

# The extrapolation of numerical eigenvalues by finite elements for differential operators <sup>☆</sup>



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## ABSTRACT

This paper discusses the extrapolation of numerical eigenvalues by finite elements for differential operators and obtains the following new results: (a) By extending a theorem of eigenvalue error estimate, which was established by Osborn, a new expansion of eigenvalue error is obtained. Many achievements, which are about the asymptotic expansions of finite element methods of differential operator eigenvalue problems, are brought into the framework of functional analysis. (b) The Richardson extrapolation of nonconforming finite elements for multiple eigenvalues and splitting extrapolation of finite elements based on domain decomposition of non-selfadjoint differential operators for multiple eigenvalues are achieved. In addition, numerical examples are provided to support the theoretical analysis.

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## 1. Introduction

The Richardson extrapolation is a well-known technique to construct high-order methods in numerical analysis. It is applicable to many problems, including ordinary or partial differential equations. All these applications are based on the existence of an error expansion for the discrete approximations in a single mesh parameter (see [16]). To produce more accurate approximations for partial differential equations, in the last 30 years, many scholars studied the Richardson extrapolation of finite element methods, e.g. see [1,2,4,9,10,13,24,25,28,33] and the references therein.

As for multidimensional problems, the Richardson extrapolation is costly since it considers just a single parameter. So, the splitting extrapolation, which is based on multivariate expansions with several mesh parameters, appears. Since 1980s, the splitting extrapolation has been developed widely in the numerical analysis community. The splitting extrapolation is a better technique to deal with the so-called curse of dimensionality and is also a highly parallel algorithm (see [20] and the book review [31]). It is especially important that the splitting extrapolation is also applied to the finite element methods, see [7,14,19,27,28], etc.

During the development of the extrapolation of finite element methods, the extrapolation for eigenvalue problems is an attractive issue, e.g. see [5,15,18,19,21,22,24,25,27,28,34]. Especially, [5] studied successfully the extrapolation of conforming finite elements for multiple eigenvalues of selfadjoint differential operator. However, to the best of our knowledge, there has no research on nonconforming finite elements extrapolations for multiple eigenvalues and the finite element extrapolations for multiple eigenvalues of non-selfadjoint differential operator. [22,24] discussed the extrapolation of nonconforming finite element eigenvalues, e.g., the asymptotic expansion of the  $EQ_1^{rot}$  element

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$$\lambda_h - \lambda = -\frac{h_1^2 + h_2^2}{3} \int_{\Omega} (\partial_1 \partial_2 u)^2 + \vartheta(h^4) \quad (1.1)$$

was proved where  $u$  is the limit of  $u_h$ , the eigenfunction corresponding to  $\lambda_h$  (see Theorem 3.1 in [22]). It is an important work. By using this expansion one can extrapolate the simple eigenvalue to obtain high-precision eigenvalues. However, in the case of multiple eigenvalues, from the spectral approximation theory we know that when  $h$  changes, the exact eigenfunction  $u$  which the finite element eigenfunction  $u_h$  approximates to also changes (e.g., see Theorem 7.4 in [3]), thus,  $u$  is related to  $h$  in the expansion. So it cannot be guaranteed that we use this expansion directly to extrapolate and get high-precision eigenvalues. As for the extrapolation of finite elements for non-selfadjoint differential operator eigenvalue problems, including the splitting extrapolation, [27] and [28] proved the asymptotic expansion for simple eigenvalues (see Theorem 2 in [27])

$$\lambda_h - \lambda = \sum_{i=1}^l \beta_i(u) \bar{h}_i^2 + \vartheta(h_0^4), \quad (1.2)$$

where  $\beta_i(u)$  is independent of  $h$ . By this expansion the splitting extrapolation for simple eigenvalues can be achieved. But for multiple eigenvalues,  $\beta_i(u)$  is related to  $h$  in (1.2). When the ascent of  $\lambda$  is larger than 1, not only is  $\beta_i(u)$  related to  $h$  but also the accuracy of  $\lambda_h$  relates to the ascent which was pointed out in [28]. So we cannot use this expansion directly to get high-precision eigenvalues. The simple eigenvalue is a strong condition since the eigenvalue of non-selfadjoint problems is not simple in general and its ascent is probably larger than 1. This paper aims to study the extrapolation of finite elements for multiple eigenvalues including the case that the ascent is equal to or larger than 1.

We develop the previous corresponding investigations and obtain the main results which are in Sections 3 and 4 in this paper. Special works of this paper are as follows:

- (a) In Section 3, we provide an eigenvalue error expansion (see (3.1)). This expansion is a simple extension of the estimate which was established by Osborn (see (2.4) in this paper). In many applications the first term on the right hand side of (3.1)/(2.4) is the dominant term and the second term is of higher order than the first one. The error is determined by the first term. Compared with (2.4), the advantage of (3.1) is that it indicates that the dominant term is effectively the size of the error. It is this feature that leads to the asymptotic formula for the error. Thus we bring the extrapolation of finite elements for differential operator eigenvalue problems into the framework of functional analysis.
- (b) In Section 4, the asymptotic expansion of finite elements for differential operator eigenvalue problems is discussed by using Theorem 3.1. In Section 4.1 we give and prove the asymptotic expansions of  $EQ_1^{rot}$  element for multiple eigenvalues. In Section 4.2, for second-order non-selfadjoint differential operator eigenvalue problems, the splitting extrapolation based on domain decomposition is discussed. We throw off the assumption that  $\lambda$  is a simple eigenvalue in Theorem 2 in [27] and realize the splitting extrapolation of finite elements for multiple eigenvalues.

Besides, in Section 5, some numerical experiments are reported to support our theory.

In this paper,  $C$  denotes a positive constant independent of  $h$ , which may stand for different values at its different occurrences.

## 2. Preliminaries

Let  $X$  be a separable complex Banach space with norm  $\|\cdot\|$  and conjugate pairs  $\langle \cdot, \cdot \rangle$ , respectively. In this section, let  $T : X \rightarrow X$  be a nonzero compact linear operator,  $T_h : X \rightarrow X$  and  $\{T_h\}_{h>0}$  be a family of compact operators, and  $\|T_h - T\| \rightarrow 0$  ( $h \rightarrow 0$ ). Consider the following eigenvalue problem:

$$Tu = \mu u, \quad (2.1)$$

and its approximation

$$T_h u_h = \mu_h u_h. \quad (2.2)$$

We use the eigenpairs of (2.2) to approximate to those of (2.1).

[12] has proved the following Lemma 2.1.

**Lemma 2.1.** *Let  $\{\mu_j\}$  be an enumeration of the nonzero eigenvalues of  $T$ , each multiple according to its multiplicity. Then there exists an enumeration  $\{\mu_{j,h}\}$  of the nonzero eigenvalues of  $T_h$ , with repetitions according to its multiplicity, such that*

$$\mu_{j,h} \rightarrow \mu_j \quad (h \rightarrow 0), \quad j = 1, 2, \dots \quad (2.3)$$

Set  $\lambda_j = \frac{1}{\mu_j}$ ,  $\lambda_{j,h} = \frac{1}{\mu_{j,h}}$ . In some papers  $\mu_j$  and  $\mu_{j,h}$  are called eigenvalues, and  $\lambda_j$  and  $\lambda_{j,h}$  are called characteristic values. In our paper all of these are called eigenvalues.

Let  $\mu_k$  be the  $k$ -th eigenvalue of (2.1) with algebraic multiplicity  $q$ ,  $\mu_k = \mu_{k+1} = \dots = \mu_{k+q-1}$ . Then  $\mu_k$  is also the eigenvalue of  $T'$  with algebraic multiplicity  $q$ , where  $T'$  is the Banach adjoint of  $T$ . Let  $\mu_{k,h}$  be the  $k$ -th eigenvalue of (2.2) and set  $\hat{\mu}_{k,h} = \frac{1}{q} \sum_{j=k}^{k+q-1} \mu_{j,h}$ . Denote by  $\rho(T)$  the resolvent set of  $T$ , and  $\sigma(T)$  the spectrum of  $T$ .

Let  $\Gamma$  be a closed Jordan curve enclosing  $\mu_k$  and  $\Delta$  be a domain enclosed by  $\Gamma$ ,  $\Delta \setminus \mu_k \subset \rho(T)$ . Let  $h$  be sufficiently small, then  $\Gamma \subset \rho(T_h)$ . The definitions of spectral projection and the ascent are as follows (see [3,8,29]):

**Spectral projection.** Denote  $R_z(T) = (z - T)^{-1}$ ,  $R_z(T_h) = (z - T_h)^{-1}$ , define

$$E = E(\mu_k) = \frac{-1}{2i\pi} \int_{\Gamma} R_z(T) dz,$$

$$E_h = E_h(\mu_k) = \frac{-1}{2i\pi} \int_{\Gamma} R_z(T_h) dz.$$

We call  $E$  the spectral projection associated with  $T$  and  $\mu_k$ , and  $E_h$  the spectral projection associated with  $T_h$  and the eigenvalues of  $T_h$  which converge to  $\mu_k$ . Let  $T'_h$  be the Banach adjoint of  $T_h$ . Similarly, we can define the spectral projection  $E'$  associated with  $T'$  and  $\mu_k$ , and  $E'_h$  associated with  $T'_h$  and the eigenvalues of  $T'_h$  which converge to  $\mu_k$ .

**Ascent, generalized eigenvector.** There exists the smallest integer  $\alpha$ , called the ascent of  $\mu_k - T$ , such that the null space  $\ker((\mu_k - T)^\alpha) = \ker((\mu_k - T)^{\alpha+1})$ . The vectors in  $\ker((\mu_k - T)^\alpha)$  are called generalized eigenvectors of  $T$  corresponding to  $\mu_k$ . Likewise, the ascent and generalized eigenvectors of  $\mu_{k,h} - T_h$ ,  $\mu_k - T'$  and  $\mu_{k,h} - T'_h$  can be defined.

We denote  $R(E)$ ,  $R(E_h)$ ,  $R(E')$  and  $R(E'_h)$  as the image spaces of  $E$ ,  $E_h$ ,  $E'$  and  $E'_h$ , respectively. Then  $R(E) = \ker((\mu_k - T)^\alpha)$ ,  $R(E_h) = \sum_{\mu_h \in \sigma(\mu_k)} \ker((\mu_h - T_h)^{l_{\mu_h}})$ , where  $\sigma(\mu_k)$  is the set of eigenvalues of  $T_h$  which converge to  $\mu_k$  and  $l_{\mu_h}$  is the ascent of  $\mu_h - T_h$ .

Let  $(T - T_h)|_{R(E)}$  and  $(T' - T'_h)|_{R(E')}$  denote the restriction of  $T - T_h$  to  $R(E)$  and  $T' - T'_h$  to  $R(E')$ , respectively.

[3,29] have proved the following error estimate:

**Lemma 2.2.** Let  $\varphi_k, \varphi_{k+1}, \dots, \varphi_{k+q-1}$  be a basis of  $R(E)$  and  $\varphi'_k, \varphi'_{k+1}, \dots, \varphi'_{k+q-1}$  be the dual basis in  $R(E')$ , then

$$|\mu_k - \hat{\mu}_{k,h}| \leq \frac{1}{q} \sum_{j=k}^{k+q-1} |\langle (T - T_h)\varphi_j, \varphi'_j \rangle| + C \|(T - T_h)|_{R(E)}\| \|(T' - T'_h)|_{R(E')}\|. \quad (2.4)$$

Lemma 2.2 is a classical result of spectral approximation. Most results about the convergence and the error estimates of finite element methods for differential operator eigenvalue problems were established on the basis of it. One of our main works is to extend (2.4), and improve ' $\leq$ ' to be ' $=$ '.

### 3. An error expansion on spectral approximation

By the argument used in Theorem 7.2 of [3], we obtain the following error expansion.

**Theorem 3.1.** Suppose that  $T, T_h : X \rightarrow X$  are compact linear operators,  $\|T_h - T\| \rightarrow 0$  ( $h \rightarrow 0$ ). Let  $\mu_k$  be a nonzero eigenvalue of  $T$  with algebraic multiplicity  $q$ . Let  $\varphi_k, \varphi_{k+1}, \dots, \varphi_{k+q-1}$  be a basis of  $R(E)$ , and let  $\varphi'_k, \varphi'_{k+1}, \dots, \varphi'_{k+q-1}$  be the dual basis in  $R(E')$ , as defined in the proof to follow. Then there exists a constant  $C$ , independent of  $h$ , such that

$$\mu_k - \hat{\mu}_{k,h} = \frac{1}{q} \sum_{j=k}^{k+q-1} \langle (T - T_h)\varphi_j, \varphi'_j \rangle + R, \quad (3.1)$$

$$|R| \leq C \|(T - T_h)|_{R(E)}\| \|(T' - T'_h)|_{R(E')}\|. \quad (3.2)$$

**Proof.** By the definitions of  $E$  and  $E_h$ , we have for any  $f \in R(E)$

$$\begin{aligned} \|E_h f - E f\| &= \left\| \frac{-1}{2i\pi} \int_{\Gamma} (R_z(T_h) - R_z(T)) dz f \right\| \\ &= \left\| \frac{-1}{2i\pi} \int_{\Gamma} R_z(T_h)(T - T_h)R_z(T) dz f \right\| \end{aligned}$$

$$\begin{aligned}
&= \left\| \frac{-1}{2i\pi} \lim_{d(\pi) \rightarrow 0} \sum_{k=1}^m R_{\xi_k}(T_h)(T - T_h)R_{\xi_k}(T)\Delta z_k f \right\| \\
&\leq \frac{1}{2\pi} \text{length}(\Gamma) \sup_{z \in \Gamma} \|R_z(T_h)\| \| (T - T_h)|_{R(E)} \| \sup_{z \in \Gamma} \|R_z(T)\| \|f\| \\
&\leq C \| (T - T_h)|_{R(E)} \| \|f\|.
\end{aligned}$$

Thus we get

$$\|(E_h - E)|_{R(E)}\| \leq C \|(T - T_h)|_{R(E)}\| \rightarrow 0 \quad (h \rightarrow 0). \quad (3.3)$$

For any  $f \in R(E)$ ,  $E_h f = 0$  implies

$$\|f\| = \|Ef - E_h f\| \leq \|(E_h - E)|_{R(E)}\| \|f\| \rightarrow 0 \quad (h \rightarrow 0),$$

i.e.,  $f = 0$ . So  $E_h|_{R(E)} : R(E) \rightarrow R(E_h)$  is one-to-one. Since  $\dim R(E_h) = \dim R(E) = q$ ,  $E_h|_{R(E)}$  is onto. Hence  $(E_h|_{R(E)})^{-1}$  exists and is defined on  $R(E_h)$ , and we write  $E_h^{-1}$  for  $(E_h|_{R(E)})^{-1}$  for simplicity. For sufficiently small  $h$  and  $f \in R(E)$  with  $\|f\| = 1$ , from (3.3) we have

$$1 - \|E_h f\| = \|Ef\| - \|E_h f\| \leq \|(E - E_h)|_{R(E)}\| \leq \frac{1}{2}.$$

Hence  $\|E_h f\| \geq \frac{1}{2} \|f\|$ . This implies  $E_h^{-1}$  is bounded, and for sufficiently small  $h$ ,  $E_h^{-1}$  is uniformly bounded in  $h$ . Define

$$\hat{T}_h = E_h^{-1} T_h E_h|_{R(E)}. \quad (3.4)$$

Clearly,

$$\hat{T}_h : R(E) \rightarrow R(E).$$

Since  $R(E_h)$  is the invariant subspace of  $T_h$ ,  $E_h E_h^{-1}$  is the identical operator on  $R(E_h)$  and  $E_h^{-1} E_h$  is the identical operator on  $R(E)$ , thus we have  $\sigma(\hat{T}_h) = \{\mu_{k,h}, \dots, \mu_{k+q-1,h}\}$ . And we can see that the algebraic and geometric multiplicity of any  $\mu_{i,h}$  as an eigenvalue of  $\hat{T}_h$  is equal to its algebraic and geometric multiplicity as an eigenvalue of  $T_h$ . Write  $\hat{T} = T|_{R(E)}$ , by the spectral decomposition theorem we have  $\sigma(\hat{T}) = \{\mu_k, \dots, \mu_{k+q-1}\}$ . Thus the traces of  $\hat{T}$  and  $\hat{T}_h$  can be obtained as follows, respectively,

$$\text{trace } \hat{T} = q\mu_k, \quad \text{trace } \hat{T}_h = q\hat{\mu}_{k,h}.$$

Since  $\hat{T}$  and  $\hat{T}_h$  are defined on the same space  $R(E)$ , we have

$$\mu_k - \hat{\mu}_{k,h} = \frac{1}{q} \text{trace}(\hat{T} - \hat{T}_h) = \frac{1}{q} \sum_{j=k}^{k+q-1} \langle (\hat{T} - \hat{T}_h)\varphi_j, \varphi'_j \rangle. \quad (3.5)$$

Here, each  $\varphi'_j$  is an element of  $R(E)'$ , the dual space of  $R(E)$ , and  $\{\varphi'_j\}$  is the dual basis of  $\{\varphi_j\}$ . We extend each  $\varphi'_j$  to  $X$  as follows. By the decomposition  $X = R(E) \oplus \ker(E)$ , for any  $f \in X$  we can write  $f = g + w$  with  $g \in R(E)$  and  $w \in \ker(E)$ . Define

$$\langle f, \varphi'_j \rangle = \langle g, \varphi'_j \rangle.$$

It is obvious that  $\varphi'_j$ , so extended, is bounded, i.e.,  $\varphi'_j \in X'$ . Then by calculation we get

$$\begin{aligned}
\langle f, (\mu - T')^\alpha \varphi'_j \rangle &= \langle (\mu - T)^\alpha f, \varphi'_j \rangle \\
&= \langle (\mu - T)^\alpha g, \varphi'_j \rangle + \langle (\mu - T)^\alpha w, \varphi'_j \rangle = 0.
\end{aligned}$$

Since  $f$  is arbitrary, we have  $(\mu - T')^\alpha \varphi'_j = 0$ . Thus  $\varphi'_j \in R(E')$ ,  $j = k, k+1, \dots, k+q-1$ . From the fact that  $T_h E_h = E_h T_h$  and  $E_h^{-1} E_h$  is the identical operator on  $R(E)$ , we can see that

$$\begin{aligned}
\langle (\hat{T} - \hat{T}_h)\varphi_j, \varphi'_j \rangle &= \langle T\varphi_j - E_h^{-1}T_h E_h \varphi_j, \varphi'_j \rangle \\
&= \langle E_h^{-1}E_h(T - T_h)\varphi_j, \varphi'_j \rangle \\
&= \langle (T - T_h)\varphi_j, \varphi'_j \rangle + \langle (E_h^{-1}E_h - I)(T - T_h)\varphi_j, \varphi'_j \rangle.
\end{aligned} \tag{3.6}$$

Let  $L_h = E_h^{-1}E_h$ , and it can be easily proved that  $L_h$  is the projection operator from  $X$  to  $R(E)$  along  $\ker(E_h)$ . Hence  $L'_h$  is the projection operator from  $X'$  to  $\ker(E_h)^\perp = R(E'_h)$  along  $R(E)^\perp = \ker(E')$ . Thus

$$\langle (L_h - I)(T - T_h)\varphi_j, E'_h \varphi'_j \rangle = \langle (T - T_h)\varphi_j, (L'_h - I)E'_h \varphi'_j \rangle = 0,$$

noting that  $E' \varphi'_j = \varphi'_j$ , we get

$$\langle (L_h - I)(T - T_h)\varphi_j, \varphi'_j \rangle = \langle (L_h - I)(T - T_h)\varphi_j, (E' - E'_h)\varphi'_j \rangle. \tag{3.7}$$

From (3.7), the boundedness of  $L_h$ , and (3.3) (applied to  $T'$  and  $T'_h$ ) we have

$$\begin{aligned}
|\langle (L_h - I)(T - T_h)\varphi_j, \varphi'_j \rangle| &\leq \left( \sup_h \|L_h - I\| \right) \| (T - T_h)|_{R(E)} \| \| (E' - E'_h)|_{R(E')} \| \|\varphi_j\| \cdot \|\varphi'_j\| \\
&\leq C \| (T - T_h)|_{R(E)} \| \| (T' - T'_h)|_{R(E')} \|.
\end{aligned} \tag{3.8}$$

Finally, denote  $R = \frac{1}{q} \sum_{j=k}^{k+q-1} \langle (L_h - I)(T - T_h)\varphi_j, \varphi'_j \rangle$ ; by using (3.5), (3.6) and (3.8), we obtain the desired result.  $\square$

**Remark 3.1.** Note that in (3.1)  $\{\varphi_j\}_k^{k+q-1}$  is a basis of  $R(E)$  and  $\{\varphi'_j\}_k^{k+q-1}$  is the dual basis in  $R(E')$ , and it can be seen from the proof that they are all independent of  $h$ . Therefore, based on Theorem 3.1, we can realize the Richardson extrapolation of multiple eigenvalues and the splitting extrapolation based on domain decomposition in the next section.

#### 4. Finite element extrapolations of differential operator eigenvalue problems

In Section 4.1, based on Theorem 3.1 and [22], the Richardson extrapolations of the nonconforming element for multiple eigenvalues are discussed. In Section 4.2, by using Theorem 3.1 and [27,28], the splitting extrapolations based on domain decomposition for multiple eigenvalues are studied. For a general theory of finite element methods we refer to [3,6,11,32].

##### 4.1. The nonconforming finite element extrapolations for multiple eigenvalues

Let  $H^s(\Omega)$  be a Sobolev space with norm  $\|\cdot\|_{s,2}$ , and  $H_0^1(\Omega)$  be the subspace of  $H^1(\Omega)$  consisting of those functions which vanish on  $\partial\Omega$ .

Consider the eigenvalue problem: Find a real number  $\lambda \in \mathbf{R}$ ,  $0 \neq u \in H_0^1(\Omega)$ , such that

$$a(u, v) = \lambda b(u, v), \quad \forall v \in H_0^1(\Omega), \tag{4.1}$$

where

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad b(u, v) = \int_{\Omega} \rho(x) u v \, dx.$$

Here  $\Omega = [0, 1] \times [0, 1]$ ,  $\rho = \rho(x_1, x_2) > 0$  and  $\rho \in C^2(\Omega)$ .

Let  $\tau_h$  be a uniform rectangular mesh of  $\Omega$ , the length and width of the element be  $2h_1$  and  $2h_2$ , respectively, and  $h = 2\sqrt{h_1^2 + h_2^2}$ . Let  $S^h$  be the nonconforming  $EQ_1^{rot}$  finite element space defined on  $\tau_h$  (see [26]), i.e.,

$$\begin{aligned}
S^h = \Big\{ v \in L_2(\Omega) : & v|_e \in \text{span}\{1, x_1, x_2, x_1^2, x_2^2\}, \forall e \in \tau_h, \\
& \int_F v|_{e_1} ds = \int_F v|_{e_2} ds \text{ if } e_1, e_2 \in \tau_h, e_1 \cap e_2 = F, \\
& \int_F v|_e ds = 0 \text{ if } e \cap \partial\Omega = F \Big\}.
\end{aligned}$$

Let

$$a_h(u, v) = \sum_{e \in \tau_h} \int_e \nabla u \cdot \nabla v \, dx, \quad \|\cdot\|_h = \sqrt{a_h(\cdot, \cdot)}.$$

Then the nonconforming  $EQ_1^{rot}$  element approximation corresponding to (4.1) is: Find a real number  $\lambda_h \in \mathbf{R}$ ,  $0 \neq u_h \in S^h$ , such that

$$a_h(u_h, v) = \lambda_h b(u_h, v), \quad \forall v \in S^h. \quad (4.2)$$

Here, the eigenpair  $(\lambda_h, u_h)$  of (4.2) approximates the eigenpair  $(\lambda, u)$  of (4.1).

In order to discuss the error estimates, we will need the form of operator equation of (4.1) and (4.2). Define the operators  $T, T_h : L_2(\Omega) \rightarrow L_2(\Omega)$ :

$$a(Tf, v) = b(f, v), \quad \forall f \in L_2(\Omega), \quad \forall v \in H_0^1(\Omega),$$

$$a_h(T_h f, v) = b(f, v), \quad \forall f \in L_2(\Omega), \quad \forall v \in S^h.$$

Set  $X = L_2(\Omega)$ ,  $\langle \cdot, \cdot \rangle = b(\cdot, \cdot)$  and  $\|\cdot\| = \|\cdot\|_b = \sqrt{b(\cdot, \cdot)}$ . It can be easily seen that (4.1) and (4.2) have the equivalent operator forms (2.1) and (2.2), respectively,  $T$  and  $T_h$  are selfadjoint completely continuous operators (see for example [36]). By the  $EQ_1^{rot}$  element error estimate of the source problem corresponding to (4.1) (see [24,26]), we can deduce that  $\|T - T_h\| \rightarrow 0$  ( $h \rightarrow 0$ ).

Let  $\lambda_k$  be the  $k$ -th eigenvalue of (4.1) with algebraic multiplicity  $q$ ,  $\lambda_k = \lambda_{k+1} = \dots = \lambda_{k+q-1}$ ,  $\hat{\lambda}_{k,h} = (\frac{1}{q} \sum_{j=k}^{k+q-1} \lambda_{j,h}^{-1})^{-1}$ . Based on [22] we obtain the following:

**Theorem 4.1.** Let  $R(E) \subset H^5(\Omega)$ . Then for the nonconforming  $EQ_1^{rot}$  element, there holds the asymptotic expansion of the error of  $\hat{\lambda}_h$ :

$$\hat{\lambda}_{k,h} - \lambda_k = -\frac{1}{q} \sum_{j=k}^{k+q-1} \frac{h_1^2 + h_2^2}{3} \int_{\Omega} \partial_1^2 \partial_2^2 \varphi_j \varphi_j + \vartheta(h^4), \quad (4.3)$$

where  $\partial_i = \frac{\partial}{\partial x_i}$ ,  $\partial_i^2 = \frac{\partial^2}{\partial x_i^2}$ ,  $i = 1, 2$ .

**Proof.** Note that  $\mu_k = \frac{1}{\lambda_k}$ ,  $\hat{\mu}_{k,h} = \frac{1}{\hat{\lambda}_{k,h}}$ ,  $T$  and  $T_h$  are selfadjoint operators; from (3.1) and (3.2) we have

$$\begin{aligned} \frac{\hat{\lambda}_{k,h} - \lambda_k}{\lambda_k \hat{\lambda}_{k,h}} &= \frac{1}{q} \sum_{j=k}^{k+q-1} b((T - T_h)\varphi_j, \varphi_j) + R, \\ |R| &\leq C \|(T - T_h)|_{R(E)}\|^2. \end{aligned} \quad (4.4)$$

From [3] and [22], we can see that

$$|\hat{\lambda}_{k,h} - \lambda_k| \leq Ch^2, \quad \|(T - T_h)|_{R(E)}\| \leq Ch^2.$$

Then (4.4) can be written as

$$\hat{\lambda}_{k,h} - \lambda_k = \frac{1}{q} \lambda_k^2 \sum_{j=k}^{k+q-1} b((T - T_h)\varphi_j, \varphi_j) + \vartheta(h^4). \quad (4.5)$$

According to the error estimate of the nonconforming  $EQ_1^{rot}$  element, we have

$$|b(T\varphi_j - T_h\varphi_j, \varphi_j - \varphi_{j,h})| \leq \|T\varphi_j - T_h\varphi_j\|_b \|\varphi_j - \varphi_{j,h}\|_b \leq Ch^4,$$

and

$$b(I_h T\varphi_j - T_h\varphi_j, \varphi_{j,h}) = \frac{1}{\lambda_j} a_h(I_h T\varphi_j - T_h\varphi_j, \varphi_{j,h}) + \vartheta(h^4),$$

where  $I_h$  is the interpolation operator of  $EQ_1^{rot}$  element. Thus

$$\begin{aligned} b((T - T_h)\varphi_j, \varphi_j) &= b(T\varphi_j - T_h\varphi_j, \varphi_j - \varphi_{j,h}) + b(T\varphi_j - I_h T\varphi_j, \varphi_{j,h}) + b(I_h T\varphi_j - T_h\varphi_j, \varphi_{j,h}) \\ &= \frac{1}{\lambda_j} b(\varphi_j - I_h\varphi_j, \varphi_{j,h}) + \frac{1}{\lambda_j} a_h(I_h T\varphi_j - T_h\varphi_j, \varphi_{j,h}) + \vartheta(h^4) \\ &= \frac{1}{\lambda_j} (b(\varphi_j - I_h\varphi_j, \varphi_{j,h}) + a_h(I_h T\varphi_j - T\varphi_j, \varphi_{j,h}) \\ &\quad + a_h(T\varphi_j - T_h\varphi_j, \varphi_{j,h})) + \vartheta(h^4). \end{aligned} \quad (4.6)$$

From (3.1) in [22], we can see that

$$a_h(I_h T \varphi_j - T \varphi_j, \varphi_{j,h}) = 0.$$

From (3.3) in [22], it follows that

$$\begin{aligned} a_h(T \varphi_j - T_h \varphi_j, \varphi_{j,h}) &= -\frac{h_1^2 + h_2^2}{3} \int_{\Omega} \partial_1^2 \partial_2^2 T \varphi_j \varphi_{j,h} + \vartheta(h^4) \|T \varphi_j\|_5 \|\varphi_{j,h}\|_h \\ &= -\frac{h_1^2 + h_2^2}{3\lambda_j} \int_{\Omega} \partial_1^2 \partial_2^2 \varphi_j \varphi_j + \vartheta(h^4) \|\varphi_j\|_5 \|\varphi_{j,h}\|_h. \end{aligned}$$

From (3.9) in [22], we can derive

$$b(\varphi_j - I_h \varphi_j, \varphi_{j,h}) = \vartheta(h^4).$$

Substituting the above three equalities into (4.6) we have

$$b((T - T_h) \varphi_j, \varphi_j) = -\frac{h_1^2 + h_2^2}{3\lambda_j^2} \int_{\Omega} \partial_1^2 \partial_2^2 \varphi_j \varphi_j + \vartheta(h^4).$$

Substituting the above equality into (4.5) we get (4.3).  $\square$

**Algorithm 1** (The Richardson extrapolations for the nonconforming elements).

**Step 1.** Under the partition with the mesh parameters  $h_1, h_2$ , solving (4.2) we can get  $\hat{\lambda}_{k,h}$ .

**Step 2.** Under the partition with the mesh parameters  $\frac{h_1}{2}, \frac{h_2}{2}$ , solving (4.2) we can get  $\hat{\lambda}_{k,\frac{h}{2}}$ .

**Step 3.** Compute the value of the extrapolation

$$\lambda_{k,h}^{re} = \frac{4}{3} \hat{\lambda}_{k,\frac{h}{2}} - \frac{1}{3} \hat{\lambda}_{k,h}.$$

**Corollary 4.1.** Under the condition of Theorem 4.1, there holds

$$\lambda_{k,h}^{re} - \lambda_k = \vartheta(h^4). \quad (4.7)$$

**Proof.** From (4.3), it can be seen that

$$\hat{\lambda}_{k,\frac{h}{2}} - \lambda_k = -\frac{1}{q} \sum_{j=k}^{k+q-1} \frac{1}{3} \left( \left( \frac{h_1}{2} \right)^2 + \left( \frac{h_2}{2} \right)^2 \right) \int_{\Omega} \partial_1^2 \partial_2^2 \varphi_j \varphi_j + \vartheta(h^4).$$

Combining the above equality and (4.3), we obtain (4.7).  $\square$

Eq. (4.7) indicates that the nonconforming  $EQ_1^{rot}$  element extrapolations for multiple eigenvalues achieve the accuracy order  $\vartheta(h^4)$ .

**Remark 4.1.** Thanks to [22], we can also study multiple eigenvalues extrapolations of the nonconforming  $Q_1^{rot}$  element proposed by Rannacher [30]. Based on [17,23,35], the extrapolation of multiple eigenvalues of the Wilson element can also be discussed.

#### 4.2. Finite element extrapolations of non-selfadjoint differential operator eigenvalue problems

Let  $\Omega \subset \mathbb{R}^d$  be a bounded domain with  $d = 2, 3$ . Consider the second-order non-selfadjoint elliptic differential operator eigenvalue problems: Find a complex number  $\lambda \in \mathbb{C}$ ,  $0 \neq u \in H_0^1(\Omega)$ , such that

$$a(u, v) = \lambda b(u, v), \quad \forall v \in H_0^1(\Omega), \quad (4.8)$$

where

$$a(u, v) = \int_{\Omega} \left( \sum_{i,j=1}^d a_{ij}(x) \partial_i u \overline{\partial_j v} + \sum_{i=1}^d b_i(x) \partial_i u \overline{v} + c(x) u \overline{v} \right) dx,$$

$$b(u, v) = \int_{\Omega} \rho(x) u \overline{v} dx.$$

Assume that  $a_{ij}(x), c(x), \rho(x) \in L_{\infty}(\Omega)$ ,  $b_i(x) \in W_{1,\infty}(\Omega)$ ,  $x = (x_1, \dots, x_d)$ , and there exists a positive constant  $a_0$  such that

$$\operatorname{Re} \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \geq a_0 \sum_{i=1}^d \xi_i^2, \quad \forall x \in \Omega, \quad \forall (\xi_1, \dots, \xi_d)^T \in \mathbb{R}^d,$$

$$\operatorname{Re} c(x) \geq \frac{1}{2} a_0 + \frac{1}{2} \max_{x \in \Omega, i=1,2} |b_i(x)|^2 / a_0, \quad \forall x \in \Omega.$$

Let  $G$  be a convex quadrilateral. A quadrilateral grid by connecting the equidistant mesh points of the opposite edges of  $G$  is called a strong regular division of  $G$ . Let  $G$  be a convex hexahedron. A three-dimensional partition by connecting the corresponding strong regular division nodes of the opposite faces is called a strong regular division of  $G$ . For two-dimensional case in strong regular division there are two independent grid parameters which are determined by the numbers of the equidistant mesh points of the opposite edges. For three-dimensional case there are three independent grid parameters in strong regular division.

Let  $\overline{\Omega} = \bigcup_{s=1}^m \overline{\Omega}_s$ , where  $\Omega_s$  is a convex quadrilateral ( $d=2$ ) or convex hexahedron ( $d=3$ ), and the initial partition satisfies the compatible condition and has no interior cross points. Then for each  $\Omega_s$ , we construct such a strong regular partition  $\tau_{s,h}$  that  $\tau_h = \bigcup_{s=1}^m \tau_{s,h}$  is a piecewise strongly regular partition of the domain  $\Omega$ .

Note that the partition  $\tau_{s,h}$  has  $d$  mesh parameters  $h_{sz}$  ( $z=1, \dots, d$ ) while the partition  $\tau_h$  only has  $l$  ( $l < m \times d$ ) independent mesh parameters denoted by  $\bar{h}_1, \dots, \bar{h}_l$ . Let  $h_0 = \max_{1 \leq i \leq l} \bar{h}_i$ .

When  $\Omega_s$  is a quadrilateral,  $h_{sz}$  is the side-length of an element on  $\partial\Omega_s$ . When  $\Omega_s$  is a hexahedron,  $h_{sz}$  is the edge-length of an element on  $\partial\Omega_s$ . Since  $\tau_h$  is a piecewise strongly regular partition, there exist constants  $C_1$  and  $C_2$  independent of the mesh diameter  $h$  such that

$$C_1 h \leq h_0 \leq C_2 h.$$

Using the standard method (see [6,11,32]) we define the isoparametric  $d$ -linear finite element space  $S^h$  under the partition  $\tau_h$ ,  $S^h \subset H_0^1(\Omega) \cap C(\Omega)$ . Then the conforming finite element approximation corresponding to (4.8) is: Find a complex number  $\lambda_h \in \mathbb{C}$ ,  $0 \neq u_h \in S^h$ , such that

$$a(u_h, v) = \lambda_h b(u_h, v), \quad \forall v \in S^h. \quad (4.9)$$

The adjoint equation corresponding to (4.8) is: Find a complex number  $\lambda \in \mathbb{C}$ ,  $0 \neq w \in H_0^1(\Omega)$ , such that

$$a(v, w) = \lambda b(v, w), \quad \forall v \in H_0^1(\Omega). \quad (4.10)$$

In order to discuss error estimates, we need the forms of operator equations of (4.8), (4.9) and (4.10). Define the operators  $T, T^*: L_2(\Omega) \rightarrow L_2(\Omega)$ :

$$a(Tf, v) = b(f, v), \quad \forall f \in L_2(\Omega), \quad \forall v \in H_0^1(\Omega),$$

$$a(v, T^*g) = b(v, g), \quad \forall g \in L_2(\Omega), \quad \forall v \in H_0^1(\Omega).$$

We also define the operators  $T_h, T_h^*: L_2(\Omega) \rightarrow S^h$ :

$$a(T_h f, v) = b(f, v), \quad \forall f \in L_2(\Omega), \quad \forall v \in S^h,$$

$$a(v, T_h^* g) = b(v, g), \quad \forall g \in L_2(\Omega), \quad \forall v \in S^h.$$

Observing that  $S^h \subset H_0^1(\Omega)$  we have

$$a(Tf - T_h f, v) = 0, \quad \forall v \in S^h, \quad (4.11)$$

$$a(v, T^*g - T_h^*g) = 0, \quad \forall v \in S^h. \quad (4.12)$$

Set  $X = L_2(\Omega)$ ,  $\langle \cdot, \cdot \rangle = b(\cdot, \cdot)$  and  $\| \cdot \| = \| \cdot \|_b = \sqrt{b(\cdot, \cdot)}$ . Let  $T' = T^*$  and  $T_h' = T_h^*$  denote the Hilbert adjoints of  $T$  and  $T_h$ , respectively. Let  $E^*$  be the spectral projection associated with  $T^*$  and  $\mu_k^*$ . In this case we would let  $\varphi_k, \dots, \varphi_{k+q-1}$  be an



orthonormal basis for  $R(E)$  and let  $\varphi_j^* = E^* \varphi_j$  (see [3, p. 691]). From [3], we can see that (4.8) and (4.9) have the equivalent operator forms (2.1) and (2.2), respectively,  $\|T - T_h\| \rightarrow 0$  ( $h \rightarrow 0$ ) and (4.10) is equivalent to the operator form  $T^* w = \mu^* w$ .

Let  $\lambda_k$  be the  $k$ -th eigenvalue of (4.8) with algebraic multiplicity  $q$ ,  $\lambda_k = \lambda_{k+1} = \dots = \lambda_{k+q-1}$ ,  $\hat{\lambda}_{k,h} = (\frac{1}{q} \sum_{j=k}^{k+q-1} \lambda_{j,h}^{-1})^{-1}$ . Based on [27] we obtain the following:

**Theorem 4.2.** Suppose that the coefficients of differential operator are piecewise smooth:  $a_{ij} \in (\prod_{s=1}^m C^5(\Omega_s)) \cap L_\infty(\Omega)$ ,  $b_i \in (\prod_{s=1}^m C^5(\Omega_s)) \cap W_{1,\infty}(\Omega)$ ,  $c \in (\prod_{s=1}^m C^4(\Omega_s)) \cap L_\infty(\Omega)$ , and  $\rho \in (\prod_{s=1}^m C^4(\Omega_s)) \cap L_\infty(\Omega)$ . Assume that  $R(E) \subset (\prod_{s=1}^m H^6(\Omega_s)) \cap H_0^1(\Omega)$ . Then there holds the asymptotic expansion of the error of  $\hat{\lambda}_{k,h}$ :

$$\hat{\lambda}_{k,h} - \lambda_k = \sum_{i=1}^l \beta_i \bar{h}_i^2 + \vartheta(h_0^4), \quad (4.13)$$

where the constants  $\beta_i$  ( $i = 1, 2, \dots, l$ ) are independent of  $h$ .

**Proof.** Noting that  $\mu_k = \frac{1}{\lambda_k}$  and  $\hat{\mu}_{k,h} = \frac{1}{\hat{\lambda}_{k,h}}$ , it follows from (3.1) and (3.2) that

$$\begin{aligned} \frac{\hat{\lambda}_{k,h} - \lambda_k}{\lambda_k \hat{\lambda}_{k,h}} &= \frac{1}{q} \sum_{j=k}^{k+q-1} \langle (T - T_h) \varphi_j, \varphi_j^* \rangle + R, \\ |R| &\leq C \|(T - T_h)|_{R(E)}\| \|(T^* - T_h^*)|_{R(E^*)}\|. \end{aligned} \quad (4.14)$$

From [3] and [27], we can see that

$$\begin{aligned} |\hat{\lambda}_{k,h} - \lambda_k| &\leq Ch_0^2, \\ \|(T - T_h)|_{R(E)}\| + \|(T^* - T_h^*)|_{R(E^*)}\| &\leq Ch_0^2. \end{aligned}$$

Substituting the above two estimates into (4.14), we obtain

$$\hat{\lambda}_{k,h} - \lambda_k = \frac{1}{q} \lambda_k^2 \sum_{j=k}^{k+q-1} \langle (T - T_h) \varphi_j, \varphi_j^* \rangle + \vartheta(h_0^4). \quad (4.15)$$

By the definition of operator  $T^*$ , (4.11) and (4.12), we deduce that

$$\begin{aligned} \langle (T - T_h) \varphi_j, \varphi_j^* \rangle &= a((T - T_h) \varphi_j, T^* \varphi_j^*) \\ &= a((T - T_h) \varphi_j, T^* \varphi_j^* - T_h^* \varphi_j^*) \\ &= a(T \varphi_j - I_h T \varphi_j, T^* \varphi_j^* - T_h^* \varphi_j^*) \\ &= a(T \varphi_j - I_h T \varphi_j, T^* \varphi_j^*) - a(T \varphi_j - I_h T \varphi_j, T_h^* \varphi_j^*). \end{aligned} \quad (4.16)$$

From (20) in [27], it can be derived that

$$\begin{aligned} a(T \varphi_j - I_h T \varphi_j, T_h^* \varphi_j^*) &= \sum_{s=1}^m \left\{ \sum_{z=1}^d \frac{1}{12} h_{sz}^2 \left[ \sum_{i=1, i \neq z}^d \sum_{j=1}^d \int_{\Omega_s} a_{ij}(x) \partial_z^2 \partial_i T \varphi_j \partial_j T_h^* \varphi_j^* dx \right. \right. \\ &\quad - \sum_{j=1}^d \int_{\Omega_s} \partial_z^2 T \varphi_j \partial_z \left( a_{zj}(x) \partial_j T_h^* \varphi_j^* \right) dx + \sum_{j=1}^d \int_{\Omega_s} \partial_i^2 \partial_z T \varphi_j b_z(x) T_h^* \varphi_j^* dx \\ &\quad \left. - \int_{\Omega_s} \partial_z^2 T \varphi_j \partial_z (b_z(x) T_h^* \varphi_j^*) dx - \int_{\Omega_s} \partial_z^2 T \varphi_j c(x) T_h^* \varphi_j^* dx \right] \\ &\quad \left. + \sum_{i,j,z}^d h_{sz}^2 h_{si}^2 \int_{\Omega_s} M_{zi} T \varphi_j \partial_z \partial_j T_h^* \varphi_j^* dx \right\} + R, \end{aligned}$$

where  $M_{zi}$  are fourth-order differential operators and

$$|R| \leq Ch_0^4 \left( \sum_{s=1}^m \|T \varphi_j\|_{6,2,\Omega_s}^2 \right)^{\frac{1}{2}} \|T_h^* \varphi_j^*\|_{1,2,\Omega}.$$

From (26) in [27] we have

$$\begin{aligned} a(T\varphi_j - I_h T\varphi_j, T^* \varphi_j^*) &= b(T\varphi_j - I_h T\varphi_j, \varphi_j^*) \\ &= \sum_{s=1}^m \sum_{z=1}^d \frac{1}{12} h_{sz}^2 \int_{\Omega_s} \rho(x) \varphi_j^* \partial_z^2 T\varphi_j dx + \vartheta(h_0^4). \end{aligned}$$

Substituting the above two equalities into (4.16), together with (4.15), and noting that there are only  $l$  parameters are independent among  $h_{s,z}$  ( $s = 1, \dots, m$ ;  $z = 1, \dots, d$ ), and packing them we get (4.13). Since  $\{\varphi_j\}_k^{k+q-1}$ ,  $\{\varphi_j^*\}_k^{k+q-1}$  are independent of  $h$ , the constants  $\beta_i$  ( $i = 1, 2, \dots, l$ ) are independent of  $h$ .  $\square$

**Algorithm 2** (The Richardson extrapolations for the conforming element).

**Step 1.** Under the partition with mesh parameters  $\bar{h}_1, \dots, \bar{h}_l$ , solve (4.9) to obtain  $\hat{\lambda}_{k,h}$ .

**Step 2.** Under the partition with mesh parameters  $\frac{\bar{h}_1}{2}, \dots, \frac{\bar{h}_l}{2}$ , solve (4.9) to obtain  $\hat{\lambda}_{k, \frac{h}{2}}$ .

**Step 3.** Compute the values of the extrapolation

$$\lambda_{k,h}^{re} = \frac{4}{3} \hat{\lambda}_{k, \frac{h}{2}} - \frac{1}{3} \hat{\lambda}_{k,h}.$$

**Corollary 4.2.** Under the condition of Theorem 4.2, there holds

$$\lambda_{k,h}^{re} - \lambda_k = \vartheta(h^4). \quad (4.17)$$

**Proof.** From (4.13), it can be seen that

$$\hat{\lambda}_{k, \frac{h}{2}} - \lambda_k = \sum_{i=1}^l \beta_i \left( \frac{\bar{h}_i}{2} \right)^2 + \vartheta(h_0^4).$$

Combining the above equality and (4.13) we get (4.17).  $\square$

Let  $\bar{h}^{(j)} = (\bar{h}_1, \dots, \bar{h}_{j-1}, \frac{\bar{h}_j}{2}, \bar{h}_{j+1}, \dots, \bar{h}_l)$ . We denote by  $\hat{\lambda}_{k,h}^{(j)}$  the  $k$ -th approximate eigenvalue obtained by  $d$ -linear finite element under the partition with mesh parameters  $\bar{h}^{(j)}$ . Based on [27], we have the following algorithm:

**Algorithm 3** (Splitting extrapolations based on domain decomposition).

**Step 1.** Under the partition with mesh parameters  $\bar{h}_1, \dots, \bar{h}_l$ , solve (4.9) to obtain  $\hat{\lambda}_{k,h}$ .

**Step 2.** Make a decomposition for the domain, and solve (4.9) in parallel on the meshes with mesh parameters  $\bar{h}^{(1)}, \dots, \bar{h}^{(l)}$ , respectively to get  $\hat{\lambda}_{k,h}^{(1)}, \dots, \hat{\lambda}_{k,h}^{(l)}$ .

**Step 3.** Compute the value of the extrapolation

$$\lambda_{k,h}^{se} = \frac{4}{3} \sum_{j=1}^l \hat{\lambda}_{k,h}^{(j)} - \frac{4l-3}{3} \hat{\lambda}_{k,h}.$$

**Corollary 4.3.** Under the condition of Theorem 4.2, there holds

$$\lambda_{k,h}^{se} - \lambda_k = \vartheta(h_0^4). \quad (4.18)$$

**Proof.** From (4.13), it can be seen that

$$\hat{\lambda}_{k,h}^{(j)} - \lambda_k = \sum_{i=1, i \neq j}^l \beta_i \bar{h}_i^2 + \beta_j \left( \frac{\bar{h}_j}{2} \right)^2 + \vartheta(h_0^4), \quad (4.19)$$

from (4.19) and (4.13) we have

**Table 1**The numerical eigenvalues  $\hat{\lambda}_{k,h}^{re}$  obtained by Algorithm 1 ( $k = 2, 5$ ).

Elements' number	$4 \times 8$	$8 \times 16$	$16 \times 32$	$32 \times 64$	$64 \times 128$
$\lambda_{2,h}$	47.037394	48.729567	49.190488	49.308450	49.338117
$\lambda_{3,h}$	47.494826	48.761872	49.192582	49.308582	49.338125
$\hat{\lambda}_{2,h}$	47.265003	48.745714	49.191535	49.308516	49.338121
$\hat{\lambda}_{2,h}^{re}$	49.239284	49.340142	49.347510	49.347989	
$\lambda_{5,h}$	93.906837	97.335849	98.343652	98.607137	98.673766
$\lambda_{6,h}$	97.616416	97.619961	98.362606	98.608342	98.673842
$\hat{\lambda}_{5,h}$	95.725701	97.477698	98.353128	98.607739	98.673804
$\hat{\lambda}_{5,h}^{re}$	98.061697	98.644938	98.692609	98.695826	

$$\begin{aligned}
\lambda_{k,h}^{se} - \lambda_k &= \frac{4}{3} \sum_{j=1}^l \hat{\lambda}_{k,h}^{(j)} - \frac{4l-3}{3} \hat{\lambda}_{k,h} - \lambda_k \\
&= \frac{4}{3} \sum_{j=1}^l (\hat{\lambda}_{k,h}^{(j)} - \lambda_k) - \frac{4l-3}{3} (\hat{\lambda}_{k,h} - \lambda_k) \\
&= \frac{4}{3} \sum_{j=1}^l \left( \sum_{i=1, i \neq j}^l \beta_i \bar{h}_i^2 + \beta_j \left( \frac{\bar{h}_j}{2} \right)^2 \right) - \frac{4l-3}{3} \left( \sum_{i=1}^l \beta_i \bar{h}_i^2 \right) + \vartheta(h_0^4) \\
&= \frac{4}{3} \sum_{j=1}^l \left( \sum_{i=1}^l \beta_i \bar{h}_i^2 - \frac{3}{4} \beta_j \bar{h}_j^2 \right) - \frac{4l-3}{3} \left( \sum_{i=1}^l \beta_i \bar{h}_i^2 \right) + \vartheta(h_0^4) \\
&= 0 + \vartheta(h_0^4).
\end{aligned}$$

In Theorem 2 of [27], the assumption of simple eigenvalue is a strong condition because the eigenvalue of non-selfadjoint problems is not simple in general. Compared with [27], we easily drop the assumption of simple eigenvalue by using Theorem 3.1.  $\square$

**Remark 4.2.** In Algorithms 1–3 we can also define

$$\hat{\lambda}_{k,h} = \frac{1}{q} \sum_{j=k}^{k+q-1} \lambda_{j,h}.$$

Theorems 4.1, 4.2 and Corollaries 4.1–4.3 also hold for such defined  $\hat{\lambda}_{k,h}$  for the reason that by a calculation we have

$$\left| \frac{1}{q} \sum_{j=k}^{k+q-1} \lambda_{j,h} - \left( \frac{1}{q} \sum_{j=k}^{k+q-1} \lambda_{j,h}^{-1} \right)^{-1} \right| \leq C \max_{k \leq j \leq k+q-1} |\lambda_{j,h} - \lambda_k|^2 \leq Ch^4.$$

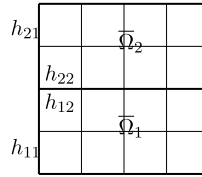
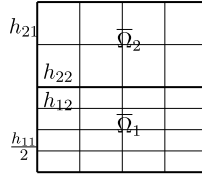
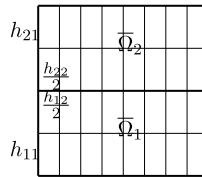
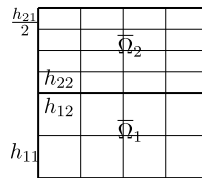
## 5. Numerical experiments

**Example 5.1.** Consider the eigenvalue problem (4.1) with  $\rho(x) = 1$ . It is well known that the multiplicities of the exact eigenvalues  $\lambda_2$  and  $\lambda_5$  of this problem are both equal to 2, and  $\lambda_2 = \lambda_3 = 5\pi^2 \approx 49.34802$ ,  $\lambda_5 = \lambda_6 = 10\pi^2 \approx 98.696044$ . We make a uniform rectangular partition for  $\Omega$  (the proportion of the mesh's length and width is 2:1). We use the non-conforming  $EQ_1^{rot}$  element to compute the approximate eigenvalues corresponding to (4.1), and set  $\hat{\lambda}_{2,h} = \frac{\lambda_{2,h} + \lambda_{3,h}}{2}$  and  $\hat{\lambda}_{5,h} = \frac{\lambda_{5,h} + \lambda_{6,h}}{2}$ . Let the approximate eigenvalues obtained by Algorithm 1 be denoted by

$$\hat{\lambda}_{k,h}^{re} \equiv \frac{4}{3} \hat{\lambda}_{k, \frac{h}{2}} - \frac{1}{3} \hat{\lambda}_{k,h} \quad (k = 2, 5).$$

Numerical results are listed in Table 1. Now we use the formula  $\lg(\frac{|\hat{\lambda}_{k,h}^{re} - \lambda_k|}{|\hat{\lambda}_{k, \frac{h}{2}}^{re} - \lambda_k|}) / \lg 2$  to calculate and obtain the orders of

$\hat{\lambda}_{k,h}^{re} \rightarrow \lambda_k$  are about  $\vartheta(h^4)$  ( $k = 2, 5$ ), which coincides with our theoretical analysis.

Fig. 1.  $(\bar{h}_1, \bar{h}_2, \bar{h}_3)$ .Fig. 2.  $(\frac{\bar{h}_1}{2}, \bar{h}_2, \bar{h}_3)$ .Fig. 3.  $(\bar{h}_1, \frac{\bar{h}_2}{2}, \bar{h}_3)$ .Fig. 4.  $(\bar{h}_1, \bar{h}_2, \frac{\bar{h}_3}{2})$ .

**Example 5.2.** Consider the non-symmetric convection diffusion problem:

$$-\Delta u + b \cdot \nabla u = \lambda u, \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega, \quad (5.1)$$

where  $b = (1, 1)$ , and  $\Omega$  is a square  $(0, 1) \times (0, 1)$ .

We know that the multiplicities of the eigenvalues  $\lambda_2 = \lambda_3$  and  $\lambda_5 = \lambda_6$  are all equal to 2, and

$$\lambda_2 = \lambda_3 \approx 49.848022005, \quad \lambda_5 = \lambda_6 \approx 99.196044011.$$

We make a uniform square partition for  $\Omega$  and use the conforming bilinear element to compute the approximate eigenvalues corresponding to (5.1). We decompose  $\bar{\Omega}$  as:  $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ , where  $\bar{\Omega}_1 = [0, 1] \times [0, \frac{1}{2}]$  and  $\bar{\Omega}_2 = [0, 1] \times [\frac{1}{2}, 1]$ . Let  $\tau_{s,h}$  be the partition on  $\bar{\Omega}_s$  ( $s = 1, 2$ ).

Note that  $\tau_{1,h}$  has 2 mesh parameters  $h_{11}$  and  $h_{12}$ ,  $\tau_{2,h}$  has 2 mesh parameters  $h_{21}$  and  $h_{22}$ , but  $\tau_h$  only has 3 independent mesh parameters  $\bar{h}_1 = h_{11}$ ,  $\bar{h}_2 = h_{12} = h_{22}$  and  $\bar{h}_3 = h_{21}$  (see Figs. 1–4). Let  $h_0 = \max_{1 \leq i \leq 3} \bar{h}_i$ .

Let  $\lambda_{k,h}$ ,  $\lambda_{k,h}^{(1)}$ ,  $\lambda_{k,h}^{(2)}$  and  $\lambda_{k,h}^{(3)}$  be the approximate eigenvalues corresponding to the bilinear finite element approximations on the partition with mesh parameters  $(\bar{h}_1, \bar{h}_2, \bar{h}_3)$ ,  $(\frac{\bar{h}_1}{2}, \bar{h}_2, \bar{h}_3)$ ,  $(\bar{h}_1, \frac{\bar{h}_2}{2}, \bar{h}_3)$  and  $(\bar{h}_1, \bar{h}_2, \frac{\bar{h}_3}{2})$ , respectively. Let  $\hat{\lambda}_{k,h} = (\frac{1}{2}(\frac{1}{\lambda_{k,h}} + \frac{1}{\lambda_{k+1,h}}))^{-1}$  ( $k = 2, 5$ ),  $\hat{\lambda}_{k,h}^{(i)} = (\frac{1}{2}(\frac{1}{\lambda_{k,h}^{(i)}} + \frac{1}{\lambda_{k+1,h}^{(i)}}))^{-1}$  ( $i = 1, 2, 3, k = 2, 5$ ). We denote the approximate eigenvalues obtained by Algorithm 2 by

$$\hat{\lambda}_{k,h}^{re} \equiv \frac{4}{3}\hat{\lambda}_{k,\frac{h}{2}} - \frac{1}{3}\hat{\lambda}_{k,h} \quad (k = 2, 5),$$

**Table 2**

The numerical eigenvalues  $\hat{\lambda}_{k,h}^{re}$  obtained by Algorithm 2, and  $\hat{\lambda}_{k,h}^{se}$  obtained by Algorithm 3 ( $k = 2, 5$ ).

$h_0$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
$\hat{\lambda}_{2,h}$	51.951541	50.365882	49.976951	49.880220	49.856069
$\hat{\lambda}_{2,h}^{(1)}, \hat{\lambda}_{2,h}^{(3)}$	51.544287	50.267955	49.952725	49.874180	49.854560
$\hat{\lambda}_{2,h}^{(2)}$	51.148823	50.170812	49.928548	49.868143	49.853051
$\hat{\lambda}_{2,h}^{re}$	49.837329	49.847307	49.847976	49.848019	
$\hat{\lambda}_{2,h}^{se}$	49.795240	49.844650	49.847811	49.848011	
$\hat{\lambda}_{5,h}$	109.81232	101.77789	99.835956	99.355663	99.235926
$\hat{\lambda}_{5,h}^{(1)}, \hat{\lambda}_{5,h}^{(3)}$	107.57658	101.27850	99.715022	99.325675	99.228444
$\hat{\lambda}_{5,h}^{(2)}$	105.64870	100.79796	99.595254	99.295760	99.220967
$\hat{\lambda}_{5,h}^{re}$	99.099747	99.188645	99.195565	99.196014	
$\hat{\lambda}_{5,h}^{se}$	98.298853	99.139610	99.192529	99.195824	

and the approximate eigenvalues by Algorithm 3 by

$$\hat{\lambda}_{k,h}^{se} \equiv \sum_{j=1}^3 \frac{4}{3} \hat{\lambda}_{k,h}^{(j)} - \left( \frac{4}{3} \times 3 - 1 \right) \hat{\lambda}_{k,h} \quad (k = 2, 5).$$

Numerical results listed in Table 2 are in accordance with our theoretical analysis.

**Remark 5.1.** We can see from Example 5.2 that the numerical eigenvalue by the splitting extrapolation has the same accuracy with that by the Richardson extrapolation. The splitting extrapolation method needs to solve four subproblems. The number of nodes of the maximum subproblem, which is illustrated in Fig. 3, are just a half of the number of nodes on the fine grid of the Richardson extrapolation method. When considering Example 5.2 in  $\Omega \subset R^3$ , we make a split  $\Omega = \Omega_1 \cup \Omega_2$ . From [27] we know that, to obtain the same accuracy, the splitting extrapolation method needs to solve five subproblems, and the number of nodes of the maximum subproblem is just a quarter of the number of nodes on the fine grid of the Richardson extrapolation method. The higher the dimensions are, the more superior the splitting extrapolation method performs.

## 6. Concluding remarks

This paper discusses the extrapolation of numerical eigenvalues by finite elements for differential operators. Theorem 3.1 in the paper provides a new error expansion. Using this error expansion we solve the extrapolation for multiple eigenvalues, which was once thought to be a complicated work. For instance, in the paper we achieve the extrapolation of nonconforming finite elements for multiple eigenvalues and the splitting extrapolation based on domain decomposition of conforming finite elements for multiple eigenvalues (including the case that the ascent is larger than 1) of non-selfadjoint differential operator. Although the proof of Theorem 3.1 is just a minor modification of that of Osborn (see Theorem 3 of [29], Theorem 7.2 of [3]), Theorem 3.1 develops the spectral approximation theory and is a general result.

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