz's inequality, we get

$$\begin{aligned}\left|\int_F \partial_i u[v]\nu_i^+\,ds\right| &= \left|\nu_i^+\int_F R_0^K(\partial_i u)R_0^F\big([v]\big)\,ds\right|\\ &\leqslant \left\{\int_F \big(R_0^K(\partial_i u)\big)^2\,ds\right\}^{\frac12}\left\{\int_F \big(R_0^F\big([v]\big)\big)^2\,ds\right\}^{\frac12}. \end{aligned} \tag{16}$$

By Lemma 2.2 with $r=1$ and (10) with $s=1$, we obtain

$$\begin{aligned}\int_F \big(R_0^K(\partial_i u)\big)^2\,ds &\leqslant C\big\{h_K^{-1}\big\|R_0^K(\partial_i u)\big\|_{0,K}^2 + h_K\big|R_0^K(\partial_i u)\big|_{1,K}^2\big\}\\ &\leqslant Ch_K|u|_{2,K}^2. \end{aligned} \tag{17}$$

From (11), Lemma 2.2 with $r=1$ and (10) with $s=1$, we derive

$$\begin{aligned}\int_F \big(R_0^F\big([v]\big)\big)^2\,ds &= \int_F \big\{[v] - P_0^F\big([v]\big)\big\}^2\,ds\\ &= \int_F \big\{\big(v^+ - P_0^F(v^+)\big) - \big(v^- - P_0^F(v^-)\big)\big\}^2\,ds\\ &\leqslant 2\left\{\int_F \big(v^+ - P_0^F(v^+)\big)^2\,ds + \int_F \big(v^- - P_0^F(v^-)\big)^2\,ds\right\}\\ &\leqslant 2\left\{\int_F \big(v^+ - P_0^{K^+}(v^+)\big)^2\,ds + \int_F \big(v^- - P_0^{K^-}(v^-)\big)^2\,ds\right\}\\ &\leqslant C\big\{\big(h_{K^+}^{-1}\big\|R_0^{K^+}(v^+)\big\|_{0,K^+}^2 + h_{K^+}\big|R_0^{K^+}(v^+)\big|_{1,K^+}^2\big) + \big(h_{K^-}^{-1}\big\|R_0^{K^-}(v^-)\big\|_{0,K^-}^2 + h_{K^-}\big|R_0^{K^-}(v^-)\big|_{1,K^-}^2\big)\big\}\\ &\leqslant C\big\{h_{K^+}|v^+|_{1,K^+}^2 + h_{K^-}|v^-|_{1,K^-}^2\big\}, \end{aligned} \tag{18}$$

where $v^+ = v|_{K^+}$, $v^- = v|_{K^-}$. Substituting (17) and (18) into (16), we get

$$\left|\int_F \partial_i u[v]\nu_i^+\,ds\right| \leqslant Ch_K|u|_{2,K}|v|_{1,K^+\cup K^-}. \tag{19}$$

Thus, substituting (19) into (14), we have

$$E_h(u,v) \leqslant Ch\sum_K |u|_{2,K}|v|_{1,K} \leqslant Ch|u|_{2,\Omega}\left(\sum_K |v|_{1,K}^2\right)^{\frac12}.$$

In other words, (12) holds. And in a similar way, we obtain (13). □

For Crouzeix–Raviart element and Q_1^{rot} element, define interpolation operator $I_h: H^1(\Omega)\to S^h$ by

$$\int_F I_h u\,ds = \int_F u\,ds \quad \forall F, \forall u\in H^1(\Omega).$$

For EQ_1^{rot} element, define interpolation operator $I_h : H^1(\Omega) \to S^h$ by the above equality and

$$\int_K I_h u \, dx = \int_K u \, dx \quad \forall K \in \pi_h, \ \forall u \in H^1(\Omega).$$

According to the interpolation theory (see [13]), we have

$$\|u_j - I_h u_j\|_0 \leqslant C h^{1+r} |u_j|_{1+r},$$
$$\|u_j - I_h u_j\|_h \leqslant C h^r |u_j|_{1+r}, \quad 0 < r \leqslant 1.$$

Theorem 2.2. *Let $u \in H^2(\Omega)$. Then*

$$\|u_h - u\|_h \leqslant C h |u|_2. \tag{20}$$

Further, assume that (6) *be valid, then*

$$\|u_h - u\|_b \leqslant C h^{\frac{3}{2}} |u|_2. \tag{21}$$

Proof. From the Strang lemma, we have

$$\|u - u_h\|_h \leqslant C\left(\inf_{v \in S^h} \|u - v\|_h + \sup_{w_h \in S^h,\, w_h \neq 0} \frac{E_h(u, w_h)}{\|w_h\|_h}\right). \tag{22}$$

According to the interpolation error estimate, we get

$$\inf_{v \in S^h} \|u - v\|_h \leqslant \|u - I_h u\|_h \leqslant C h |u|_2.$$

By (12) we obtain

$$\sup_{w_h \in S^h,\, w_h \neq 0} \frac{E_h(u, w_h)}{\|w_h\|_h} \leqslant C h |u|_2.$$

Substituting the above two inequalities into (22), we get (20).

According to Nitsche (1974), Lascaux and Lesaint (1975) (see [13]), we have

$$\|u - u_h\|_b \leqslant \sup_{g \in L_2(\partial\Omega)} \frac{1}{\|g\|_b} \inf_{v \in S^h} \left\{ C\|u - u_h\|_h \|\varphi - v\|_h + E_h(u, \varphi - v) + E_h(\varphi, u - u_h) \right\}, \tag{23}$$

where for each $g \in L_2(\partial\Omega)$, $\varphi \in H^{\frac{3}{2}}(\Omega)$ is the unique solution of the following variational problem

$$a(v, \varphi) = (g, v), \quad \forall v \in H^1(\Omega).$$

Using the interpolation error estimate, we derive

$$\|\varphi - I_h \varphi\|_h \leqslant C h^{\frac{1}{2}} \|\varphi\|_{\frac{3}{2}} \leqslant C h^{\frac{1}{2}} \|g\|_b,$$

and thus, by (12), we get

$$E_h(u, \varphi - I_h \varphi) \leqslant C h |u|_2 \|\varphi - I_h \varphi\|_h \leqslant C h^{\frac{3}{2}} |u|_2 \|g\|_b.$$

Combining (13) and (20), we have

$$E_h(\varphi, u - u_h) \leqslant C h^{\frac{3}{2}} |u|_2 \|g\|_b.$$

Taking $v = I_h \varphi$ in (23), and using (20) and the above three inequalities, we obtain (21). □

Theorem 2.3. *Let $u \in H^{\frac{3}{2}}(\Omega)$. Then*

$$\|u_h - u\|_h \leqslant C h^{\frac{1}{2}} |u|_{\frac{3}{2}}. \tag{24}$$

Further, assume that (6) *be valid, then*

$$\|u_h - u\|_b \leqslant C h |u|_{\frac{3}{2}}. \tag{25}$$

Proof. The proof is exactly the same as that given for Theorem 2.2 except that we use (13) instead of (12). □

3. Nonconforming element approximations of the Steklov eigenvalue problem

At first, we transform (2) and (3) into the operator forms.

Note that $a(\cdot,\cdot)$ is coercive. Using the source problem (4) associated with (2), we define the operator $A: L_2(\partial\Omega) \to H^{\frac{3}{2}}(\Omega) \subset H^1(\Omega)$ by $a(Af, v) = b(f, v)$, $\forall v \in H^1(\Omega)$. Define $T: L_2(\partial\Omega) \to H^1(\partial\Omega)$ by $Tf = (Af)'$, where the prime denotes the restriction to $\partial\Omega$.

Bramble and Osborn [10] proved that (2) has the operator form:

$$Tw = \mu w. \tag{26}$$

Namely, if $(\mu, w) \in \mathbb{R} \times L_2(\partial\Omega)$ is an eigenpair of (26), then (λ, Aw) is an eigenpair of (2), $\lambda = \frac{1}{\mu}$; conversely, if (λ, u) is an eigenpair of (2), then (μ, u') is an eigenpair of (26), $\mu = \frac{1}{\lambda}$.

Since $a_h(\cdot,\cdot)$ is uniformly elliptic with respect to h, the approximate source problem (5) associated with (3) is uniquely solvable. Thus, define $A_h: L_2(\partial\Omega) \to S^h$ by $a_h(A_h f, v) = b(f, v)$, $\forall v \in S^h$. Define $T_h: L_2(\partial\Omega) \to \delta S^h \subset L_2(\partial\Omega)$ by $T_h f = (A_h f)'$. Let (λ_h, w_h) be an eigenpair of (3). According to the definition of A_h and (3), we derive

$$a_h(A_h w_h', v) = b(w_h', v) = b(w_h, v) = \frac{1}{\lambda_h} a_h(w_h, v), \quad \forall v \in S^h.$$

Using the fact that $v \in S^h$ is arbitrary, we infer that $A_h w_h' = \frac{1}{\lambda_h} w_h$. And using the definition of T_h, we obtain $\lambda_h (A_h w_h')' = \lambda_h T_h w_h'$. Then,

$$T_h w_h' = \frac{1}{\lambda_h} w_h'; \tag{27}$$

i.e. (μ_h, w_h') is an eigenpair of (27), $\mu_h = \frac{1}{\lambda_h}$. Conversely, suppose that (μ_h, w_h) is an eigenpair of (27), then

$$\begin{aligned} a_h(A_h w_h, v) &= b(w_h, v) = \frac{1}{\mu_h} b(T_h w_h, v) \\ &= \frac{1}{\mu_h} b\big((A_h w_h)', v\big) = \frac{1}{\mu_h} b(A_h w_h, v) \quad \forall v \in S^h; \end{aligned}$$

i.e. $(\lambda_h, A_h w_h)$ is an eigenpair of (3), $\mu_h = \frac{1}{\lambda_h}$. Consequently, (3) has the operator form (27).

Next, we prove that T and $T_h: L_2(\partial\Omega) \to L_2(\partial\Omega)$ are self-adjoint operators and $\|T_h - T\|_b \to 0$ as $h \to 0$.

Since $\forall f, g \in L_2(\partial\Omega)$,

$$b(Tf, g) = b(g, Tf) = a(Ag, Af) = a(Af, Ag) = b(f, Ag) = b(f, Tg),$$

$T: L_2(\partial\Omega) \to L_2(\partial\Omega)$ is a self-adjoint operator. In a similar way, we can also prove that $T_h: L_2(\partial\Omega) \to L_2(\partial\Omega)$ is a self-adjoint operator.

Moreover, because of (25) we have

$$\begin{aligned} \|T_h - T\|_b &= \sup_{g \in L_2(\partial\Omega)} \frac{\|T_h g - Tg\|_b}{\|g\|_b} = \sup_{g \in L_2(\partial\Omega)} \frac{\|A_h g - Ag\|_b}{\|g\|_b} \\ &\leqslant \sup_{g \in L_2(\partial\Omega)} \frac{Ch\|Ag\|_{\frac{3}{2}}}{\|g\|_b} \leqslant Ch \to 0 \quad (h \to 0). \end{aligned} \tag{28}$$

This shows that $T_h \to T$ in norm as $h \to 0$. Notice that T_h is a finite rank operator. Thus T is a compact operator.

We are now in a position to prove the main results of this section.

Let λ_j denote the j-th eigenvalue of T, and $M(\lambda_j)$ be the space spanned by eigenfunctions of T corresponding to λ_j and $\delta M(\lambda_j)$ denote the space of restrictions to $\partial\Omega$ of functions in $M(\lambda_j)$.

The following properties of the order-preserving convergence and error estimates were discussed in [7], [16] and [27].

Lemma 3.1. *Suppose that T and T_h are self-adjoint and completely continuous, and $\|T - T_h\|_b \to 0$ $(h \to 0)$. Let λ_j be the j-th eigenvalue of T with algebraic multiplicity q and $\lambda_{j,h}$ be the j-th eigenvalue of T_h. Then*

$$\lambda_{j,h} \to \lambda_j \quad (h \to 0). \tag{29}$$

Let $u_{j,h}$ be an eigenfunction of T_h corresponding to $\lambda_{j,h}$ with $\|u_{j,h}\|_b = 1$, then there exists $u_j \in M(\lambda_j)$ with $\|u_j\|_b = 1$, such that

$$\|u_{j,h} - u_j\|_b \leqslant C \big\| (T - T_h)|_{M(\lambda_j)} \big\|_b. \tag{30}$$

Proof. See, for example, [27]. □

Theorem 3.1. *Let λ_j be the j-th eigenvalue of* (2)*, and $\lambda_{j,h}$ be the j-th eigenvalue of* (3)*. Let $u_{j,h}$ be an eigenfunction corresponding to $\lambda_{j,h}$ with $\|u_{j,h}\|_b = 1$. Then there exists $u_j \in M(\lambda_j)$ with $\|u_j\|_b = 1$, such that*

$$\lambda_{j,h} - \lambda_j = \frac{\lambda_j \lambda_{j,h}}{b(u_j, u_{j,h})} b\big((T - T_h)u_j, u_j\big) + R_1, \tag{31}$$

$$\|u_{j,h} - u_j\|_b \leqslant C\lambda_j^2 \big\|(T - T_h)u_j\big\|_b, \tag{32}$$

$$\|u_{j,h} - u_j\|_h = \lambda_j \|Au_j - A_h u_j\|_h + R_2, \tag{33}$$

where $|R_1| \leqslant C\|(T - T_h)u_j\|_b^2$, $|R_2| \leqslant C\|(T - T_h)u_j\|_b$.

Proof. Combining the above Lemma 3.1 and Lemma 1 in [26], we have

$$\|u_{j,h} - u_j\|_b \leqslant C\lambda_j^2 \big\|(T - T_h)u_{j,h}\big\|_b.$$

A simple calculation shows that

$$\begin{aligned}\|u_{j,h} - u_j\|_b &\leqslant C\lambda_j^2 \big\|(T - T_h)(u_{j,h} - u_j + u_j)\big\|_b \\ &\leqslant C\lambda_j^2 \big(\big\|(T - T_h)u_j\big\|_b + \|T - T_h\|_b \|u_{j,h} - u_j\|_b\big),\end{aligned}$$

which, together with $\|T - T_h\|_b \to 0 (h \to 0)$, yields (32). Since

$$\begin{aligned}b(Tu_{j,h} - T_h u_{j,h}, u_j) &= b(Tu_{j,h}, u_j) - b\big(\lambda_{j,h}^{-1} u_{j,h}, u_j\big) \\ &= b(u_{j,h}, Tu_j) - b\big(\lambda_{j,h}^{-1} u_{j,h}, u_j\big) = \big(\lambda_j^{-1} - \lambda_{j,h}^{-1}\big) b(u_{j,h}, u_j),\end{aligned}$$

we have

$$\begin{aligned}\lambda_{j,h} - \lambda_j &= \frac{\lambda_j \lambda_{j,h}}{b(u_j, u_{j,h})} b\big((T - T_h)u_{j,h}, u_j\big) \\ &= \frac{\lambda_j \lambda_{j,h}}{b(u_j, u_{j,h})} \big(b\big((T - T_h)u_j, u_j\big) + b\big((T - T_h)(u_{j,h} - u_j), u_j\big)\big) \\ &\equiv \frac{\lambda_j \lambda_{j,h}}{b(u_j, u_{j,h})} b\big((T - T_h)u_j, u_j\big) + R_1.\end{aligned}$$

Moreover, using the facts that T and T_h are symmetric, $\lambda_{j,h} \to \lambda_j$, and (32) is valid, we can infer that

$$\begin{aligned}|R_1| &= \left|\frac{\lambda_j \lambda_{j,h}}{b(u_j, u_{j,h})} b\big((T - T_h)(u_{j,h} - u_j), u_j\big)\right| \leqslant C\big|b\big(u_{j,h} - u_j, (T - T_h)u_j\big)\big| \\ &\leqslant C\|u_{j,h} - u_j\|_b \big\|(T - T_h)u_j\big\|_b \leqslant C\big\|(T - T_h)u_j\big\|_b^2.\end{aligned}$$

Therefore, we get (31) immediately.

From (3) and the definitions of T_h and A_h, we get

$$\begin{aligned}\|u_{j,h} - \lambda_j A_h u_j\|_h^2 &= a_h(u_{j,h} - \lambda_j A_h u_j, u_{j,h} - \lambda_j A_h u_j) \\ &= b(\lambda_{j,h} u_{j,h} - \lambda_j u_j, u_{j,h} - \lambda_j A_h u_j) \leqslant \|\lambda_{j,h} u_{j,h} - \lambda_j u_j\|_b \|u_{j,h} - \lambda_j A_h u_j\|_b \\ &\leqslant \big(\|\lambda_{j,h} u_{j,h} - \lambda_j u_j\|_b + \|u_{j,h} - \lambda_j A_h u_j\|_b\big)^2,\end{aligned}$$

which, together with (31) and (32), yields

$$\|u_{j,h} - \lambda_j A_h u_j\|_h \leqslant C\big\|(T - T_h)u_j\big\|_b.$$

Let $R_2 = \|u_{j,h} - u_j\|_h - \|\lambda_j A_h u_j - \lambda_j A u_j\|_h$. Then, using the triangle inequality and the above inequality, we deduce that

$$\begin{aligned}|R_2| &\leqslant \big\|u_{j,h} - u_j - (\lambda_j A_h u_j - \lambda_j A u_j)\big\|_h \\ &= \|u_{j,h} - \lambda_j A_h u_j\big\|_h \leqslant C\big\|(T - T_h)u_j\big\|_b.\end{aligned}$$

In other words, we obtain (33). □

Remark 3.1. Theorem 3.1 estimates the errors in nonconforming finite element method approximation of eigenpairs in terms of error estimates for the corresponding source problems.

Remark 3.2. The proof of Theorem 3.1 shows that: for spectral approximation T_h of selfadjoint completely continuous operator T in Hilbert space, if $\|T_h - T\|_b \to 0$ as $h \to 0$, then (31) and (32) hold; for second or fourth order eigenvalue problems of selfadjoint elliptic differential operators, if T_h converges to T in norm as $h \to 0$, then (31)–(33) are valid.

Theorem 3.2. *Under the assumptions of Theorem* 3.1, *the following error estimates hold*:

$$|\lambda_{j,h} - \lambda_j| \leqslant C\lambda_j^2 h^2 \|u_j\|^2_{\frac{1}{2},\partial\Omega}, \tag{34}$$

$$\|u_j - u_{j,h}\|_b \leqslant C\lambda_j^2 h^{\frac{3}{2}} \|u_j\|_{\frac{1}{2},\partial\Omega}, \tag{35}$$

$$\|u_j - u_{j,h}\|_h \leqslant C\lambda_j h \|u_j\|_{\frac{1}{2},\partial\Omega}, \tag{36}$$

where C is a constant independent of h and λ_j.

Proof. From (20), (21) and (7), we have

$$\|Au_j - A_h u_j\|_h \leqslant Ch\|u_j\|_{\frac{1}{2},\partial\Omega}, \tag{37}$$

$$\|Tu_j - T_h u_j\|_b \leqslant Ch^{\frac{3}{2}} \|u_j\|_{\frac{1}{2},\partial\Omega}. \tag{38}$$

By an easy calculation, we get

$$\begin{aligned} b(Tu_j - T_h u_j, u_j) &= b(Tu_j, u_j) - b(T_h u_j, u_j) \\ &= a_h(Au_j, Au_j) - a_h(A_h u_j, A_h u_j) \\ &= a_h(Au_j - A_h u_j, Au_j) + a_h(A_h u_j, Au_j - A_h u_j) \\ &= 2a_h(Au_j - A_h u_j, Au_j) - a_h(Au_j - A_h u_j, Au_j - A_h u_j). \end{aligned}$$

And we have

$$\begin{aligned} a_h(Au_j - A_h u_j, Au_j) &= a_h(Au_j - A_h u_j, Au_j) - b(Au_j - A_h u_j, u_j) + b(Tu_j - T_h u_j, u_j) \\ &= E_h(Au_j - A_h u_j, Au_j) + b(Tu_j - T_h u_j, u_j). \end{aligned}$$

Then we have

$$b(Tu_j - T_h u_j, u_j) = -2E_h(Au_j - A_h u_j, Au_j) + a_h(Au_j - A_h u_j, Au_j - A_h u_j),$$

which, together with (12), (20) and (7), yields

$$b(Tu_j - T_h u_j, u_j) \leqslant Ch^2 \|u_j\|^2_{\frac{1}{2},\partial\Omega}. \tag{39}$$

Therefore, substituting (38) and (39) into (31), we obtain (34). Similarly, substituting (38) into (32), and (37) and (38) into (33), we obtain (35) and (36), respectively. □

4. Lower bounds of eigenvalues

Armentano and Duran [5], Lin and Lin [21], Yang [26] and Zhang et al. [28] analyzed the lower bounds for eigenvalues of the vibrating membrane problem and the vibrating plate problem by nonconforming finite element methods. Whereas, in this section, we study the lower bounds of eigenvalues of the Steklov eigenvalue problem using nonconforming finite element methods.

Consider eigenvalue problem (1).

From Armentano and Duran [5] and Zhang et al. [28], the following lemma holds.

Lemma 4.1. *Let* $(\lambda_j, u_j) \in \mathbb{R} \times H^1(\Omega)$ *be an eigenpair of* (2) *and* $(\lambda_{j,h}, u_{j,h}) \in \mathbb{R} \times S^h$ *be an eigenpair of* (3). *Then*

$$\lambda_j - \lambda_{j,h} = \|u_j - u_{j,h}\|_h^2 - \lambda_{j,h}\|v - u_{j,h}\|_b^2 + \lambda_{j,h}\big(\|v\|_b^2 - \|u_j\|_b^2\big) + 2a_h(u_j - v, u_{j,h}), \quad \forall v \in S^h. \tag{40}$$

Proof. Note that $\|u_j\|_b = \|u_{j,h}\|_b = 1$, $a_h(u_j, u_j) = \lambda_j$ and $a_h(u_{j,h}, u_{j,h}) = \lambda_{j,h}$. Then

$$\begin{aligned} \lambda_j + \lambda_{j,h} &= a_h(u_j - u_{j,h}, u_j - u_{j,h}) + 2a_h(u_j, u_{j,h}) \\ &= \|u_j - u_{j,h}\|_h^2 + 2a_h(v, u_{j,h}) + 2a_h(u_j - v, u_{j,h}) \\ &= \|u_j - u_{j,h}\|_h^2 + 2\lambda_{j,h} b(v, u_{j,h}) + 2a_h(u_j - v, u_{j,h}) \\ &= \|u_j - u_{j,h}\|_h^2 - \lambda_{j,h}\|v - u_{j,h}\|_b^2 + \lambda_{j,h}\|u_{j,h}\|_b^2 + \lambda_{j,h}\|v\|_b^2 + 2a_h(u_j - v, u_{j,h}) \\ &= \|u_j - u_{j,h}\|_h^2 - \lambda_{j,h}\|v - u_{j,h}\|_b^2 + 2\lambda_{j,h} + \lambda_{j,h}\big(\|v\|_b^2 - \|u_j\|_b^2\big) + 2a_h(u_j - v, u_{j,h}), \end{aligned}$$

subtracting $2\lambda_{j,h}$ by parts, we obtain (40). □

Lemma 4.2. *Let $u \in H^2(\Omega)$. Then*

$$\left|a_h(u - I_h u, v)\right| = \left|\int_\Omega (u - I_h u) v \, dx\right| \leqslant Ch^s |u|_2 \|v\|_d, \quad \forall v \in S^h, \tag{41}$$

where $s = 2$, $d = 0$ for Crouzeix–Raviart element and Q_1^{rot} element, and $s = 3$, $d = h$ for EQ_1^{rot} element.

Proof. Using Green's formula, we obtain (see [5], [21], pp. 88–89)

$$\sum_K \int_K \nabla(u - I_h u) \nabla v \, dx = 0, \quad \forall v \in S^h.$$

Thus,

$$\begin{aligned} a_h(u - I_h u, v) &= \sum_K \int_K \nabla(u - I_h u) \nabla v \, dx + \int_\Omega (u - I_h u) v \, dx \\ &= \int_\Omega (u - I_h u) v \, dx \quad \forall v \in S^h. \end{aligned} \tag{42}$$

For Crouzeix–Raviart element and Q_1^{rot} element, we have

$$\left|\int_\Omega (u - I_h u) v \, dx\right| \leqslant Ch^2 |u|_2 \|v\|_0.$$

This shows that (41) holds with $s = 2$ and $d = 0$.

On the other hand, for EQ_1^{rot} element, we introduce a piecewise constant interpolation operator I_0. Then

$$\begin{aligned} \left|\int_\Omega (u - I_h u) v \, dx\right| &= \left|\int_\Omega (u - I_h u) I_0 v \, dx + \int_\Omega (u - I_h u)(v - I_0 v) \, dx\right| \\ &= \left|\int_\Omega (u - I_h u)(v - I_0 v) \, dx\right| \leqslant Ch^3 |u|_2 \|v\|_h. \end{aligned}$$

This shows that (41) holds with $s = 3$ and $d = h$. □

Lemma 4.3. *Let $u_j \in H^2(\Omega)$ be an eigenfunction of* (1)*. Then the following estimate holds*:

$$\|u_j - I_h u_j\|_b \leqslant Ch^{\frac{3}{2}} \|u_j\|_2. \tag{43}$$

Proof. For simplicity, one can assume that $f = \lambda_j u_j$.

Thus, $\forall g \in L_2(\partial\Omega)$,

$$\begin{aligned} b(g, u_j - I_h u_j) &= a(Ag, u_j) - a_h(A_h g, I_h u_j) \\ &= a_h(Ag, u_j - I_h u_j) + a_h(Ag - A_h g, I_h u_j) \\ &= -a_h(Ag - A_h g, u_j - I_h u_j) + a_h(Ag, u_j - I_h u_j) + a_h(Ag - A_h g, u_j) \\ &= -a_h(Ag - A_h g, u_j - I_h u_j) + a_h(Ag, u_j - I_h u_j) \\ &\quad + a_h(Ag - A_h g, u_j) - b(Tg - T_h g, f) + b(Tg - T_h g, f) \\ &= -a_h(Ag - A_h g, u_j - I_h u_j) + a_h(Ag, u_j - I_h u_j) + E_h(Ag - A_h g, u_j) + b(Tg - T_h g, f). \end{aligned} \tag{44}$$

Further, we obtain

$$\begin{aligned} b(g, u_j - I_h u_j) &= -a_h(Ag - A_h g, u_j - I_h u_j) + E_h(Ag, u_j - I_h u_j) + b(g, u_j - I_h u_j) \\ &\quad + E_h(Ag - A_h g, u_j) + b(Tg - T_h g, f). \end{aligned}$$

Consequently,

$$\left|b(Tg - T_h g, f)\right| = \left|-E_h(Ag - A_h g, u_j) - E_h(Ag, u_j - I_h u_j) + a_h(Ag - A_h g, u_j - I_h u_j)\right|$$
$$\leqslant Ch^{\frac{3}{2}} \|u_j\|_2 \|g\|_b.$$

From (24), (42) and (41), we get

$$\left|a_h(Ag, u_j - I_h u_j)\right| = \left|a_h(Ag - A_h g, u_j - I_h u_j) + a_h(A_h g, u_j - I_h u_j)\right|$$
$$\leqslant Ch^{\frac{3}{2}} \|u_j\|_2 \|g\|_b.$$

After we substitute the above two inequalities into (44), and estimate the first term and the third term on the right-hand side of (44) using (12), (24) and (42), we obtain

$$b(g, u_j - I_h u_j) \leqslant Ch^{\frac{3}{2}} \|u_j\|_2 \|g\|_b, \quad \forall g \in L_2(\partial\Omega).$$

Then, we easily see that (43) holds. □

Theorem 4.1. *Under the assumptions of Theorem* 3.1, *if h is sufficiently small, then*

$$\lambda_j - \lambda_{j,h} = \|u_j - u_{j,h}\|_h^2 + 2\int_\Omega (u_j - I_h u_j) u_{j,h}\, dx + R, \quad |R| \leqslant Ch^{\frac{5}{2}}. \tag{45}$$

Proof. Taking $v = I_h u_j$ in (40), we estimate the second, the third and the fourth terms on the right-hand side of (40). From (43) and (35), we have

$$\|I_h u_j - u_{j,h}\|_b \leqslant \|I_h u_j - u_j\|_b + \|u_j - u_{j,h}\|_b \leqslant C\lambda_j^2 h^{\frac{3}{2}} \|u_j\|_{\frac{1}{2},\partial\Omega}.$$

In addition, we introduce the piecewise constant interpolation operator I_0 on $\partial\Omega$.

Then, from (43), we have

$$\left|\|I_h u_j\|_b^2 - \|u_j\|_b^2\right| = \left|\int_{\partial\Omega} (u_j - I_h u_j)\big((u_j + I_h u_j) - I_0(u_j + I_h u_j)\big)\, ds + \int_{\partial\Omega} (u_j - I_h u_j) I_0(u_j + I_h u_j)\, ds\right|$$
$$= \left|\int_{\partial\Omega} (u_j - I_h u_j)\big((u_j + I_h u_j) - I_0(u_j + I_h u_j)\big)\, ds\right|$$
$$\leqslant \|u_j - I_h u_j\|_b \left\|(u_j + I_h u_j) - I_0(u_j + I_h u_j)\right\|_b$$
$$\leqslant Ch^{\frac{3}{2}} \|u_j\|_2 h \|u_j + I_h u_j\|'_{1,\partial\Omega}$$
$$\leqslant Ch^{\frac{5}{2}} \|u_j\|_2^2.$$

Thus, from the previous estimates and (42), we obtain (45). □

Corollary 4.1. *For EQ_1^{rot} element, if $\|u_j - u_{j,h}\|_h \geqslant Ch^2$, then, for h small enough, we have that*

$$\lambda_{j,h} \leqslant \lambda_j. \tag{46}$$

Proof. From (41) and $|R| \leqslant Ch^{\frac{5}{2}}$, we know that the second and the third terms on the right-hand side of (45) are infinitesimals of higher order than the order of the first term. So the sign of the right-hand side of (45) is determined by the first term. Thus, (46) holds. □

Corollary 4.2. *For Crouzeix–Raviart element and Q_1^{rot} element, we assume that there exists a positive constant C_1 independent of h and λ_j such that $\|u_j - u_{j,h}\|_h \geqslant C_1\lambda_j^2 h^2$. Then, for the large eigenvalues λ_j and h small enough, we have that*

$$\lambda_{j,h} \leqslant \lambda_j. \tag{47}$$

Proof. From the assumption and (36), we have

$$C_1\lambda_j^2 h^2 \leqslant \|u_j - u_{j,h}\|_h^2 \leqslant C\lambda_j^2 h^2 \|u_j\|_{\frac{1}{2},\partial\Omega}^2.$$

According to the interpolation error estimate and (7), we have

$$\left|\int_\Omega (u - I_h u) u_{j,h}\, dx\right| \leqslant Ch^2 |u_j|_2 \leqslant Ch^2 \lambda_j \|u_j\|_{\frac{1}{2},\partial\Omega}.$$

Table 1
The result for the Crouzeix–Raviart element.

h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
$\frac{\sqrt{2}}{8}$	0.2401790890	1.482482005	1.484090572	2.058113927
$\frac{\sqrt{2}}{16}$	0.2401040676	1.489779325	1.489999113	2.075646883
$\frac{\sqrt{2}}{32}$	0.2400853285	1.491663183	1.491691799	2.080782122
$\frac{\sqrt{2}}{64}$	0.2400806459	1.492141960	1.492145608	2.082166026
$\frac{\sqrt{2}}{128}$	0.2400794755	1.492262687	1.492263147	2.082524916
$\frac{\sqrt{2}}{256}$	0.2400791829	1.492293003	1.492293061	2.082616282

Table 2
The result for the Q_1^{rot} element.

h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
$\frac{\sqrt{2}}{8}$	0.2402292155	1.490291098	1.490291098	2.058707332
$\frac{\sqrt{2}}{16}$	0.2401165859	1.491625241	1.491625241	2.075731971
$\frac{\sqrt{2}}{32}$	0.2400884571	1.492109485	1.492109485	2.080793868
$\frac{\sqrt{2}}{64}$	0.2400814280	1.492251517	1.492251517	2.082167669
$\frac{\sqrt{2}}{128}$	0.2400796710	1.492289816	1.492289816	2.082525158
$\frac{\sqrt{2}}{256}$	0.2400792318	1.492299752	1.492299752	2.082616321

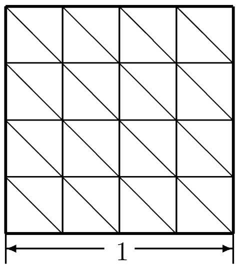

Fig. 1.

Now, we consider the terms on the right-hand side of (45). Clearly, the third term is an infinitesimal of higher order than the order of the first term. As a general rule, the second term is an infinitesimal of the same order as that of the first term. Then, we can conclude that the sign of $\lambda_j - \lambda_{j,h}$ is determined by the coefficients of the first and the second terms. However, the coefficient of the first term can be larger for the large eigenvalues than that of the second term. So, we can obtain the lower bounds of the large eigenvalues. Note that if the second term and the third one are infinitesimals of higher order than the order of the first term, it is easy to see that the sign of $\lambda_j - \lambda_{j,h}$ is determined by the first term. Then we also get the desired result. □

5. Numerical tests

Consider the problem (1), where $\Omega \subset \mathbb{R}^2$ is a unit square domain.

In Fig. 1, we show the initial triangulation of the domain Ω for the Crouzeix–Raviart element. We refine the initial triangulation in a uniform way (each triangle is divided into four similar triangle). For the EQ_1^{rot} element and the Q_1^{rot} element. We set a uniformly square mesh of Ω. Then we compute the first four eigenvalues with Matlab 7.1. Tables 1–3 show the numerical results.

Note that the exact eigenvalues of the problem have not been known yet. Motivated by Armentano and Padra [6], we program using conforming P1 element method with $h = \frac{\sqrt{2}}{576}$ to solve the problem, and obtain the upper bounds of the first four eigenvalues. Similarly, using EQ_1^{rot} element method with $h = \frac{\sqrt{2}}{400}$ and Q_1^{rot} element method with $h = \frac{\sqrt{2}}{512}$, we obtain the lower bounds of the first eigenvalue and the other three eigenvalues respectively. Therefore, the first exact eigenvalue $\lambda_1 \in [0.2400790855, 0.2400791144]$; the second exact eigenvalue $\lambda_2 \in [1.492302282, 1.492305003]$; the third exact eigenvalue $\lambda_3 \in [1.492302282, 1.492305388]$; the fourth exact eigenvalue $\lambda_4 \in [2.082639338, 2.082659329]$.

Table 3
The result for the EQ_1^{rot} element.

h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
$\frac{\sqrt{2}}{8}$	0.2400789992	1.490217735	1.490217735	2.058655383
$\frac{\sqrt{2}}{16}$	0.2400790443	1.491606733	1.491606733	2.075718729
$\frac{\sqrt{2}}{32}$	0.2400790725	1.492104846	1.492104846	2.080790540
$\frac{\sqrt{2}}{64}$	0.2400790819	1.492250357	1.492250357	2.082166835
$\frac{\sqrt{2}}{128}$	0.2400790845	1.492289526	1.492289526	2.082524950
$\frac{\sqrt{2}}{256}$	0.2400790852	1.492299680	1.492299680	2.082616268

Now we use the formula $\lg(\frac{|\lambda_{j,h}-\lambda_j|}{|\lambda_{j,\frac{h}{2}}-\lambda_j|})/\lg 2$ where λ_j are the midpoints of the above intervals, and obtain from Tables 1–3 that the orders of $\lambda_{j,h} \to \lambda_j$ $(h \to 0)$ are about $O(h^2)$ for the nonconforming Crouzeix–Raviart element, the Q_1^{rot} element, and the EQ_1^{rot} element. It is also shown that the EQ_1^{rot} element gives lower bounds for the exact eigenvalues, whereas the nonconforming Crouzeix–Raviart element and the Q_1^{rot} element provide lower bounds for the exact eigenvalues except the smallest eigenvalue. The numerical results coincide with Theorem 3.2, Corollary 4.1, and Corollary 4.2.

Remark 5.1. Alonso and Russo [2] discussed nonconforming Crouzeix–Raviart element approximation of Steklov eigenvalue problems and obtained some important results. However, special features of our work are: firstly, the Q_1^{rot} element and EQ_1^{rot} element are also discussed. Secondly, our results in Section 3 are derived by a different path of [2]. And the optimal order error estimate $\|u_j - u_{j,h}\|_b \leqslant C\lambda_j^2 h^{\frac{3}{2}} \|u_j\|_{\frac{1}{2},\partial\Omega}$ is obtained when Ω is a bounded convex polygonal domain. Thirdly, our results in Section 4 seem to be new.

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