

LOCAL WELL-POSEDNESS AND SMALL DEBORAH LIMIT OF A MOLECULE-BASED Q -TENSOR SYSTEM

SIRUI LI

School of Mathematical Sciences and LMAM, Peking University
Beijing, 100871, China

WEI WANG

Beijing International Center for Mathematical Research, Peking University
Beijing, 100871, China

Current address: Department of Mathematics, Zhejiang University
Hangzhou, 310027, China

PINGWEN ZHANG

School of Mathematical Sciences and LMAM, Peking University
Beijing, 100871, China

(Communicated by Yuan Lou)

ABSTRACT. In this paper, we consider a hydrodynamic Q -tensor system for nematic liquid crystal flow, which is derived from Doi-Onsager molecular theory by the Bingham closure. We first prove the existence and uniqueness of local strong solution. Furthermore, by taking Deborah number goes to zero and using the Hilbert expansion method, we present a rigorous derivation from the molecule-based Q -tensor theory to the Ericksen-Leslie theory.

CONTENTS

1. Introduction	2612
1.1. The Ericksen-Leslie theory	2612
1.2. The Q -tensor theory	2613
1.3. Motivations and main results	2617
2. The Bingham closure and the critical point	2620
2.1. The Bingham closure and Bingham map	2620
2.2. The energy functional and critical points	2624
3. Existence and uniqueness of the local strong solution for the dynamical Q -tensor systems	2626
3.1. Linearized system and iteration scheme	2626
3.2. Convergence of the sequence	2630
4. Some linearized operators	2632
5. Rigorous derivation from the Q -tensor theory to the Ericksen-Leslie theory	2636
5.1. The Hilbert expansion	2636

2010 *Mathematics Subject Classification.* Primary: 35Q35; Secondary: 35A01, 76A15, 76D05.

Key words and phrases. Liquid crystals, Q -tensor theory, Ericksen-Leslie theory, local strong solution.

5.2. Existence of the Hilbert expansion	2641
5.3. The system for the remainder	2645
5.4. Uniform estimates for the remainder	2648
5.5. Proof of Theorem 1.2	2652
6. Appendix	2653
6.1. Some basic estimates in Sobolev spaces	2653
Acknowledgments	2654
REFERENCES	2654

1. Introduction. Liquid crystals are a state of matter whose properties are intermediate between those of the conventional isotropic liquid and the crystalline solid. The nematic, composed by rod-like molecules, is the simplest liquid crystal phase, exhibiting long-range orientational order but no positional order. We refer [6] for a comprehensive elaboration of the physics of liquid crystals. There are three main theories to model nematic liquid crystal flow: the Doi-Onsager theory, the Landau-de Gennes theory and the Ericksen-Leslie theory. The first is microscopic theory derived from viewpoints of statistical mechanics, and the later two are macroscopic theories based on continuum mechanics.

Notations and conventions. The Einstein convention will be assumed throughout the paper. We introduce the following notations for the space of symmetric traceless tensors

$$\mathbb{Q} \stackrel{\text{def}}{=} \{Q \in \mathbb{R}^{3 \times 3} : Q_{ij} = Q_{ji}, Q_{ii} = 0\}, \quad (1.1)$$

$$\mathbb{Q}_{phy} \stackrel{\text{def}}{=} \{Q \in \mathbb{Q} : \text{the eigenvalues of } Q \in (-\frac{1}{3}, \frac{2}{3})\}. \quad (1.2)$$

The space \mathbb{Q} is endowed with the inner product $\langle Q_1, Q_2 \rangle \stackrel{\text{def}}{=} Q_1 : Q_2 = Q_{1ij}Q_{2ij}$. The set \mathbb{Q} is a five-dimensional linear subspace of $\mathbb{R}^{3 \times 3}$. We define the matrix norm on \mathbb{Q} as $|Q| \stackrel{\text{def}}{=} \sqrt{\text{tr}Q^2} = \sqrt{Q_{ij}Q_{ij}}$. In terms of this norm, the Sobolev space is defined as

$$H^k(\mathbb{R}^3) \stackrel{\text{def}}{=} \left\{ f : \int_{\mathbb{R}^3} \sum_{|\alpha'| \leq k} |\partial^{\alpha'} f(\mathbf{x})|^2 d\mathbf{x} < \infty \right\}$$

with k being a non-negative integer and α' being a multi-index. For two tensors $A, B \in \mathbb{Q}$ we denote $(A \cdot B)_{ij} = A_{ik}B_{kj}$ and $A : B = A_{ij}B_{ij}$. We denote $(M : Q)_{ij} = M_{ijkl}Q_{kl}$ where M is the fourth-order tensor and $Q \in \mathbb{Q}$. In addition, $\mathbf{n}_1 \otimes \mathbf{n}_2 \otimes \cdots \otimes \mathbf{n}_k$ denotes the tensor product of k vectors $\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_k$, and we usually omit the symbol \otimes for simplicity. We use $f_{,i}$ to denote $\partial_i f$ for simplicity and \mathbf{I} to denote the 3×3 order identity tensor.

1.1. The Ericksen-Leslie theory. The hydrodynamic theory of liquid crystals, established by Ericksen [8] and Leslie [16] in the 1960's, is a system coupling the time evolution equation of the fluid velocity $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ with the director equation describing the motion of the director field $\mathbf{n} = \mathbf{n}(t, \mathbf{x}) \in \mathbb{S}^2$. The general Ericksen-Leslie system takes the form

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nabla \cdot \sigma, \quad (1.3)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (1.4)$$

$$\mathbf{n} \times (\mathbf{h} - \gamma_1 \mathbf{N} - \gamma_2 \mathbf{D} \cdot \mathbf{n}) = 0, \quad (1.5)$$

where \mathbf{v} is the velocity of the fluid and p is the pressure. The stress σ is modeled by the phenomenological constitutive relation

$$\sigma = \sigma^L + \sigma^E,$$

where σ^L is the viscous (Leslie) stress

$$\sigma^L = \alpha_1(\mathbf{nn} : \mathbf{D})\mathbf{nn} + \alpha_2\mathbf{nN} + \alpha_3\mathbf{Nn} + \alpha_4\mathbf{D} + \alpha_5\mathbf{nn} \cdot \mathbf{D} + \alpha_6\mathbf{D} \cdot \mathbf{nn} \quad (1.6)$$

with $\kappa = (\nabla \mathbf{v})^T$, $\mathbf{D} = \frac{1}{2}(\kappa + \kappa^T)$ and

$$\mathbf{N} = \mathbf{n}_t + \mathbf{v} \cdot \nabla \mathbf{n} - \mathbf{\Omega} \cdot \mathbf{n}, \quad \mathbf{\Omega} = \frac{1}{2}(\kappa^T - \kappa).$$

The six constants $\alpha_1, \dots, \alpha_6$ are said to be the Leslie coefficients. Moreover, σ^E is the elastic (Ericksen) stress given by

$$\sigma_{ij}^E = -\frac{\partial E_F}{\partial n_{k,j}} n_{k,i}, \quad (1.7)$$

where $E_F = E_F(\mathbf{n}, \nabla \mathbf{n})$ is the Oseen-Frank energy with the form

$$\begin{aligned} E_F = & \frac{k_1}{2}(\nabla \cdot \mathbf{n})^2 + \frac{k_2}{2}(\mathbf{n} \cdot (\nabla \times \mathbf{n}))^2 + \frac{k_3}{2}|\mathbf{n} \times (\nabla \times \mathbf{n})|^2 \\ & + \frac{k_2 + k_4}{2}(\text{tr}(\nabla \mathbf{n})^2 - (\nabla \cdot \mathbf{n})^2). \end{aligned} \quad (1.8)$$

Here k_1, k_2, k_3, k_4 are elastic constants. The molecular field \mathbf{h} is given by

$$\mathbf{h} = -\frac{\delta E_F}{\delta \mathbf{n}} = \nabla \cdot \frac{\partial E_F}{\partial (\nabla \mathbf{n})} - \frac{\partial E_F}{\partial \mathbf{n}}.$$

The Leslie coefficients and material dependent coefficients γ_1, γ_2 satisfy the following relations

$$\alpha_2 + \alpha_3 = \alpha_6 - \alpha_5, \quad (1.9)$$

$$\gamma_1 = \alpha_3 - \alpha_2, \quad \gamma_2 = \alpha_6 - \alpha_5, \quad (1.10)$$

where (1.9) is called Parodi's relation derived from the Onsager reciprocal relation [23]. These two relations will ensure that the system (1.3)–(1.5) has a basic energy law:

$$\begin{aligned} -\frac{d}{dt} \left(\int_{\mathbb{R}^3} \frac{1}{2} |\mathbf{v}|^2 d\mathbf{x} + E_F \right) = & \int_{\mathbb{R}^3} \left(\left(\alpha_1 + \frac{\gamma_2^2}{\gamma_1} \right) (\mathbf{D} : \mathbf{nn})^2 + \alpha_4 |\mathbf{D}|^2 \right. \\ & \left. + \left(\alpha_5 + \alpha_6 - \frac{\gamma_2^2}{\gamma_1} \right) |\mathbf{D} \cdot \mathbf{n}|^2 + \frac{1}{\gamma_1} |\mathbf{n} \times \mathbf{h}|^2 \right) d\mathbf{x}. \end{aligned} \quad (1.11)$$

For the well-posedness results of the Ericksen-Leslie system, we can refer to [17, 28]. In addition, under a natural physical condition on the Leslie coefficients, [28] proved the well-posedness of the system, and the global existence of weak solution in two-dimensional case was showed in [14, 26].

1.2. The Q -tensor theory. The most general continuum theory for the nematic liquid crystals is the celebrated Landau-de Gennes theory which can describe uniaxial and biaxial liquid phases. In this phenomenological theory, the detailed nature of molecular interactions and molecular structures is ignored, and the state of the nematic liquid crystals is described by a macroscopic tensor value order parameter $Q(\mathbf{x})$, which is a symmetric and traceless 3×3 matrix, i.e. $Q \in \mathbb{Q}$. Physically, it can

be interpreted as the second-order traceless moment of the orientational distribution function f , that is,

$$Q(\mathbf{x}) = \int_{\mathbb{S}^2} (\mathbf{m}\mathbf{m} - \frac{1}{3}\mathbf{I})f(\mathbf{x}, \mathbf{m})d\mathbf{m}. \quad (1.12)$$

Under this interpretation, the so-called physical constraint is that the eigenvalues of Q should satisfy

$$\lambda_i(Q) \in (-\frac{1}{3}, \frac{2}{3}), \text{ for } 1 \leq i \leq 3, \quad (1.13)$$

namely, $Q \in \mathbb{Q}_{phy}$.

The nematic liquid crystal is called isotropic at \mathbf{x} when $Q(\mathbf{x}) = 0$. When $Q(\mathbf{x})$ has two equal non-zero eigenvalues, it is called uniaxial and $Q(\mathbf{x})$ can be written as

$$Q(\mathbf{x}) = s(\mathbf{n}\mathbf{n} - \frac{1}{3}\mathbf{I}), \quad s \in \mathbb{R}, \quad \mathbf{n}(\mathbf{x}) \in \mathbb{S}^2.$$

When $Q(\mathbf{x})$ has three distinct eigenvalues, it is called biaxial and $Q(\mathbf{x})$ can be written as

$$Q(\mathbf{x}) = s(\mathbf{n}\mathbf{n} - \frac{1}{3}\mathbf{I}) + r(\mathbf{n}'\mathbf{n}' - \frac{1}{3}\mathbf{I}), \quad \mathbf{n}, \mathbf{n}' \in \mathbb{S}^2, \quad \mathbf{n} \cdot \mathbf{n}' = 0, \quad s, r \in \mathbb{R}.$$

The classic Landau-de Gennes energy functional, being a nonlinear functional of Q and its spatial derivatives, takes the following general form

$$\begin{aligned} & \mathcal{F}_{LG}(Q, \nabla Q) \\ &= \int_{\mathbb{R}^3} \left\{ \underbrace{-\frac{a}{2}tr(Q^2) - \frac{b}{3}tr(Q^3) + \frac{c}{4}(tr(Q^2))^2}_{\text{bulk energy}} \right. \\ & \quad \left. + \frac{1}{2} \underbrace{\left(L_1 |\nabla Q|^2 + L_2 Q_{ij,j} Q_{ik,k} + L_3 Q_{ij,k} Q_{ik,j} + L_4 Q_{ij} Q_{kl,i} Q_{kl,j} \right)}_{\text{elastic energy}} \right\} d\mathbf{x}, \quad (1.14) \end{aligned}$$

where a, b, c are material-dependent and temperature-dependent non-negative constants and $L_i (i = 1, 2, 3, 4)$ are material dependent elastic constants. We refer to [6, 20] for more details. The energy (1.14) can not ensure Q to satisfy the natural physical constraint (1.13). For this reason, based on the mean-field Maier-Saupe energy, Ball-Majumdar [4] proposed an energy functional, which will diverge if $Q \notin \mathbb{Q}_{phy}$. There are many works to study the equilibrium solutions of the classic Landau-de Gennes model, for example, one may see [4, 19] and the references therein.

So far, there are two types of dynamic Q -tensor theories to describe the flow of nematic liquid crystal. The first type models are obtained by variational methods under physical considerations, such as Beris-Edwards model [5] and Qian-Sheng's model [24]. Let $\mathcal{F}(Q, \nabla Q)$ be the total free energy, and define

$$\mu_Q = \frac{\delta \mathcal{F}(Q, \nabla Q)}{\delta Q}.$$

The dynamical Q -tensor model of this types can be written in the following general form:

$$\frac{\partial Q}{\partial t} + \mathbf{v} \cdot \nabla Q = D^{rot}(\mu_Q) + F(Q, \mathbf{D}) + \boldsymbol{\Omega} \cdot Q - Q \cdot \boldsymbol{\Omega}, \quad (1.15)$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nabla \cdot (\sigma^{dis} + \sigma^s + \sigma^a + \sigma^d), \quad (1.16)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (1.17)$$

where \mathbf{v} is the fluid velocity, $D^{rot}(\mu_Q)$ is the rotational diffusion term, $F(Q, \mathbf{D})$ and $\boldsymbol{\Omega} \cdot Q - Q \cdot \boldsymbol{\Omega}$ are induced by the deformation part and rotation part of the velocity gradient, respectively. In addition, σ^d is the distortion stress, σ^a is the anti-symmetric part of orientational-induced stress, $\sigma^s = \gamma F(Q, \mu_Q)$ which conjugates to $F(Q, \mathbf{D})$ (γ is a constant), is the symmetric stress induced by the orientation of molecules, and σ^{dis} is an additional dissipation stress.

In Beris-Edwards's model and Qian-Sheng's model, module some constants, σ^a and σ^d are the same, i.e.,

$$\sigma_{ij}^d = \frac{\partial \mathcal{F}}{\partial (Q_{kl,j})} Q_{kl,i}, \quad \sigma^a = Q \cdot \mu_Q - \mu_Q \cdot Q. \quad (1.18)$$

In Beris-Edwards's model, the other terms are given by

$$D_{BE}^{rot} = -\Gamma \mu_Q, \quad \sigma_{BE}^{dis} = \eta \mathbf{D}, \quad \sigma_{BE}^s = F_{BE}(Q, \mu_Q),$$

$$F_{BE}(Q, A) = \xi \left(\left(Q + \frac{1}{3} \mathbf{I} \right) \cdot A + A \cdot \left(Q + \frac{1}{3} \mathbf{I} \right) - 2 \left(Q + \frac{1}{3} \mathbf{I} \right) (A : Q) \right).$$

In Qian-Sheng's model, they are given by

$$D_{QS}^{rot} = -\Gamma \mu_Q, \quad \sigma_{QS}^s = -\frac{1}{2} \frac{\mu_2^2}{\mu_1} \mu_Q, \quad F_{QS}(Q, \mathbf{D}) = -\frac{1}{2} \frac{\mu_2}{\mu_1} \mathbf{D},$$

$$\sigma_{QS}^{dis} = \beta'_1 Q(Q : \mathbf{D}) + \beta'_2 \mathbf{D} + \beta'_3 (Q \cdot \mathbf{D} + \mathbf{D} \cdot Q).$$

When taking $\mathcal{F}(Q, \nabla Q) = \mathcal{F}_{LG}(Q, \nabla Q)$, for the well-posedness results of the Beris-Edwards's model on whole space and bounded domain, we refer to [21, 22, 13] and [1, 2].

The second type is derived from the molecular kinetic theory by closure approximations. In such models, the evolution of Q is derived from the evolution of probability density function f by relation (1.12). However, one have to approximate the higher order moment such as

$$\int_{\mathbb{S}^2} \mathbf{m} \mathbf{m} \mathbf{m} \mathbf{m} f(\mathbf{x}, \mathbf{m}) d\mathbf{m} \quad (1.19)$$

by using Q . This process is called closure approximation. There are various kinds of closure approximation and then they lead to different models in Q -tensor form, which are summarized in [10, 11]. However, these models do not obey energy dissipation law. In [12], based on Doi's kinetic theory, the authors proposed a Q -tensor model with energy dissipation law by using the Bingham closure. In this paper, we are mainly concerned this model. Before introducing it, we first give a brief description of the Bingham closure.

For a given configuration distribution function $f(\mathbf{m})$ satisfying

$$\int_{\mathbb{S}^2} f(\mathbf{m}) d\mathbf{m} = 1, \quad \int_{\mathbb{S}^2} (\mathbf{m} \mathbf{m} - \frac{1}{3} \mathbf{I}) f(\mathbf{m}) d\mathbf{m} = Q,$$

the Bingham closure is to use the quasi-equilibrium distribution (also called the Bingham distribution)

$$f_Q = \frac{1}{Z_Q} \exp(B_Q : \mathbf{m} \mathbf{m}), \quad Z_Q = \int_{\mathbb{S}^2} \exp(B_Q : \mathbf{m} \mathbf{m}) d\mathbf{m},$$

to approximate f . Here, $B_Q \in \mathbb{Q}$ depends on Q and is determined by the following relation

$$\int_{\mathbb{S}^2} (\mathbf{m}\mathbf{m} - \frac{1}{3}\mathbf{I})f_Q d\mathbf{m} = Q.$$

By Proposition 1, B_Q can be uniquely determined for $Q \in \mathbb{Q}_{phy}$. Then, the fourth-order moment and the sixth-order moment of f are approximated by

$$M_Q^{(4)} = \int_{\mathbb{S}^2} \mathbf{m}\mathbf{m}\mathbf{m}\mathbf{m}f_Q d\mathbf{m}, \quad \text{and} \quad M_Q^{(6)} = \int_{\mathbb{S}^2} \mathbf{m}\mathbf{m}\mathbf{m}\mathbf{m}\mathbf{m}\mathbf{m}f_Q d\mathbf{m}.$$

Now we introduce the dynamic Q -tensor model presented in [12]. For given free energy functional $\mathcal{F}(Q, \nabla Q)$, define

$$\mu_Q = \frac{\delta \mathcal{F}(Q, \nabla Q)}{\delta Q}.$$

We introduce the following two operators

$$\begin{aligned} \mathcal{M}_Q(A) &= \frac{1}{3}A + Q \cdot A - A : M_Q^{(4)}, \\ \mathcal{N}_Q(A)_{\alpha\beta} &= \partial_i \left\{ \left[\gamma_{\perp} (M_{\alpha\beta kl}^{(4)} \delta_{ij} - \frac{1}{3} \delta_{\alpha\beta} Q_{kl} \delta_{ij}) \right. \right. \\ &\quad \left. \left. + (\gamma_{\parallel} - \gamma_{\perp}) (M_{\alpha\beta kl ij}^{(6)} - \frac{1}{3} \delta_{\alpha\beta} M_{kl ij}^{(4)}) \right] \partial_j A_{kl} \right\}. \end{aligned}$$

Based on the Doi-Onsager's molecular theory, making use of the aforementioned Bingham closure approximation, the new Q -tensor model is given as following [12]:

$$\begin{aligned} \frac{\partial Q}{\partial t} + \mathbf{v} \cdot \nabla Q &= \frac{\varepsilon}{De} \mathcal{N}_Q(\mu_Q) - \frac{2}{De} \left(\mathcal{M}_Q(\mu_Q) + \mathcal{M}_Q^T(\mu_Q) \right) \\ &\quad + \mathcal{M}_Q(\nabla \mathbf{v}) + \mathcal{M}_Q^T(\nabla \mathbf{v}), \end{aligned} \quad (1.20)$$

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla p + \frac{\gamma}{Re} \Delta \mathbf{v} + \frac{1-\gamma}{2Re} \nabla \cdot (\mathbf{D} : M_Q^{(4)}) \\ &\quad + \frac{1-\gamma}{DeRe} \left(2\nabla \cdot \mathcal{M}_Q(\mu_Q) + \mu_Q : \nabla Q \right), \end{aligned} \quad (1.21)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (1.22)$$

where De and Re are called Deborah number and Reynolds number, respectively, and $\gamma \in (0, 1)$ is a constant. The small parameter $\sqrt{\varepsilon}$ characterizes the typical interaction distance, which is usually at the scale of molecule length. The term $\mathcal{N}_Q(\mu_Q)$ represents the translational diffusion. An important feature of this model is that (1.20)-(1.22) obeys the following basic energy dissipative law (see [12])

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{v}|^2 d\mathbf{x} + \frac{1-\gamma}{ReDe} \mathcal{F}(Q, \nabla Q) \right) &= - \int_{\mathbb{R}^3} \left(\frac{\gamma}{Re} |\nabla \mathbf{v}|^2 + \frac{1-\gamma}{2Re} \mathbf{D} : M_Q^{(4)} : \mathbf{D} \right. \\ &\quad \left. - \frac{\varepsilon(1-\gamma)}{ReDe^2} \mu_Q : \mathcal{N}_Q(\mu_Q) + \frac{4(1-\gamma)}{ReDe^2} \mu_Q : \mathcal{M}_Q(\mu_Q) \right) d\mathbf{x}. \end{aligned} \quad (1.23)$$

In [12], the energy functional is also derived from Onsager's molecular theory:

$$\mathcal{F}(Q, \nabla Q) = \mathcal{F}_b(Q) + \mathcal{F}_e(Q, \nabla Q),$$

where the bulk energy $\mathcal{F}_b(Q)$ and the elastic distortion energy \mathcal{F}_e are respectively given by

$$\begin{aligned}\mathcal{F}_b(Q) &= L_0 \int \left(-\ln Z_Q + Q : B_Q - \frac{\alpha}{2} |Q|^2 \right) d\mathbf{x}, \\ \mathcal{F}_e(Q, \nabla Q) &= \frac{\varepsilon}{2} \int \left\{ L_1 |\nabla Q|^2 + L_2 \left(Q_{ik,i} Q_{jk,j} + Q_{jk,i} Q_{ik,j} \right) + L_3 |\nabla Q^{(4)}|^2 \right. \\ &\quad \left. + L_4 \left(Q_{iklm,i}^{(4)} Q_{jklm,j}^{(4)} + Q_{jklm,i}^{(4)} Q_{iklm,j}^{(4)} \right) + L_5 Q_{ijkl,i}^{(4)} Q_{kl,j}^{(4)} \right\} d\mathbf{x},\end{aligned}$$

where $Q^{(4)} = Q^{(4)}(Q)$ is the fourth order symmetric traceless moment of the Bingham distribution f_Q . Namely,

$$\begin{aligned}Q_{ijkl}^{(4)} &:= \int_{\mathbb{S}^2} \left\{ m_i m_j m_k m_l - \frac{1}{7} \left(m_i m_j \delta_{kl} + m_k m_l \delta_{ij} + m_i m_k \delta_{jl} + m_j m_l \delta_{ik} \right. \right. \\ &\quad \left. \left. + m_i m_l \delta_{jk} + m_j m_k \delta_{il} \right) + \frac{1}{35} \left(\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \right\} f_Q d\mathbf{m}.\end{aligned}$$

The difference between $Q^{(4)}$ and $M_Q^{(4)}$ is that $Q^{(4)}$ is traceless, i.e. $Q_{ijkk}^{(4)} = 0$, while $M_Q^{(4)}$ is not. The bulk energy \mathcal{F}_b is equivalent to the penalized energy derived by Ball-Majumdar in [4]. Thus, the order parameter tensor Q should satisfy the physical constraint (1.13).

The parameters appearing in the system (1.20)-(1.22) have clear physical significance but not are phenomenological. In [12], the coefficients $L_i (i = 0, 1, \dots, 5)$ are also explicitly calculated in terms of physical molecular parameters. The parameter ε appears in the elastic energy \mathcal{F}_e due to the fact that the ratios between the coefficients of \mathcal{F}_e and the ones in \mathcal{F}_b are at the order of square of molecule length. Another important feature of the molecule-based Q -tensor system (1.20)-(1.22) is that the translational and rotational diffusions are still maintained.

1.3. Motivations and main results. The connection between different level of liquid crystal theories is a problem of both physical and mathematical importance. Based on a formal asymptotical expansion, Kuzuu-Doi [15] and E-Zhang [7] derived the Ericksen-Leslie equation from the Doi-Onsager equations by taking small Deborah number limit for spacial homogeneous case and inhomogeneous case, respectively. Wang-Zhang-Zhang rigorously justified this limit in [27] before the first singularity time of the Ericksen-Leslie system. In [29], they also presented a rigorous derivation from Beris-Edwards model to Ericksen-Leslie model. In [12], it is proposed a systematic study on the modeling for liquid crystals in both static and dynamic cases. They derived a Q -tensor model from Onsager's molecular theory and Doi's kinetic theory, which is introduced in the previous subsection, and also derived Oseen-Frank model and Ericksen-Leslie model.

The main aim of this paper is to prove the local well-posedness for strong solution of the molecule-based Q -tensor model, and also to show that the strong solution will converges to the solution of Ericksen-Leslie system under the limit of Deborah number $De \rightarrow 0$.

In this paper, to avoid some tedious technical difficulties, we will only consider the case when the translational diffusion $\mathcal{N}_Q(\mu_Q) = 0$ and the coefficients $L_0 =$

1, $L_3 = L_4 = L_5 = 0$. Then

$$\mu_Q = \frac{\delta \mathcal{F}_b(Q)}{\delta Q} + \frac{\delta \mathcal{F}_e(\nabla Q)}{\delta Q}, \quad (1.24)$$

$$\frac{\delta \mathcal{F}_b(Q)}{\delta Q} = B_Q - \alpha Q, \quad (1.25)$$

$$\left(\frac{\delta \mathcal{F}_e(Q)}{\delta Q} \right)_{ij} = -\varepsilon \left(L_1 \Delta Q_{ij} + L_2 (Q_{ik,jk} + Q_{jk,ik}) \right) =: \varepsilon \mathcal{L}(Q). \quad (1.26)$$

Then, the corresponding molecule-based Q -tensor system becomes :

$$\begin{aligned} \frac{\partial Q}{\partial t} + \mathbf{v} \cdot \nabla Q &= -\frac{2}{De} \left(\mathcal{M}_Q(B_Q - \alpha Q + \varepsilon \mathcal{L}(Q)) + \mathcal{M}_Q^T(B_Q - \alpha Q + \varepsilon \mathcal{L}(Q)) \right) \\ &\quad + \mathcal{M}_Q(\nabla \mathbf{v}) + \mathcal{M}_Q^T(\nabla \mathbf{v}), \end{aligned} \quad (1.27)$$

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla p + \frac{\gamma}{Re} \Delta \mathbf{v} + \frac{1-\gamma}{2Re} \nabla \cdot (\mathbf{D} : M_Q^{(4)}) \\ &\quad + \frac{1-\gamma}{DeRe} \nabla \cdot \left(2\mathcal{M}_Q(B_Q - \alpha Q + \varepsilon \mathcal{L}(Q)) + \varepsilon \sigma^d(Q, Q) \right), \end{aligned} \quad (1.28)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (1.29)$$

where σ^d is defined by

$$\sigma_{ji}^d(Q, \tilde{Q}) \stackrel{\text{def}}{=} -\frac{\partial \mathcal{F}_e}{\partial Q_{kl,j}} \tilde{Q}_{kl,i} = -(L_1 Q_{kl,j} \tilde{Q}_{kl,i} + L_2 Q_{km,m} \tilde{Q}_{kj,i} + L_2 Q_{kj,l} \tilde{Q}_{kl,i}).$$

It not hard to see that $\nabla \cdot \sigma^d(Q, Q)$ differs from $\mu_Q : \nabla Q$ with only pressure terms.

When $\alpha > \alpha^*$, the bulk energy function \mathcal{F}_b has stable uniaxial critical points $Q = S_2(\mathbf{n}\mathbf{n} - \frac{1}{3}\mathbf{I})$ for any $\mathbf{n} \in \mathbb{S}^2$, which correspond to nematic phase. Here, $S_2 = S_2(\alpha)$ is a increasing function of α for $\alpha > \alpha^*$, see the precise definition in (2.19). Throughout this paper, we always assume $\alpha > \alpha^*$ and $L_1 > 0, L_1 + 2L_2 > 0$. Thus, it is known from Lemma 2.2 in [29] that

$$\int \mathcal{L}(Q) : Q d\mathbf{x} \geq c_0 \int |\nabla Q|^2 d\mathbf{x},$$

for some constant $c_0 > 0$.

We first state the following the local well-posedness result.

Theorem 1.1. *Let $s \geq 2$ be an integer. $\mathbf{n}^* \in \mathbb{S}^2$ is a constant vector and $Q^* = S_2(\mathbf{n}^*\mathbf{n}^* - \frac{1}{3}\mathbf{I})$. If the initial data satisfies*

$$\mathbf{v}_I(\mathbf{x}) \in H^s(\mathbb{R}^3), \quad Q_I(\mathbf{x}) - Q^* \in H^{s+1}(\mathbb{R}^3), \quad (1.30)$$

with

$$Q_I(\mathbf{x}) \in \mathbb{Q}_{phy,\delta} := \left\{ Q \in \mathbb{Q} : \text{all the eigenvalues of } Q \text{ belong to } \left[-\frac{1}{3} + \delta, \frac{2}{3} - \delta \right] \right\}, \quad (1.31)$$

for all $\mathbf{x} \in \mathbb{R}^3$, then there exists $T > 0$ and a unique solution (\mathbf{v}, Q) of the Q -tensor system (1.27)-(1.29) on $[0, T]$, such that $\mathbf{v}(0, \mathbf{x}) = \mathbf{v}_I(\mathbf{x}), Q(0, \mathbf{x}) = Q_I(\mathbf{x})$, and

$$\mathbf{v}(t, \mathbf{x}) \in C([0, T]; H^s(\mathbb{R}^3)) \cap L^2(0, T; H^{s+1}(\mathbb{R}^3)), \quad (1.32)$$

$$Q(t, \mathbf{x}) - Q^* \in C([0, T]; H^{s+1}(\mathbb{R}^3)), \quad (1.33)$$

and $Q(t, \mathbf{x}) \in \mathbb{Q}_{phy,\delta/2}$.

Next, we consider the small Deborah number limit $De \rightarrow 0$. To obtain the full Ericksen-Leslie system, we have to take $De = O(\varepsilon)$ as in [12]. For simplicity, we choose $De = \varepsilon$. Then the system can be written as:

$$\begin{aligned} \frac{\partial Q^\varepsilon}{\partial t} + \mathbf{v}^\varepsilon \cdot \nabla Q^\varepsilon = & -\frac{2}{\varepsilon} \left(\mathcal{M}_{Q^\varepsilon} (B_{Q^\varepsilon} - \alpha Q^\varepsilon + \varepsilon \mathcal{L}(Q^\varepsilon)) \right. \\ & \left. + \mathcal{M}_{Q^\varepsilon}^T (B_{Q^\varepsilon} - \alpha Q^\varepsilon + \varepsilon \mathcal{L}(Q^\varepsilon)) \right) + \mathcal{M}_{Q^\varepsilon} (\nabla \mathbf{v}^\varepsilon) + \mathcal{M}_{Q^\varepsilon}^T (\nabla \mathbf{v}^\varepsilon), \end{aligned} \quad (1.34)$$

$$\begin{aligned} \frac{\partial \mathbf{v}^\varepsilon}{\partial t} + \mathbf{v}^\varepsilon \cdot \nabla \mathbf{v}^\varepsilon = & -\nabla p^\varepsilon + \frac{\gamma}{Re} \Delta \mathbf{v}^\varepsilon + \frac{1-\gamma}{2Re} \nabla \cdot (\mathbf{D} : M_{Q^\varepsilon}^{(4)}) \\ & + \frac{1-\gamma}{\varepsilon Re} \nabla \cdot \left(2\mathcal{M}_{Q^\varepsilon} (B_{Q^\varepsilon} - \alpha Q^\varepsilon + \varepsilon \mathcal{L}(Q^\varepsilon)) + \varepsilon \sigma^d(Q^\varepsilon, Q^\varepsilon) \right), \end{aligned} \quad (1.35)$$

$$\nabla \cdot \mathbf{v}^\varepsilon = 0. \quad (1.36)$$

We define the coefficient in Ericksen-Leslie theory as:

$$\begin{aligned} \alpha_1 = -\frac{S_4}{2}, \quad \alpha_2 = -\frac{S_2}{2} \left(1 + \frac{1}{\zeta}\right), \quad \alpha_3 = -\frac{S_2}{2} \left(1 - \frac{1}{\zeta}\right), \\ \alpha_4 = \frac{4}{15} - \frac{5}{21} S_2 - \frac{1}{35} S_4, \quad \alpha_5 = \frac{1}{7} S_4 + \frac{6}{7} S_2, \quad \alpha_6 = \frac{1}{7} S_4 - \frac{1}{7} S_2, \end{aligned} \quad (1.37)$$

and

$$\gamma_1 = \frac{1}{\frac{1}{3S_2} + \frac{2}{3S_2^2} - \frac{2}{S_2^2\alpha}}, \quad \gamma_2 = -S_2, \quad \zeta \stackrel{\text{def}}{=} -\frac{\gamma_2}{\gamma_1} = \frac{1}{3} + \frac{2}{3S_2} - \frac{2}{S_2\alpha}, \quad (1.38)$$

and the elastic constants in Oseen-Frank energy are given by

$$k_1 = k_3 = 2(L_1 + L_2)S_2^2, \quad k_2 = 2L_1S_2^2, \quad k_4 = L_2S_2^2. \quad (1.39)$$

Here $S_4 = S_4(\alpha)$ is also a constant related to α , see the definition in (2.19).

For a given direction field $\mathbf{n}(t, \mathbf{x})$, we define

$$\mathcal{P}^{out}(\mathbf{Q}) = \mathbf{Q} - (\mathbf{nn} \cdot \mathbf{Q} + \mathbf{Q} \cdot \mathbf{nn}) - 2(\mathbf{Q} : \mathbf{nn})\mathbf{nn}, \quad (1.40)$$

$$\mathcal{H}_{\mathbf{n}}(Q) = \psi_1(\mathbf{nn} - \frac{1}{3}\mathbf{I})(\mathbf{nn} : Q) + \psi_2(-Q + \mathbf{nn} \cdot Q + Q \cdot \mathbf{nn} - \frac{2}{3}\mathbf{I}(\mathbf{nn} : Q)), \quad (1.41)$$

where ψ_1 and ψ_2 are constants depending on α . $\mathcal{H}_{\mathbf{n}}(Q)$ is the linearized operator of $B_Q - \alpha Q$ around the local critical point $S_2(\mathbf{nn} - \frac{1}{3}\mathbf{I})$. The detailed motivation of the above definitions will be explained in Section 4.

The second main result of this paper is stated as follows.

Theorem 1.2. *Let $(\mathbf{n}(t, \mathbf{x}), \mathbf{v}(t, \mathbf{x}))$ be a solution of the Ericksen-Leslie system (1.3)-(1.5) on $[0, T]$ with the coefficients given by (1.37)-(1.39), which satisfies*

$$\mathbf{v} \in C([0, T]; H^k), \quad \nabla \mathbf{n} \in C([0, T]; H^k) \quad \text{for } k \geq 20.$$

Let $Q_0(t, x) = S_2(\mathbf{n}(t, \mathbf{x})\mathbf{n}(t, \mathbf{x}) - \mathbf{I})$ and the functions $(Q_1, Q_2, Q_3, \mathbf{v}_1, \mathbf{v}_2)$ are determined by Proposition 8. Assume that the initial data $(Q_I^\varepsilon, \mathbf{v}_I^\varepsilon)$ takes the form

$$Q_I^\varepsilon(\mathbf{x}) = \sum_{k=0}^3 \varepsilon^k Q_3(0, \mathbf{x}) + \varepsilon^3 Q_{I,R}^\varepsilon(\mathbf{x}), \quad \mathbf{v}_I^\varepsilon(\mathbf{x}) = \sum_{k=0}^3 \varepsilon^k \mathbf{v}_k(0, \mathbf{x}) + \varepsilon^3 \mathbf{v}_{I,R}^\varepsilon(\mathbf{x}),$$

where $(Q_{I,R}^\varepsilon, \mathbf{v}_{I,R}^\varepsilon)$ satisfies

$$\|\mathbf{v}_{I,R}^\varepsilon\|_{H^2} + \|Q_{I,R}^\varepsilon\|_{H^3} + \varepsilon^{-1} \|\mathcal{P}^{out}(Q_{I,R}^\varepsilon)\|_{L^2} \leq E_0.$$

Then there exists $\varepsilon_0 > 0$ and $E_1 > 0$ such that for all $\varepsilon < \varepsilon_0$, the system (1.34)–(1.36) has a unique solution $(Q^\varepsilon(t, \mathbf{x}), \mathbf{v}^\varepsilon(t, \mathbf{x}))$ on $[0, T]$ which has the expansion

$$Q^\varepsilon(t, \mathbf{x}) = \sum_{k=0}^3 \varepsilon^k Q_k(t, \mathbf{x}) + \varepsilon^3 Q_R(t, \mathbf{x}), \quad \mathbf{v}^\varepsilon(t, \mathbf{x}) = \sum_{k=0}^3 \varepsilon^k \mathbf{v}_k(t, \mathbf{x}) + \varepsilon^3 \mathbf{v}_R(t, \mathbf{x}),$$

where (Q_R, \mathbf{v}_R) satisfies

$$\mathfrak{E}(Q_R(t), \mathbf{v}_R(t)) \leq E_1.$$

Here $\mathfrak{E}(Q, \mathbf{v})$ is defined by

$$\begin{aligned} \mathfrak{E}(Q, \mathbf{v}) \stackrel{\text{def}}{=} & \frac{1}{2} \int \left\{ (|\mathbf{v}|^2 + \mathcal{J}_{\mathbf{n}}^{-1}(Q) : Q + \frac{1-\gamma}{\varepsilon Re} \mathcal{H}_{\mathbf{n}}^\varepsilon(Q) : Q) \right. \\ & \left. + \varepsilon^2 (|\nabla \mathbf{v}|^2 + \frac{1-\gamma}{\varepsilon Re} \mathcal{H}_{\mathbf{n}}^\varepsilon(\nabla Q) : \nabla Q) + \varepsilon^4 (|\Delta \mathbf{v}|^2 + \frac{1-\gamma}{\varepsilon Re} \mathcal{H}_{\mathbf{n}}^\varepsilon(\Delta Q) : \Delta Q) \right\} d\mathbf{x}, \end{aligned}$$

and $\mathcal{H}_{\mathbf{n}}^\varepsilon(Q) = \mathcal{H}_{\mathbf{n}}(Q) + \varepsilon \mathcal{L}(Q)$.

Remark 1. It can be observed from [7] that the Leslie coefficients of Ericksen-Leslie system derived from the Doi-Onsager system have same forms as (1.37)–(1.38) except for γ_1 . The only difference is due to the Bingham closure approximation.

The remaining sections of this paper are organized as follows. In Section 2, the important properties of the Bingham closure and the critical point are presented. Section 3 is devoted to the proof for the existence of the local strong solution of the molecule-based Q -tensor system. In Section 4, we present some important linearized operators which will be used in deriving the Ericksen-Leslie system from the molecule-based Q -tensor system. In Section 5, by making the so-called Hilbert expansion, we present a rigorous derivation from the molecule-based Q -tensor theory to the Ericksen-Leslie theory.

2. The Bingham closure and the critical point. This section is mainly concerned to the important properties of the Bingham closure and the critical points.

2.1. The Bingham closure and Bingham map. The Bingham closure plays an important role in the system (1.20)–(1.22). For this, one should find $B_Q \in \mathbb{Q}$ such that

$$\int_{\mathbb{S}^2} (\mathbf{m}\mathbf{m} - \frac{1}{3}\mathbf{I}) \frac{\exp(B_Q : \mathbf{m}\mathbf{m})}{\int_{\mathbb{S}^2} \exp(B_Q : \mathbf{m}'\mathbf{m}') d\mathbf{m}'} d\mathbf{m} = Q, \quad (2.1)$$

for a given $Q \in \mathbb{Q}_{phy}$. The following proposition tells us that B_Q can be uniquely defined for any $Q \in \mathbb{Q}_{phy}$. We call this map from $Q \in \mathbb{Q}_{phy}$ to $B_Q \in \mathbb{Q}$ Bingham map.

Proposition 1 (Existence and uniqueness of B_Q). *For a given $Q \in \mathbb{Q}_{phy}$, there exists a unique $B_Q \in \mathbb{Q}$ such that (2.1) holds.*

Proof. A sketched proof is given in [4]. Here we give a detailed proof for completeness.

Define $\omega : \mathbb{Q} \rightarrow \mathbb{R}$ as:

$$\omega(B) = \ln \int_{\mathbb{S}^2} e^{\mathbf{m}\mathbf{m} : B} d\mathbf{m}. \quad (2.2)$$

Obviously, $\omega(B)$ depends only on its eigenvalues. From the fact that

$$\int_{\mathbb{S}^2} e^{\mathbf{m}\mathbf{m}:B_1} d\mathbf{m} \int_{\mathbb{S}^2} e^{\mathbf{m}\mathbf{m}:B_2} d\mathbf{m} \geq \left(\int_{\mathbb{S}^2} e^{\mathbf{m}\mathbf{m}:(B_1+B_2)/2} d\mathbf{m} \right)^2, \quad (2.3)$$

we know $\omega(B)$ is convex. Then we can define its *convex conjugate* by Legendre transformation: $\omega^*(Q) : X \rightarrow \mathbb{R}$ as

$$\omega^*(Q) = \sup_{B \in \mathbb{Q}} (B : Q - \omega(B)) \quad (2.4)$$

with domain X defined by

$$X = \{Q : \sup_{B \in \mathbb{Q}} (B : Q - \omega(B)) < +\infty\}. \quad (2.5)$$

We will prove that $X = \mathbb{Q}_{phy}$. For this, we need an elementary inequality:

Claim. Let $b_1 \leq b_2 \leq b_3$ and $q_1 \leq q_2 \leq q_3$ are the eigenvalues of B and Q respectively, then $B : Q \leq b_1 q_1 + b_2 q_2 + b_3 q_3$.

To prove it, we can assume B is diagonal without loss of generality. Suppose $Q = q_1 \mathbf{n}_1 \otimes \mathbf{n}_1 + q_2 \mathbf{n}_2 \otimes \mathbf{n}_2 + q_3 \mathbf{n}_3 \otimes \mathbf{n}_3$ with $\mathbf{n}_i \cdot \mathbf{n}_j = \delta_j^i$. Then $B : Q = \sum_{i,j=1,2,3} b_i q_j n_{ji}^2$, where $\mathbf{n}_i = (n_{i1}, n_{i2}, n_{i3})^T$. A direct computation shows that

$$\begin{aligned} b_1 q_1 + b_2 q_2 + b_3 q_3 - B : Q &= (q_1 - q_2)(b_2 - b_3)n_{13}^2 + (q_1 - q_2)(b_1 - b_2)(1 - n_{11}^2) \\ &\quad + (q_2 - q_3)(b_1 - b_2)n_{31}^2 + (q_2 - q_3)(b_2 - b_3)(1 - n_{33}^2) \\ &\geq 0, \end{aligned}$$

which yields our claim.

For $Q \in \mathbb{Q}_{phy}$ with eigenvalues $q_1 \leq q_2 \leq q_3$ and $q_i \in (-\frac{1}{3}, \frac{2}{3})$. For any $\Lambda(B) = \{b_1, b_2, b_3\}$, we can assume that $b_1 \leq b_2 \leq 0 \leq b_3$ or $b_1 \leq 0 \leq b_2 \leq b_3$. Consider the first case

$$A = \left\{ \mathbf{m} : m_1^2 - \frac{1}{3} - q_1 < 0, m_2^2 - \frac{1}{3} - q_2 < 0, m_3^2 - \frac{1}{3} - q_3 > 0 \right\},$$

or the later case

$$A = \left\{ \mathbf{m} : m_1^2 - \frac{1}{3} - q_1 < 0, m_2^2 - \frac{1}{3} - q_2 > 0, m_3^2 - \frac{1}{3} - q_3 > 0 \right\}.$$

We know that the measure of A is positive in both cases. Therefore,

$$\begin{aligned} \exp(\omega(B) - B : Q) &\geq \exp(\omega(B) - q_1 b_1 - q_2 b_2 - q_3 b_3) \\ &= \int_{\mathbb{S}^2} \exp(b_1 m_1^2 + b_2 m_2^2 + b_3 m_3^2 - q_1 b_1 - q_2 b_2 - q_3 b_3) d\mathbf{m} \\ &= \int_{\mathbb{S}^2} \exp\left(b_1\left(m_1^2 - \frac{1}{3} - q_1\right) + b_2\left(m_2^2 - \frac{1}{3} - q_2\right) + b_3\left(m_3^2 - \frac{1}{3} - q_3\right)\right) d\mathbf{m} \\ &\geq \int_A 1 d\mathbf{m} = \text{meas}(A). \end{aligned}$$

This implies that $B : Q - \omega(B) \leq -\ln(\text{meas}(A))$ is bounded. Hence $Q \in X$, i.e. $\mathbb{Q}_{phy} \subseteq X$.

On the other hand, if $q_1 \leq -\frac{1}{3}$, we take $b_1 = 2b \rightarrow -\infty, b_2 = b_3 = -b$, thus we have

$$\begin{aligned} \exp(B : Q - \omega(B)) &= \int_{\mathbb{S}^2} \exp\left(b(-m_1^2 + \frac{1}{3} + q_1)\right) d\mathbf{m} \\ &\geq 4\pi e^{b(\frac{1}{3} + q_1)} \rightarrow +\infty. \end{aligned} \quad (2.6)$$

If $q_3 > \frac{2}{3}$, taking $b_3 = 2b \rightarrow +\infty, b_1 = b_2 = -b$, then we can also obtain that $\exp(B : Q - \omega(B))$ is unbounded, which implies $X \subseteq \mathbb{Q}_{phy}$. Therefore, $X = \mathbb{Q}_{phy}$.

For $Q \in \mathbb{Q}_{phy}$, then there exists $B \in \mathbb{Q}$ such that

$$B : Q - \omega(B) = \sup_{B_1 \in \mathbb{Q}} (B_1 : Q - \omega(B_1)).$$

Thus

$$Q = (\nabla_B \omega)(B) = \frac{\int_{S^2} (\mathbf{m}\mathbf{m} - \frac{1}{3}\mathbf{I}) \exp(B : \mathbf{m}\mathbf{m}) d\mathbf{m}}{\int_{S^2} \exp(B : \mathbf{m}\mathbf{m}) d\mathbf{m}}. \quad (2.7)$$

We let $B_Q = B$, then the existence of B_Q is proved. Since $\omega(B)$ is convex, we can deduce that $(\nabla_B \omega)(B_1) \neq (\nabla_B \omega)(B_2)$ for $B_1 \neq B_2$, which implies the uniqueness. \square

The map from \mathbb{Q}_{phy} to \mathbb{Q} which satisfies (2.7) is a diffeomorphism, and so is its inverse. We denote them by $B = B(Q) : \mathbb{Q}_{phy} \rightarrow \mathbb{Q}$ and $Q = Q(B) : \mathbb{Q} \rightarrow \mathbb{Q}_{phy}$ respectively. For $\Lambda, \delta > 0$, we introduce compact subsets of \mathbb{Q} as

$$\mathbb{Q}_\Lambda = \{Q \in \mathbb{Q} : \text{all the eigenvalues of } Q \text{ belong to } [-\Lambda, \Lambda]\}, \quad (2.8)$$

$$\mathbb{Q}_{phy, \delta} = \{Q \in \mathbb{Q} : \text{all the eigenvalues of } Q \text{ belong to } [-\frac{1}{3} + \delta, \frac{2}{3} - \delta]\}. \quad (2.9)$$

The next proposition tells us that $B(Q)$ maps a compact subset of \mathbb{Q}_{phy} to a compact subset of \mathbb{Q} .

Proposition 2. *For any $\delta > 0$, there is a positive constant $\Lambda = \Lambda(\delta)$ such that, for all $Q \in \mathbb{Q}_{phy, \delta}$, $B_Q \in \mathbb{Q}_\Lambda$.*

Proof. We only need to consider the case that Q and B are both diagonal. Assume $Q = \text{diag}\{q_1, q_2, q_3\}$ and $B = \text{diag}\{b_1, b_2, b_3\}$ with $b_1 \geq b_2 \geq b_3$. Let

$$U = \left\{ \mathbf{m} : m_3^2 < \frac{\delta}{8}, m_2^2 < \frac{\delta}{4} \right\}, \quad V = \left\{ \mathbf{m} : m_3^2 > \frac{\delta}{2} \right\}. \quad (2.10)$$

Then $U \cap V = \emptyset$, and

$$\int_U e^{(b_2-b_1)m_2^2 + (b_3-b_1)(m_3^2 - \frac{\delta}{4})} d\mathbf{m} \geq \int_U e^{(b_2-b_1)\frac{\delta}{8} - (b_3-b_1)\frac{\delta}{8}} d\mathbf{m} \geq \text{meas}(U), \quad (2.11)$$

$$\int_V e^{(b_2-b_1)m_2^2 + (b_3-b_1)(m_3^2 - \frac{\delta}{4})} d\mathbf{m} \leq \int_V e^{(b_3-b_1)\frac{\delta}{4}} d\mathbf{m} = e^{(b_3-b_1)\frac{\delta}{4}} \text{meas}(V). \quad (2.12)$$

Therefore, we have

$$\begin{aligned} & q_3 + \frac{1}{3} \\ &= \frac{1}{\int_{S^2} e^{b_1 m_1^2 + b_2 m_2^2 + b_3 m_3^2} d\mathbf{m}} \left(\int_{S^2} m_3^2 e^{b_1 m_1^2 + b_2 m_2^2 + b_3 m_3^2} d\mathbf{m} \right) \\ &= \frac{1}{\int_{S^2} e^{(b_2-b_1)m_2^2 + (b_3-b_1)(m_3^2 - \frac{\delta}{4})} d\mathbf{m}} \left(\int_{S^2 \setminus V} + \int_V m_3^2 e^{(b_2-b_1)m_2^2 + (b_3-b_1)(m_3^2 - \frac{\delta}{4})} d\mathbf{m} \right) \\ &\leq \frac{\delta}{2} + \frac{1}{\int_{S^2} e^{(b_2-b_1)m_2^2 + (b_3-b_1)(m_3^2 - \frac{\delta}{4})} d\mathbf{m}} \left(\int_V e^{(b_2-b_1)m_2^2 + (b_3-b_1)(m_3^2 - \frac{\delta}{4})} d\mathbf{m} \right) \\ &\leq \frac{\delta}{2} + \frac{\text{meas}(V)}{\text{meas}(U)} e^{(b_3-b_1)\frac{\delta}{4}}, \end{aligned}$$

which implies

$$b_1 - b_3 \leq \frac{4}{\delta} \ln \left(\frac{2 \operatorname{meas}(V)}{\delta \operatorname{meas}(U)} \right) \triangleq \Lambda(\delta). \quad (2.13)$$

This concludes the proof of the proposition. \square

Proposition 3. *The Jacobian matrix $\nabla_B Q(B)$ is positive definite for any $B \in \mathbb{Q}$. Consequently, $B(Q)$ is a smooth map from \mathbb{Q}_{phy} to \mathbb{Q} .*

Proof. It is straightforward to calculate that for any non-zero $E \in \mathbb{Q}$, it holds

$$\begin{aligned} & \langle \nabla_B Q(B) E, E \rangle \\ &= \frac{\int_{\mathbb{S}^2} (\mathbf{m} \mathbf{m} : E)^2 \exp(B : \mathbf{m} \mathbf{m}) d\mathbf{m}}{\int_{\mathbb{S}^2} \exp(B : \mathbf{m} \mathbf{m}) d\mathbf{m}} - \frac{\left(\int_{\mathbb{S}^2} (\mathbf{m} \mathbf{m} : E) \exp(B : \mathbf{m} \mathbf{m}) d\mathbf{m} \right)^2}{\left(\int_{\mathbb{S}^2} \exp(B : \mathbf{m} \mathbf{m}) d\mathbf{m} \right)^2} \\ &= \frac{\int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \left[(\mathbf{m} \mathbf{m} : E) \exp(B : \mathbf{m} \mathbf{m}) - (\mathbf{m}' \mathbf{m}' : E) \exp(B : \mathbf{m}' \mathbf{m}') \right]^2 d\mathbf{m} d\mathbf{m}'}{\left(\int_{\mathbb{S}^2} \exp(B : \mathbf{m} \mathbf{m}) d\mathbf{m} \right)^2} > 0. \end{aligned}$$

Thus, the Jacobian $\nabla_B Q(B)$ is positive definite. Together with the fact that $Q(B)$ is a smooth function of B , we know the inverse $B(Q)$ is also smooth. \square

We give some estimates related to the Bingham map.

Lemma 2.1. *For any $\delta > 0$, $k \in N^*$ and constant matrix $Q^* \in \mathbb{Q}_{phy}$, there is a positive constant C_δ depending on δ such that if $Q(\mathbf{x}) \in \mathbb{Q}_{phy, \delta}$, then*

$$\|B(Q) - B(Q^*)\|_{H^k} \leq C_\delta \|Q - Q^*\|_{H^k}.$$

The above lemma is a direct consequence of Proposition 3 and Lemma 6.2 by using change of variables.

Lemma 2.2. *For any $\delta > 0$, there is a positive constant C_δ depending on δ such that if $Q_1, Q_2 \in \mathbb{Q}_{phy, \delta}$, then*

$$|B(Q_1) - B(Q_2)| \leq C_\delta |Q_1 - Q_2|.$$

Thus

$$|\partial_i B(Q)| \leq C_\delta |\partial_i Q|.$$

Moreover, for $k \in N^*$, there exists a constant $C = C(\delta, \|Q_1 - Q^*\|_{H^k}, \|Q_2 - Q^*\|_{H^k})$ such that

$$\|B(Q_1) - B(Q_2)\|_{H^k} \leq C(\delta, \|Q_1 - Q^*\|_{H^k}, \|Q_2 - Q^*\|_{H^k}) \|Q_1 - Q_2\|_{H^k}.$$

Proof. The first assertion is a direct consequence of Proposition 3. The second one can be induced by Proposition 3 and Lemma 6.4. \square

Remark 2. Since $M_Q^{(4)}$ is a smooth function of B_Q , it shares the same estimates with B_Q .

Now we give some properties for the operator $\mathcal{M}_Q : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}$

$$\mathcal{M}_Q(A) = \frac{1}{3} A + Q \cdot A - A : M_Q^{(4)}.$$

Note that \mathcal{M}_Q is defined not only for the symmetric matrix, and $\mathcal{M}_Q(A)$ is not necessarily symmetric even if A is symmetric. The following Lemma 2.3 gives some basic properties of \mathcal{M}_Q , which proof can be found in [12].

Lemma 2.3. (i) For any $Q \in \mathbb{Q}_{phy}$, it holds that

$$\mathcal{M}_Q(B_Q) = \frac{3}{2}Q.$$

(ii) \mathcal{M}_Q is self-adjoint on $\mathbb{R}^{3 \times 3}$, i.e., $\mathcal{M}_Q(A) : B = \mathcal{M}_Q(B) : A$ for $A, B \in \mathbb{R}^{3 \times 3}$;

(iii) For any $Q \in \mathbb{Q}_{phy}$ and $A \in \mathbb{R}^{3 \times 3}$, the operator $\mathcal{M}_Q(A)$ is positive, i.e.,

$$\mathcal{M}_Q(A) : A \geq 0.$$

Lemma 2.4. For any $\delta > 0$, there is a positive constant C_δ depending on δ such that if $Q(\mathbf{x}) \in \mathbb{Q}_{phy, \delta}$, $A \in \mathbb{R}^{3 \times 3}$, then it holds for any multiple index a ,

$$\|\partial^a \mathcal{M}_Q(A) - \mathcal{M}_Q(\partial^a A)\|_{L^2} \leq C_\delta (\|\nabla Q\|_{L^\infty} \|A\|_{H^{|a|-1}} + \|\nabla Q\|_{H^{|a|-1}} \|A\|_{L^\infty}). \quad (2.14)$$

Consequently, if $|a| \geq 2$, we have

$$\|\partial^a \mathcal{M}_Q(A) - \mathcal{M}_Q(\partial^a A)\|_{L^2} \leq C_\delta \|\nabla Q\|_{H^{|a|}} \|A\|_{H^{|a|-1}}. \quad (2.15)$$

Proof. With Lemma 6.3, Lemma 2.2 and Remark 2, direct computation shows that

$$\begin{aligned} \|\partial^a (A : M_Q^{(4)}) - \partial^a A : M_Q^{(4)}\|_{L^2} &\leq C(\|\nabla M_Q^{(4)}\|_{L^\infty} \|A\|_{H^{|a|-1}} + \|\nabla M_Q^{(4)}\|_{H^{|a|-1}} \|A\|_{L^\infty}) \\ &\leq C_\delta (\|\nabla B\|_{L^\infty} \|A\|_{H^{|a|-1}} + \|\nabla B\|_{H^{|a|-1}} \|A\|_{L^\infty}) \\ &\leq C_\delta (\|\nabla Q\|_{L^\infty} \|A\|_{H^{|a|-1}} + \|\nabla Q\|_{H^{|a|-1}} \|A\|_{L^\infty}), \end{aligned}$$

which concludes the lemma. \square

Lemma 2.5. For any $\delta > 0$ and $k \in N^*$, there exist constants $C_1 = C_1(\delta)$ and $C_2 = C_2(\delta, \|Q_1 - Q^*\|_{H^k}, \|Q_2 - Q^*\|_{H^k})$ such that

$$\|\mathcal{M}_{Q_1}(A) - \mathcal{M}_{Q_2}(A)\|_{H^k} \leq C_1 \|A\|_{H^k} \|Q_1 - Q_2\|_{L^\infty} + C_2 \|A\|_{L^\infty} \|Q_1 - Q_2\|_{H^k}.$$

If $0 \leq k \leq 2$, there exist constant $C = C(\delta, \|Q_1 - Q^*\|_{H^2}, \|Q_2 - Q^*\|_{H^2})$ such that

$$\begin{aligned} \|\mathcal{M}_{Q_1}(A) - \mathcal{M}_{Q_2}(A)\|_{H^k} &\leq C \|A\|_{H^2} \|Q_1 - Q_2\|_{H^k}, \\ \|\mathcal{M}_{Q_1}(A) - \mathcal{M}_{Q_2}(A)\|_{H^k} &\leq C \|A\|_{H^k} \|Q_1 - Q_2\|_{H^2}. \end{aligned}$$

Proof. From Lemma 6.1, we have that

$$\begin{aligned} \|\mathcal{M}_{Q_1}(A) - \mathcal{M}_{Q_2}(A)\|_{H^k} &\leq C(\|Q_1 - Q_2\|_{L^\infty} \|A\|_{H^k} + \|Q_1 - Q_2\|_{H^k} \|A\|_{L^\infty}) \\ &\quad + C(\|M_{Q_1}^{(4)} - M_{Q_2}^{(4)}\|_{L^\infty} \|A\|_{H^k} + \|M_{Q_1}^{(4)} - M_{Q_2}^{(4)}\|_{H^k} \|A\|_{L^\infty}). \end{aligned}$$

Then the conclusion can be deduced from Lemma 2.2 and Remark 2. \square

2.2. The energy functional and critical points. The bulk part of free energy density functional takes the following form

$$f_{bulk}(Q) \stackrel{\text{def}}{=} -\ln Z_Q + B_Q : Q - \frac{1}{2}\alpha|Q|^2.$$

A direct calculation yields that

$$\frac{\partial f_{bulk}(Q)}{\partial Q} = 0 \Rightarrow B_Q - \alpha Q = 0.$$

We say that a tensor Q_0 is a critical point of the bulk free energy density functional $f_{bulk}(Q)$ if Q_0 satisfies $B_{Q_0} - \alpha Q_0 = 0$. The critical points are completely classified in [18, 9].

Proposition 4. *Let η be a solution of the equation*

$$\frac{3e^\eta}{\int_0^1 e^{\eta x^2} dx} = 3 + 2\eta + \frac{\eta^2}{\alpha}. \quad (2.16)$$

Then there holds

$$B_Q - \alpha Q = 0 \iff B_Q = \eta(\mathbf{nn} - \frac{1}{3}\mathbf{I}), \quad \mathbf{n} \in \mathbb{S}^2, \quad (2.17)$$

and there exists a critical number $\alpha^ > 0$ such that*

- (i) when $\alpha < \alpha^*$, $\eta = 0$ is the only solution of (2.16);*
- (ii) when $\alpha = \alpha^*$, besides $\eta = 0$ there is another solution $\eta = \eta^*$ of (2.16);*
- (iii) when $\alpha > \alpha^*$, besides $\eta = 0$ there are other two solutions $\eta_1 > \eta^* > \eta_2$ of (2.16).*

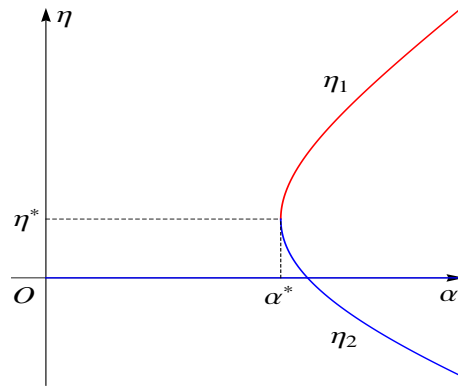


FIGURE 1. The $\alpha - \eta$ curve of the critical point.

In the sequel, we always choose $\alpha > \alpha^*$, and $\eta = \eta_1(\alpha)$ corresponding the stable equilibrium solution. We also introduce some important constants used in this paper. All of them only depend on the parameter α .

We define

$$A_k = \int_{-1}^1 x^k e^{\eta x^2} dx, \quad S_k = \frac{\int_{-1}^1 P_k(x) e^{\eta x^2} dx}{\int_{-1}^1 e^{\eta x^2} dx}, \quad (2.18)$$

where $P_k(x)$ is the k -th order Legendre polynomial. Particularly,

$$P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

Then we have

$$S_2 = \frac{3A_2 - A_0}{2A_0}, \quad S_4 = \frac{35A_4 - 30A_2 + 3A_0}{8A_0}. \quad (2.19)$$

An important fact induced by Proposition 4 is that

$$B_Q - \alpha Q = 0 \iff Q = S_2(\mathbf{nn} - \frac{1}{3}\mathbf{I}), \quad \mathbf{n} \in \mathbb{S}^2. \quad (2.20)$$

The relation

$$\alpha = \frac{A_0}{A_2 - A_4}, \quad (2.21)$$

and the inequalities

$$3A_2^2 + 2A_0A_2 - 5A_0A_4 > 0, \quad 6A_2 - 5A_4 - A_0 > 0, \quad (2.22)$$

will play important roles in Section 4. Their proofs can be found in [27], and we omit them here.

3. Existence and uniqueness of the local strong solution for the dynamical Q -tensor systems. This section is devoted to the proof for the existence of the local strong solution of the system (1.27)–(1.29). For $s \geq 2$, we define the space:

$$X(\delta, T, C_0) = \left\{ (Q, \mathbf{v}) : Q \in \mathbb{Q}_{phy, \delta/2}, \|Q - Q^*\|_{H^{s+1}} + \|\mathcal{L}(Q)\|_{L_t^2(H_x^s)} \right. \\ \left. + \|\mathbf{v}\|_{H^s} + \|\nabla \mathbf{v}\|_{L_t^2(H_x^s)} \leq C_0, \text{ a.e. } t \in [0, T] \right\}.$$

If $(Q, \mathbf{v}) \in X$, then by Sobolev imbedding, we have

$$\|Q\|_{L^\infty} + \|\nabla Q\|_{L^\infty} + \|\mathbf{v}\|_{L^\infty} \leq C(C_0).$$

The proof of Theorem 1.1 is based on iterative argument and a closed energy estimate.

3.1. Linearized system and iteration scheme. First of all, we take

$$(Q^{(0)}(t, \mathbf{x}), \mathbf{v}^{(0)}(t, \mathbf{x})) = (Q_I(\mathbf{x}), \mathbf{v}_I(\mathbf{x})) \in X(\delta, T, C_0).$$

Assuming that $(Q^{(n)}, \mathbf{v}^{(n)}) \in X(\delta, T, C_0)$ has been constructed, we construct $(Q^{(n+1)}, \mathbf{v}^{(n+1)})$ by solving the following linearized system:

$$\frac{\partial Q^{(n+1)}}{\partial t} + \mathbf{v}^{(n)} \cdot \nabla Q^{(n+1)} = -\frac{2}{De} \left(\mathcal{M}_{Q^{(n)}}(B_{Q^{(n)}} - \alpha Q^{(n)} + \varepsilon \mathcal{L}(Q^{(n+1)})) \right. \\ \left. + \mathcal{M}_{Q^{(n)}}^T(B_{Q^{(n)}} - \alpha Q^{(n)} + \varepsilon \mathcal{L}(Q^{(n+1)})) \right) + \mathcal{M}_{Q^{(n)}}(\nabla \mathbf{v}^{(n+1)}) + \mathcal{M}_{Q^{(n)}}^T(\nabla \mathbf{v}^{(n+1)}), \quad (3.1)$$

$$\frac{\partial \mathbf{v}^{(n+1)}}{\partial t} + \mathbf{v}^{(n)} \cdot \nabla \mathbf{v}^{(n+1)} = -\nabla p^{(n+1)} + \frac{\gamma}{Re} \Delta \mathbf{v}^{(n+1)} + \frac{1-\gamma}{2Re} \nabla \cdot (\mathbf{D}^{(n+1)} : M_{Q^{(n)}}^{(4)}) \\ + \frac{1-\gamma}{DeRe} \nabla \cdot \left(2\mathcal{M}_{Q^{(n)}}(B_{Q^{(n)}} - \alpha Q^{(n)} + \varepsilon \mathcal{L}(Q^{(n+1)})) + \varepsilon \sigma^d(Q^{(n)}, Q^{(n+1)}) \right), \quad (3.2)$$

$$\nabla \cdot \mathbf{v}^{(n+1)} = 0, \quad (3.3)$$

with initial data:

$$(Q^{(n+1)}(0, \mathbf{x}), \mathbf{v}^{(n+1)}(0, \mathbf{x})) = (Q_I(\mathbf{x}), \mathbf{v}_I(\mathbf{x})). \quad (3.4)$$

The existence of $(Q^{(n+1)}, \mathbf{v}^{(n+1)})$ is ensured by the classical parabolic theory, see [3] for example. Now we prove that $(Q^{(n+1)}, \mathbf{v}^{(n+1)}) \in X$, for a suitably chosen $T > 0$.

Define the energy functional

$$E_s(\mathbf{v}, Q) = \int_{\mathbb{R}^3} \left(|Q - Q^*|^2 + \frac{(1-\gamma)\varepsilon}{2DeRe} (L_1 |\nabla Q|^2 + 2L_2 |Q_{ij,j}|^2) + \frac{1}{2} |\mathbf{v}|^2 \right. \\ \left. + \frac{(1-\gamma)\varepsilon}{2DeRe} (L_1 |\nabla^{s+1} Q|^2 + 2L_2 |\nabla^s Q_{ij,j}|^2) + \frac{1}{2} |\nabla^s \mathbf{v}|^2 \right) d\mathbf{x}, \\ F_s(\mathbf{v}, Q) = \int_{\mathbb{R}^3} \left(|\mathcal{L}(Q)|^2 + |\nabla^s \mathcal{L}(Q)|^2 + |\nabla \mathbf{v}|^2 + |\nabla^{s+1} \mathbf{v}|^2 \right) d\mathbf{x}.$$

Obviously, we have

$$E_s \sim \|Q - Q^*\|_{L^2}^2 + \|\nabla Q\|_{H^s}^2 + \|\mathbf{v}\|_{H^s}^2, \quad F_s \sim \|\nabla Q\|_{H^{s+1}}^2 + \|\nabla \mathbf{v}\|_{H^s}^2.$$

Let $E_s^{(n)} = E_s(Q^{(n)}, \mathbf{v}^{(n)})$. We will prove the following closed energy estimates:

$$\frac{1}{2} \frac{d}{dt} E_s^{(n+1)} + \nu F_s^{(n+1)} \leq C(\delta, C_0, \nu)(1 + E_s^{(n+1)}), \quad (3.5)$$

for some small $\nu > 0$. The proof is split into three steps.

Step 1. L^2 energy estimate for $Q^{(n+1)} - Q^*$. From Lemma 2.2, we have

$$\begin{aligned} \|B_{Q^{(n)}} - \alpha Q^{(n)}\|_{L^2} &= \|B_{Q^{(n)}} - \alpha Q^{(n)} - B_{Q^*} + \alpha Q^*\|_{L^2} \\ &\leq C_\delta \|Q^{(n)} - Q^*\|_{L^2} \leq C(\delta, C_0). \end{aligned}$$

Therefore, by making L^2 inner product to (3.1) with $Q^{(n+1)} - Q^*$, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|Q^{(n+1)} - Q^*\|_{L^2}^2 &= \langle \partial_t Q^{(n+1)}, Q^{(n+1)} - Q^* \rangle \\ &= \frac{4}{De} \langle \mathcal{M}_{Q^{(n)}}(-B_{Q^{(n)}} + \alpha Q^{(n)}), Q^{(n+1)} - Q^* \rangle \\ &\quad - \frac{4\varepsilon}{De} \langle \mathcal{M}_{Q^{(n)}}(\mathcal{L}(Q^{(n+1)})), Q^{(n+1)} - Q^* \rangle + 2 \langle \mathcal{M}_{Q^{(n)}}(\nabla \mathbf{v}^{(n+1)}), Q^{(n+1)} - Q^* \rangle \\ &\leq \left(C(\delta, C_0) + C_\delta \|\mathcal{L}(Q^{(n+1)})\|_{L^2} + C_\delta \|\nabla \mathbf{v}^{(n+1)}\|_{L^2} \right) \|Q^{(n+1)} - Q^*\|_{L^2} \\ &\leq C(\delta, C_0) ((E_s^{(n+1)})^{1/2} + E_s^{(n+1)}). \end{aligned} \quad (3.6)$$

Step 2. L^2 estimates for $(\nabla Q^{(n+1)}, \mathbf{v}^{(n+1)})$. In this step and the next step, a key point is that we will use the self-adjointness of $\mathcal{M}_{Q^{(n)}}$. By making L^2 inner product to (3.1) with $\mathcal{L}(Q^{(n+1)})$, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \left(L_1 |\nabla Q|^2 + 2L_2 |Q_{ij,j}|^2 \right) d\mathbf{x} &= \langle \partial_t Q^{(n+1)}, \mathcal{L}(Q^{(n+1)}) \rangle \\ &= - \langle \mathbf{v}^{(n)} \cdot \nabla Q^{(n+1)}, \mathcal{L}(Q^{(n+1)}) \rangle - \frac{4\varepsilon}{De} \langle \mathcal{M}_{Q^{(n)}}(\mathcal{L}(Q^{(n+1)})), \mathcal{L}(Q^{(n+1)}) \rangle \\ &\quad + \frac{4}{De} \langle \mathcal{M}_{Q^{(n)}}(-B_{Q^{(n)}} + \alpha Q^{(n)}), \mathcal{L}(Q^{(n+1)}) \rangle + 2 \langle \mathcal{M}_{Q^{(n)}}(\nabla \mathbf{v}^{(n+1)}), \mathcal{L}(Q^{(n+1)}) \rangle \\ &\leq \|\mathbf{v}^{(n)}\|_{L^\infty} \|\nabla Q^{(n+1)}\|_{L^2} \|\mathcal{L}(Q^{(n+1)})\|_{L^2} - \nu \|\mathcal{L}(Q^{(n+1)})\|_{L^2}^2 \\ &\quad + C(\delta, C_0) \|\mathcal{L}(Q^{(n+1)})\|_{L^2} + 2 \langle \mathcal{M}_{Q^{(n)}}(\nabla \mathbf{v}^{(n+1)}), \mathcal{L}(Q^{(n+1)}) \rangle \\ &\leq C(\delta, C_0, \nu)(1 + E_s^{(n+1)}) - \nu \|\mathcal{L}(Q^{(n+1)})\|_{L^2}^2 + 2 \langle \mathcal{M}_{Q^{(n)}}(\nabla \mathbf{v}^{(n+1)}), \mathcal{L}(Q^{(n+1)}) \rangle. \end{aligned} \quad (3.7)$$

From (3.2), we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{v}^{(n+1)}\|_{L^2}^2 &= \langle \partial_t \mathbf{v}^{(n+1)}, \mathbf{v}^{(n+1)} \rangle \\ &= \frac{\gamma}{Re} \langle \Delta \mathbf{v}^{(n+1)}, \mathbf{v}^{(n+1)} \rangle + \frac{1-\gamma}{2Re} \langle \nabla \cdot (\mathbf{D}^{(n+1)} : M_{Q^{(n)}}^{(4)}), \mathbf{v}^{(n+1)} \rangle \\ &\quad + \frac{2(1-\gamma)}{DeRe} \langle \mathcal{M}_{Q^{(n)}}(-B_{Q^{(n)}} + \alpha Q^{(n)}), \nabla \mathbf{v}^{(n+1)} \rangle \end{aligned}$$

$$\begin{aligned}
& -\frac{2(1-\gamma)\varepsilon}{DeRe}\langle \mathcal{M}_{Q^{(n)}}(\mathcal{L}(Q^{(n+1)})), \nabla \mathbf{v}^{(n+1)} \rangle \\
& -\frac{(1-\gamma)\varepsilon}{DeRe}\langle \sigma^d(Q^{(n)}, Q^{(n+1)}), \nabla \mathbf{v}^{(n+1)} \rangle \\
\leq & -\frac{\gamma}{Re}\|\nabla \mathbf{v}^{(n+1)}\|_{L^2}^2 - \frac{1-\gamma}{2Re}\langle \mathbf{D}^{(n+1)} : M_{Q^{(n)}}^{(4)}, \mathbf{D}^{(n+1)} \rangle + C_\delta \|\nabla \mathbf{v}^{(n+1)}\|_{L^2} \\
& -\frac{2(1-\gamma)\varepsilon}{DeRe}\langle \mathcal{M}_{Q^{(n)}}(\nabla \mathbf{v}^{(n+1)}), \mathcal{L}(Q^{(n+1)}) \rangle \\
& + C\|\nabla Q^{(n)}\|_{H^2}\|\nabla Q^{(n+1)}\|_{L^2}\|\nabla \mathbf{v}^{(n+1)}\|_{L^2} \\
\leq & -\frac{\gamma}{2Re}\|\nabla \mathbf{v}^{(n+1)}\|_{L^2}^2 + C_\delta \|\nabla \mathbf{v}^{(n+1)}\|_{L^2} \\
& -\frac{2(1-\gamma)\varepsilon}{DeRe}\langle \mathcal{M}_{Q^{(n)}}(\nabla \mathbf{v}^{(n+1)}), \mathcal{L}(Q^{(n+1)}) \rangle + C(C_0)\|\nabla Q^{(n+1)}\|_{L^2}^2. \quad (3.8)
\end{aligned}$$

Thus, we obtain from (3.7)-(3.8) that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\|\mathbf{v}^{(n+1)}\|_{L^2}^2 + \frac{(1-\gamma)\varepsilon}{DeRe} \int_{\mathbb{R}^3} (L_1 |\nabla Q|^2 + 2L_2 |Q_{ij,j}|^2) d\mathbf{x} \right) \\
& + \frac{\gamma}{2Re} \|\nabla \mathbf{v}^{(n+1)}\|_{L^2}^2 + \nu \|\mathcal{L}(Q^{(n+1)})\|_{L^2}^2 \leq C(\delta, C_0, \nu) (1 + E_s^{(n+1)}). \quad (3.9)
\end{aligned}$$

Step 3. L^2 estimates for $(\nabla^{s+1}Q^{(n+1)}, \nabla^s \mathbf{v}^{(n+1)})$. We now turn to the estimate of the higher order derivative for $Q^{(n+1)}$,

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (L_1 |\nabla^{s+1}Q|^2 + 2L_2 |\nabla^s Q_{ij,j}|^2) d\mathbf{x} = \langle \nabla^s \partial_t Q^{(n+1)}, \nabla^s \mathcal{L}(Q^{(n+1)}) \rangle \\
& = \underbrace{-\langle \nabla^s(\mathbf{v}^{(n)} \cdot \nabla Q^{(n+1)}), \nabla^s \mathcal{L}(Q^{(n+1)}) \rangle}_I \\
& + \frac{4\varepsilon}{De} \underbrace{\langle -\nabla^s \mathcal{M}_{Q^{(n)}}(\mathcal{L}(Q^{(n+1)})), \nabla^s \mathcal{L}(Q^{(n+1)}) \rangle}_{II} \\
& + \frac{4}{De} \underbrace{\langle \nabla^s \mathcal{M}_{Q^{(n)}}(-B_{Q^{(n)}} + \alpha Q^{(n)}), \nabla^s \mathcal{L}(Q^{(n+1)}) \rangle}_{III} \\
& + 2 \underbrace{\langle \nabla^s \mathcal{M}_{Q^{(n)}}(\nabla \mathbf{v}^{(n+1)}), \nabla^s \mathcal{L}(Q^{(n+1)}) \rangle}_{IV}.
\end{aligned}$$

These terms can be estimated as following:

$$\begin{aligned}
I & \leq C\|\mathbf{v}^{(n)}\|_{H^s}\|\nabla Q^{(n+1)}\|_{H^s}\|\nabla^s \mathcal{L}(Q^{(n+1)})\|_{L^2} \leq C(\delta, C_0)(E_s^{(n+1)}F_s^{(n+1)})^{1/2}, \\
II & \leq -\langle \mathcal{M}_{Q^{(n)}}(\nabla^s \mathcal{L}(Q^{(n+1)})), \nabla^s \mathcal{L}(Q^{(n+1)}) \rangle \\
& + \langle [\nabla^s, \mathcal{M}_{Q^{(n)}}]\mathcal{L}(Q^{(n+1)}), \nabla^s \mathcal{L}(Q^{(n+1)}) \rangle \\
& \leq -\nu \|\nabla^s \mathcal{L}(Q^{(n+1)})\|_{L^2}^2 \\
& + C(\delta)\|Q^{(n)} - Q^*\|_{H^s}\|\mathcal{L}(Q^{(n+1)})\|_{H^{s-1}}\|\nabla^s \mathcal{L}(Q^{(n+1)})\|_{L^2} \\
& \leq -\nu \|\nabla^s \mathcal{L}(Q^{(n+1)})\|_{L^2}^2 + C(\delta, C_0)(E_s^{(n+1)}F_s^{(n+1)})^{1/2}, \\
III & \leq C_\delta \|Q^{(n)} - Q^*\|_{H^{s+1}}\|\nabla^{s-1} \mathcal{L}(Q^{(n+1)})\|_{L^2} \leq C(\delta, C_0)(E_s^{(n+1)})^{1/2},
\end{aligned}$$

$$\begin{aligned}
IV &= 2\langle \mathcal{M}_{Q^{(n)}}(\nabla^{s+1}\mathbf{v}^{(n+1)}), \nabla^s \mathcal{L}(Q^{(n+1)}) \rangle \\
&\quad + 2\langle [\nabla^s, \mathcal{M}_{Q^{(n)}}] \nabla \mathbf{v}^{(n+1)}, \nabla^s \mathcal{L}(Q^{(n+1)}) \rangle \\
&\leq 2\langle \mathcal{M}_{Q^{(n)}}(\nabla^{s+1}\mathbf{v}^{(n+1)}), \nabla^s \mathcal{L}(Q^{(n+1)}) \rangle \\
&\quad + C(\delta) \|Q^{(n)}\|_{H^s} \|\nabla \mathbf{v}^{(n+1)}\|_{H^{s-1}} \|\nabla^s \mathcal{L}(Q^{(n+1)})\|_{L^2} \\
&\leq 2\langle \mathcal{M}_{Q^{(n)}}(\nabla^{s+1}\mathbf{v}^{(n+1)}), \nabla^s \mathcal{L}(Q^{(n+1)}) \rangle + C(\delta, C_0)(E_s^{(n+1)} F_s^{(n+1)})^{1/2}.
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \left(L_1 |\nabla^{s+1} Q|^2 + 2L_2 |\nabla^s Q_{ij,j}|^2 \right) d\mathbf{x} \\
&\leq 2\langle \mathcal{M}_{Q^{(n)}}(\nabla^{s+1}\mathbf{v}^{(n+1)}), \nabla^s \mathcal{L}(Q^{(n+1)}) \rangle - \nu \|\nabla^s \mathcal{L}(Q^{(n+1)})\|_{L^2}^2 \\
&\quad + C(\delta, C_0)(E_s^{(n+1)})^{1/2} \left(1 + (F_s^{(n+1)})^{1/2} \right). \tag{3.10}
\end{aligned}$$

For the estimate of the higher order derivative for $\mathbf{v}^{(n+1)}$, we have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \frac{DeRe}{1-\gamma} \|\nabla^s \mathbf{v}^{(n+1)}\|_{L^2}^2 = \frac{DeRe}{1-\gamma} \langle \partial_t \nabla^s \mathbf{v}^{(n+1)}, \nabla^s \mathbf{v}^{(n+1)} \rangle \\
&= - \frac{DeRe}{1-\gamma} \langle \nabla^s (\mathbf{v}^{(n)} \cdot \nabla \mathbf{v}^{(n+1)}), \nabla^s \mathbf{v}^{(n+1)} \rangle + \frac{\gamma De}{1-\gamma} \langle \nabla^s \Delta \mathbf{v}^{(n+1)}, \nabla^s \mathbf{v}^{(n+1)} \rangle \\
&\quad - \frac{De}{2} \langle \nabla^s (\mathbf{D}^{(n+1)} : M_{Q^{(n)}}^{(4)}), \nabla^{s+1} \mathbf{v}^{(n+1)} \rangle \\
&\quad - 2\langle \nabla^s \mathcal{M}_{Q^{(n)}}(B_{Q^{(n)}} - \alpha Q^{(n)}), \nabla^{s+1} \mathbf{v}^{(n+1)} \rangle \\
&\quad - 2\varepsilon \langle \nabla^s \mathcal{M}_{Q^{(n)}}(\mathcal{L}(Q^{(n+1)})), \nabla^{s+1} \mathbf{v}^{(n+1)} \rangle \\
&\quad - \varepsilon \langle \nabla^s (\sigma^d(Q^{(n)}, Q^{(n+1)})), \nabla^{s+1} \mathbf{v}^{(n+1)} \rangle \\
&\triangleq I + II + III + IV + V + VI.
\end{aligned}$$

Estimating them term by term, we obtain

$$\begin{aligned}
I &\leq C(\|\mathbf{v}^{(n)}\|_{L^\infty} \|\nabla \mathbf{v}^{(n+1)}\|_{H^s} + \|\mathbf{v}^{(n)}\|_{H^s} \|\nabla \mathbf{v}^{(n+1)}\|_{L^\infty}) \|\nabla^s \mathbf{v}^{(n+1)}\|_{L^2} \\
&\leq C_\delta \left(E_s^{(n+1)} (E_s^{(n+1)} + F_s^{(n+1)}) \right)^{1/2}, \\
III &= - \frac{De}{2} \langle (\nabla^s \mathbf{D}^{(n+1)} : M_{Q^{(n)}}^{(4)}), \nabla^s \mathbf{D}^{(n+1)} \rangle \\
&\quad + \frac{De}{2} \langle [\nabla^s, M_{Q^{(n)}}^{(4)}] \mathbf{D}^{(n+1)}, \nabla^s \mathbf{D}^{(n+1)} \rangle \\
&\leq - \frac{De}{2} \langle (\nabla^s \mathbf{D}^{(n+1)} : M_{Q^{(n)}}^{(4)}), \nabla^s \mathbf{D}^{(n+1)} \rangle \\
&\quad + C(\delta, C_0) \|\nabla Q^{(n)}\|_{H^s} \|\mathbf{D}^{(n+1)}\|_{H^{s-1}} \|\nabla^{s+1} \mathbf{v}^{(n+1)}\|_{L^2} \\
&\leq - \frac{De}{2} \langle (\nabla^s \mathbf{D}^{(n+1)} : M_{Q^{(n)}}^{(4)}), \nabla^s \mathbf{D}^{(n+1)} \rangle + C(\delta, C_0)(E_s^{(n+1)} F_s^{(n+1)})^{1/2}, \\
IV &\leq C(\delta) \|\nabla Q^{(n)}\|_{H^s} \|\nabla^s \mathbf{v}^{(n+1)}\|_{L^2} \leq C(\delta, C_0)(E_s^{(n+1)})^{1/2}, \\
V &= - 2\varepsilon \langle \mathcal{M}_{Q^{(n)}}(\nabla^s \mathcal{L}(Q^{(n+1)})), \nabla^{s+1} \mathbf{v}^{(n+1)} \rangle \\
&\quad + 2\varepsilon \langle [\nabla^s, \mathcal{M}_{Q^{(n)}}] \mathcal{L}(Q^{(n+1)}), \nabla^{s+1} \mathbf{v}^{(n+1)} \rangle
\end{aligned}$$

$$\begin{aligned}
&\leq -2\varepsilon \langle \mathcal{M}_{Q^{(n)}}(\nabla^s \mathcal{L}(Q^{(n+1)})), \nabla^{s+1} \mathbf{v}^{(n+1)} \rangle \\
&\quad + C(\delta) \|Q^{(n)} - Q^*\|_{H^s} \|\mathcal{L}(Q^{(n+1)})\|_{H^{s-1}} \|\nabla^{s+1} \mathbf{v}^{(n+1)}\|_{L^2} \\
&\leq -2\varepsilon \langle \mathcal{M}_{Q^{(n)}}(\nabla^s \mathcal{L}(Q^{(n+1)})), \nabla^{s+1} \mathbf{v}^{(n+1)} \rangle + C(\delta, C_0) (E_s^{(n+1)} F_s^{(n+1)})^{1/2}, \\
VI &\leq C \|\nabla Q^{(n)}\|_{H^s} \|\nabla Q^{(n+1)}\|_{H^s} \|\nabla \mathbf{v}^{(n+1)}\|_{H^s} \leq C(C_0) (E_s^{(n+1)} F_s^{(n+1)})^{1/2}.
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \frac{DeRe}{1-\gamma} \|\nabla^s \mathbf{v}^{(n+1)}\|_{L^2}^2 \\
&\leq -\frac{\gamma De}{1-\gamma} \|\nabla^{s+1} \mathbf{v}^{(n+1)}\|_{L^2}^2 - 2\varepsilon \langle \mathcal{M}_{Q^{(n)}}(\nabla^s \mathcal{L}(Q^{(n+1)})), \nabla^{s+1} \mathbf{v}^{(n+1)} \rangle \\
&\quad + C(\delta, C_0) (E_s^{(n+1)})^{1/2} \left(1 + (F_s^{(n+1)})^{1/2}\right). \tag{3.11}
\end{aligned}$$

Combining (3.6), (3.9), (3.10) and (3.11), we know that it holds

$$\frac{1}{2} \frac{d}{dt} E_s^{(n+1)} + \nu F_s^{(n+1)} \leq C(\delta, C_0) (1 + E_s^{(n+1)}), \tag{3.12}$$

for $\nu > 0$ small enough. By Gronwall's inequality, we get

$$E_s^{(n+1)}(t) \leq (1 + E_s^{(n+1)}(0)) e^{C(\delta, C_0)t} - 1 = (1 + E_s(Q_I, \mathbf{v}_I)) e^{C(\delta, C_0)t} - 1, \tag{3.13}$$

for any $t \in [0, T]$. Then if we take $T_0 > 0$ such that $C(\delta, C_0)T_0 \leq \ln(1 + C_0) - \ln(1 + E_s(Q_I, \mathbf{v}_I))$, then $\sup_{0 \leq t \leq T_0} E_s^{(n+1)}(t) \leq C_0$. In addition,

$$\begin{aligned}
&\left\| \int_0^t \partial_t Q^{(n+1)}(t, \mathbf{x}) dt \right\|_{L^\infty} \leq \int_0^t \|\partial_t Q^{(n+1)}(t, \mathbf{x})\|_{H^2} dt \\
&\leq C(\delta, C_0) \int_0^t \left(\|\mathcal{L}(Q^{(n+1)})\|_{H^2} + \|\nabla \mathbf{v}^{(n+1)}\|_{H^2} + \|Q^{(n+1)} - Q^*\|_{H^3} + 1 \right) dt \\
&\leq C(\delta, C_0)t.
\end{aligned}$$

Thus, together with the assumption $Q_I \in \mathbb{Q}_{phy, \delta}$, it yields that $Q^{(n+1)} \in \mathbb{Q}_{phy, \delta/2}$ for $t \in [0, T_0]$, if we choose T_0 sufficiently small. Then we obtain $(Q^{(n+1)}, \mathbf{v}^{(n+1)}) \in X(\delta, T, C_0)$ for $T \leq T_0$.

3.2. Convergence of the sequence. In this subsection, we are going to show that the approximate solution sequence $\{(\mathbf{v}^{(\ell)}, Q^{(\ell)})\}_{\ell \in \mathbb{N}}$ is a Cauchy sequence.

We set

$$\begin{aligned}
\delta_Q^{\ell+1} &= Q^{(\ell+1)} - Q^{(\ell)}, \quad \delta_B^\ell = B_{Q^{(\ell)}} - B_{Q^{(\ell-1)}}, \quad \delta_{M^{(4)}}^\ell = M_{Q^{(\ell)}}^{(4)} - M_{Q^{(\ell-1)}}^{(4)}, \\
\delta_{\mathbf{v}}^{\ell+1} &= \mathbf{v}^{(\ell+1)} - \mathbf{v}^{(\ell)}, \quad \delta_{\mathbf{D}}^{\ell+1} = \mathbf{D}^{(\ell+1)} - \mathbf{D}^{(\ell)}, \quad \delta_p^{\ell+1} = p^{(\ell+1)} - p^{(\ell)}.
\end{aligned}$$

By taking the difference between the equations for $(\mathbf{v}^{(\ell+1)}, Q^{(\ell+1)})$ and $(\mathbf{v}^{(\ell)}, Q^{(\ell)})$, we find that

$$\frac{\partial \delta_Q^{\ell+1}}{\partial t} + \mathbf{v}^{(\ell)} \cdot \nabla \delta_Q^{\ell+1} = \frac{2\varepsilon}{De} \left(\mathcal{M}_{Q^{(\ell)}}(\mathcal{L}(\delta_Q^{\ell+1})) + \mathcal{M}_{Q^{(\ell)}}^T(\mathcal{L}(\delta_Q^{\ell+1})) \right)$$

$$+ \mathcal{M}_{Q^{(\ell)}}(\nabla \delta_{\mathbf{v}}^{\ell+1}) + \mathcal{M}_{Q^{(\ell)}}^T(\nabla \delta_{\mathbf{v}}^{\ell+1}) + \delta F_1^\ell, \quad (3.14)$$

$$\begin{aligned} \frac{\partial \delta_{\mathbf{v}}^{\ell+1}}{\partial t} + \mathbf{v}^{(\ell)} \cdot \nabla \delta_{\mathbf{v}}^{\ell+1} = & -\nabla \delta_p^{\ell+1} + \frac{\gamma}{Re} \Delta \delta_{\mathbf{v}}^{\ell+1} + \frac{1-\gamma}{2Re} \nabla \cdot (\delta_{\mathbf{D}}^{\ell+1} : M_{Q^{(\ell)}}^{(4)}) + \nabla \cdot \delta F_2^\ell \\ & - \frac{(1-\gamma)\varepsilon}{DeRe} \nabla \cdot \left(2\mathcal{M}_{Q^{(\ell)}}(\mathcal{L}(\delta_Q^{\ell+1})) + \sigma^d(Q^{(\ell)}, \delta_Q^{\ell+1}) \right), \end{aligned} \quad (3.15)$$

$$\nabla \cdot \delta_{\mathbf{v}}^{\ell+1} = 0, \quad (3.16)$$

where

$$\begin{aligned} \delta F_1^\ell = & \frac{2\varepsilon}{De} \left(\delta_Q^\ell \cdot \mathcal{L}(Q^{(\ell)}) + \mathcal{L}(Q^{(\ell)}) \cdot \delta_Q^\ell + 2\mathcal{L}(Q^{(\ell)}) : \delta_{M^{(4)}}^\ell \right) \\ & + \delta_Q^\ell \cdot \nabla \mathbf{v}^{(\ell)} + (\nabla \mathbf{v}^{(\ell)})^T \cdot \delta_Q^\ell - 2\mathbf{D}^{(\ell)} : \delta_{M^{(4)}}^\ell - \delta_{\mathbf{v}}^\ell \cdot \nabla Q^{(\ell)} \\ & + \frac{2}{De} \left(\mathcal{M}_{Q^{(\ell)}}(-B_{Q^{(\ell)}} + \alpha Q^{(\ell)}) + \mathcal{M}_{Q^{(\ell)}}^T(-B_{Q^{(\ell)}} + \alpha Q^{(\ell)}) \right. \\ & \left. - \mathcal{M}_{Q^{(\ell-1)}}(-B_{Q^{(\ell-1)}} + \alpha Q^{(\ell-1)}) - \mathcal{M}_{Q^{(\ell-1)}}^T(-B_{Q^{(\ell-1)}} + \alpha Q^{(\ell-1)}) \right), \\ \delta F_2^\ell = & -\frac{1-\gamma}{DeRe} \left(2\mathcal{M}_{Q^{(\ell)}}(-B_{Q^{(\ell)}} + \alpha Q^{(\ell)}) - 2\mathcal{M}_{Q^{(\ell-1)}}(-B_{Q^{(\ell-1)}} + \alpha Q^{(\ell-1)}) \right. \\ & \left. + \varepsilon(2\delta_Q^\ell \cdot \mathcal{L}(Q^{(\ell)}) - 2\mathcal{L}(Q^{(\ell)}) : \delta_{M^{(4)}}^\ell + \sigma^d(\delta_Q^\ell, Q^{(\ell)})) \right) \\ & + \frac{1-\gamma}{2Re} \mathbf{D}^{(\ell)} : \delta_{M^{(4)}}^\ell - \delta_{\mathbf{v}}^\ell \otimes \mathbf{v}^{(\ell)}. \end{aligned}$$

From Lemma 2.5, we have

$$\begin{aligned} \|\delta F_1^\ell\|_{L^2} & \leq C(\delta, C_0)(\|\delta_Q^\ell\|_{H^1} + \|\delta_{\mathbf{v}}^\ell\|_{L^2}), \\ \|\delta F_2^\ell\|_{L^2} & \leq C(\delta, C_0)(\|\delta_Q^\ell\|_{H^1} + \|\delta_{\mathbf{v}}^\ell\|_{L^2}). \end{aligned}$$

Similar to the proof of (3.9), we can deduce that there exist $\nu > 0$ small enough and $C(\delta, C_0, \nu) > 0$, such that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|\delta_{\mathbf{v}}^{\ell+1}\|_{L^2}^2 + \|\delta_Q^{\ell+1}\|_{L^2}^2 + \frac{(1-\gamma)\varepsilon}{DeRe} \int_{\mathbb{R}^3} (L_1 |\nabla \delta_Q^{\ell+1}|^2 + 2L_2 |(\delta_Q^{\ell+1})_{ij,j}|^2) d\mathbf{x} \right) \\ & + \frac{\gamma}{2Re} \|\nabla \delta_{\mathbf{v}}^{\ell+1}\|_{L^2}^2 + \nu \|\mathcal{L}(\delta_Q^{\ell+1})\|_{L^2}^2 \\ & \leq C(\delta, C_0, \nu) (\|\delta_{\mathbf{v}}^{\ell+1}\|_{L^2}^2 + \|\delta_Q^{\ell+1}\|_{H^1}^2 + \|\delta_{\mathbf{v}}^\ell\|_{L^2}^2 + \|\delta_Q^\ell\|_{H^1}^2). \end{aligned} \quad (3.17)$$

We denote

$$\tilde{E}_0^{(\ell)}(t) = \|\delta_{\mathbf{v}}^\ell\|_{L^2}^2 + \|\delta_Q^\ell\|_{L^2}^2 + \frac{(1-\gamma)\varepsilon}{DeRe} \int_{\mathbb{R}^3} (L_1 |\nabla \delta_Q^{\ell+1}|^2 + 2L_2 |(\delta_Q^{\ell+1})_{ij,j}|^2) d\mathbf{x}.$$

Then (3.17) implies

$$\frac{d}{dt} \tilde{E}_0^{(\ell+1)}(t) \leq C \left(\tilde{E}_0^{(\ell)}(t) + \tilde{E}_0^{(\ell+1)}(t) \right).$$

Thus, we get

$$\tilde{E}_0^{(\ell+1)}(t) \leq C \int_0^t e^{C(t-\tau)} \tilde{E}_0^{(\ell)}(\tau) d\tau \leq C \int_0^T e^{C(T-\tau)} d\tau \sup_{t \in (0, T]} \tilde{E}_0^{(\ell)}(t).$$

Taking $T < T_0$ small enough such that $C \int_0^T e^{C(T-\tau)} d\tau \leq \frac{1}{2}$, we obtain

$$\sup_{t \in (0, T]} \tilde{E}_0^{(\ell+1)}(t) \leq \frac{1}{2} \sup_{t \in (0, T]} \tilde{E}_0^{(\ell)}(t).$$

Therefore, there exist $Q - Q^* \in C([0, T], H^1)$ and $\mathbf{v} \in C([0, T], L^2)$, such that

$$Q^{(n)} - Q^* \rightarrow Q - Q^* \in C([0, T], H^1), \quad \mathbf{v}^{(n)} \rightarrow \mathbf{v} \in C([0, T], L^2). \quad (3.18)$$

By the uniform bounds and interpolation, we have for any $s' \in (0, s)$,

$$Q^{(n)} - Q^* \rightarrow Q - Q^* \in C([0, T], H^{s'+1}), \quad \mathbf{v}^{(n)} \rightarrow \mathbf{v} \in C([0, T], H^{s'}). \quad (3.19)$$

Thus we (Q, \mathbf{v}) is a classical solution of (1.27)-(1.29). The uniqueness of (Q, \mathbf{v}) is guaranteed by the same energy estimate as we have done to prove the convergence of $\{(Q^{(n)}, \mathbf{v}^{(n)})\}$. Moreover, by the standard regularity argument for parabolic system, we have that

$$Q - Q^* \in C([0, T], H^{s+1}), \quad \mathbf{v} \in C([0, T], H^s) \cap L^2([0, T], H^{s+1}). \quad (3.20)$$

We omit the details here. This complete the proof of Theorem 1.1.

4. Some linearized operators. In this section, we study some important linearized operators which will be used in deriving the Ericksen-Leslie system from the molecule-based Q -tensor system (1.27)-(1.29).

For a given $\bar{Q} = Q(\bar{B})$, the linearized operator of $Q(B)$ is defined as:

$$\begin{aligned} \mathcal{Q}_{\bar{Q}}(B) &:= \frac{d}{dt}(Q(\bar{B} + tB) - Q(\bar{B})) \\ &= M_{\bar{Q}}^{(4)} : B - \left(\frac{1}{3}\mathbf{I} + \bar{Q}\right)(\bar{Q} : B), \quad \text{for } B \in \mathbb{Q}. \end{aligned}$$

We can also introduce the linearized operator of $B(Q)$ around \bar{Q} , which is actually $\mathcal{Q}_{\bar{Q}}^{-1}$, since $Q(B)$ and $B(Q)$ are inverse functions of each other.

The following proposition shows that $\mathcal{Q}_{\bar{Q}}$ is a self-adjoint and positive operator.

Proposition 5. For $\bar{Q} \in \mathbb{Q}_{phy, \delta}$ and $B_1, B_2 \in \mathbb{Q}$, we have $\mathcal{Q}_{\bar{Q}}(B_1) : B_2 = \mathcal{Q}_{\bar{Q}}(B_2) : B_1$. Moreover, if $B_1 \neq 0$, then $\mathcal{Q}_{\bar{Q}}(B_1) : B_1 \geq 0$.

Proof. By the definition of $M_{\bar{Q}}^{(4)}$ and the fact that $\int_{\mathbb{S}^2} f_{\bar{Q}} d\mathbf{m} = 1$, it is direct to check

$$\mathcal{Q}_{\bar{Q}}(B_1) : B_2 = \int_{\mathbb{S}^2} (\mathbf{m}\mathbf{m} : B_1)(\mathbf{m}\mathbf{m} : B_2) f_{\bar{Q}} d\mathbf{m} - (\bar{Q} : B_1)(\bar{Q} : B_2) = \mathcal{Q}_{\bar{Q}}(B_2) : B_1,$$

$$\mathcal{Q}_{\bar{Q}}(B_1) : B_1 = \int_{\mathbb{S}^2} (\mathbf{m}\mathbf{m} : B_1)^2 f_{\bar{Q}} d\mathbf{m} - \left(\int_{\mathbb{S}^2} (\mathbf{m}\mathbf{m} : B_1) f_{\bar{Q}} d\mathbf{m} \right)^2 \geq 0,$$

which concludes the proof. \square

We are particularly interested in the linearized operators around the equilibrium tensor $Q_0 = S_2(\mathbf{nn} - \frac{1}{3}\mathbf{I})$, where S_2 are introduced in Section 2. We denote $\mathcal{Q}_{Q_0}(B)$ by $\mathcal{Q}_{\mathbf{n}}(B)$ for $B_0 = \eta(\mathbf{nn} - \frac{1}{3}\mathbf{I})$. For use of convenience, we calculate $\mathcal{Q}_{\mathbf{n}}$ explicitly.

For the equilibrium tensor Q_0 , the distribution function f_{Q_0} and the order parameter tensor $M_{Q_0}^{(4)}$ can be written as

$$f_{Q_0} = \frac{e^{\eta(\mathbf{m} \cdot \mathbf{n})^2}}{\int_{\mathbb{S}^2} e^{\eta(\mathbf{m} \cdot \mathbf{n})^2} d\mathbf{m}}, \quad (4.1)$$

$$\begin{aligned} M_{Q_0,ijkl}^{(4)} &= S_4 n_i n_j n_k n_l + \frac{S_2 - S_4}{7} (n_i n_j \delta_{kl} + n_i n_k \delta_{jl} + n_i n_l \delta_{jk} + n_j n_k \delta_{il} \\ &\quad + n_j n_l \delta_{ik} + n_k n_l \delta_{ij}) + \left(\frac{S_4}{35} - \frac{2S_2}{21} + \frac{1}{15} \right) (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \end{aligned} \quad (4.2)$$

Substituting (4.1) and (4.2) to the linear operator

$$\mathcal{Q}_{\mathbf{n}}(Q) = M_{Q_0}^{(4)} : Q - \frac{1}{3}\mathbf{I}(Q_0 : Q) - Q_0(Q_0 : Q),$$

then we have

$$\mathcal{Q}_{\mathbf{n}}(Q) = \xi_1(\mathbf{nn} - \frac{1}{3}\mathbf{I})(\mathbf{nn} : Q) + \xi_2(\mathbf{nn} \cdot Q + Q \cdot \mathbf{nn} - \frac{2}{3}\mathbf{I}(\mathbf{nn} : Q)) + \xi_3 Q, \quad (4.3)$$

where

$$\xi_1 = S_4 - S_2^2, \quad \xi_2 = \frac{2(S_2 - S_4)}{7}, \quad \xi_3 = 2\left(\frac{S_4}{35} - \frac{2S_2}{21} + \frac{1}{15}\right). \quad (4.4)$$

To calculate $\mathcal{Q}_{\mathbf{n}}^{-1}$ explicitly, we may assume that

$$\mathcal{Q}_{\mathbf{n}}^{-1}(Q) = \psi_1(\mathbf{nn} - \frac{1}{3}\mathbf{I})(\mathbf{nn} : Q) + \psi_2(\mathbf{nn} \cdot Q + Q \cdot \mathbf{nn} - \frac{2}{3}\mathbf{I}(\mathbf{nn} : Q)) + \psi_3 Q, \quad (4.5)$$

where $\psi_i (1 \leq i \leq 3)$ are constants. Then we have

$$\begin{aligned} & \mathcal{Q}_{\mathbf{n}}^{-1}(\mathcal{Q}_{\mathbf{n}}(Q)) \\ &= \psi_1(\mathbf{nn} - \frac{1}{3}\mathbf{I})(\mathbf{nn} : \mathcal{Q}_{\mathbf{n}}(Q)) + \psi_2(\mathbf{nn} \cdot \mathcal{Q}_{\mathbf{n}}(Q) + \mathcal{Q}_{\mathbf{n}}(Q) \cdot \mathbf{nn} \\ & \quad - \frac{2}{3}\mathbf{I}(\mathbf{nn} : \mathcal{Q}_{\mathbf{n}}(Q))) + \psi_3 \mathcal{Q}_{\mathbf{n}}(Q) \\ &= \psi_1(\frac{2}{3}\xi_1 + \frac{4}{3}\xi_2 + \xi_3)(\mathbf{nn} - \frac{1}{3}\mathbf{I})(\mathbf{nn} : Q) + \psi_2(\frac{4}{3}\xi_1 + \frac{2}{3}\xi_2)(\mathbf{nn} - \frac{1}{3}\mathbf{I})(\mathbf{nn} : Q) \\ & \quad + \psi_2(\xi_2 + \xi_3)(\mathbf{nn} \cdot Q + Q \cdot \mathbf{nn} - \frac{2}{3}\mathbf{I}(\mathbf{nn} : Q)) + \psi_3\xi_1(\mathbf{nn} - \frac{1}{3}\mathbf{I})(\mathbf{nn} : Q) \\ & \quad + \psi_3\xi_2(\mathbf{nn} \cdot Q + Q \cdot \mathbf{nn} - \frac{2}{3}\mathbf{I}(\mathbf{nn} : Q)) + \psi_3\xi_3 Q \\ &= \left(\psi_1(\frac{2}{3}\xi_1 + \frac{4}{3}\xi_2 + \xi_3) + \psi_2(\frac{4}{3}\xi_1 + \frac{2}{3}\xi_2) + \psi_3\xi_1\right)(\mathbf{nn} - \frac{1}{3}\mathbf{I})(\mathbf{nn} : Q) \\ & \quad + \left(\psi_2(\xi_2 + \xi_3) + \psi_3\xi_2\right)(\mathbf{nn} \cdot Q + Q \cdot \mathbf{nn} - \frac{2}{3}\mathbf{I}(\mathbf{nn} : Q)) + \psi_3\xi_3 Q. \end{aligned}$$

Therefore, the coefficients $\psi_i (1 \leq i \leq 3)$ satisfy

$$\begin{aligned} & \psi_1(\frac{2}{3}\xi_1 + \frac{4}{3}\xi_2 + \xi_3) + \psi_2(\frac{4}{3}\xi_1 + \frac{2}{3}\xi_2) + \psi_3\xi_1 = 0, \\ & \psi_2(\xi_2 + \xi_3) + \psi_3\xi_2 = 0, \quad \psi_3\xi_3 = 1. \end{aligned} \quad (4.6)$$

By (4.4) and the definitions of S_2 and S_4 (see (2.19)), we get that

$$\begin{aligned} \xi_2 &= \frac{2(S_2 - S_4)}{7} = \frac{6A_2 - 5A_4 - A_0}{4A_0} > 0, \\ \xi_3 &= 2\left(\frac{S_4}{35} - \frac{2S_2}{21} + \frac{1}{15}\right) = \frac{A_4 - 2A_2 + A_0}{4A_0} > 0, \\ \frac{2}{3}\xi_1 + \frac{4}{3}\xi_2 + \xi_3 &= \frac{3(A_0A_4 - A_2^2)}{2A_0^2} > 0. \end{aligned}$$

Thus, the coefficients ψ_1, ψ_2, ψ_3 can be uniquely determined.

Another important linear operator is the linearized operator $\mathcal{H}_{\mathbf{n}}(Q)$ of $B(Q) - \alpha Q$ around Q_0 , which is given by

$$\mathcal{H}_{\mathbf{n}}(Q) = \mathcal{Q}_{\mathbf{n}}^{-1}(Q) - \alpha Q, \quad (4.7)$$

plays an important role in next sections.

First, we introduce a two-dimensional subspace of \mathbb{Q} as

$$\mathbb{Q}_{\mathbf{n}}^{in} = \{\mathbf{nn}^\perp + \mathbf{n}^\perp \mathbf{n} : \mathbf{n}^\perp \in \mathbb{V}_{\mathbf{n}}\},$$

where $\mathbb{V}_{\mathbf{n}} := \{\mathbf{n} \in \mathbb{R}^3 | \mathbf{n}^\perp \cdot \mathbf{n} = 0\}$, and let $\mathbb{Q}_{\mathbf{n}}^{out}$ be the orthogonal complement of $\mathbb{Q}_{\mathbf{n}}^{in}$ in \mathbb{Q} . The following proposition gives a characterization on the kernel space and non-negativity of $\mathcal{H}_{\mathbf{n}}$:

Proposition 6. (i) $\mathcal{H}_{\mathbf{n}}\mathbb{Q}_{\mathbf{n}}^{in} = 0$, i.e. $\mathcal{H}_{\mathbf{n}}(Q) \in \mathbb{Q}_{\mathbf{n}}^{out}$.
(ii) There exists a positive constant c_0 such that for any $Q \in \mathbb{Q}_{\mathbf{n}}^{out}$,

$$\langle \mathcal{H}_{\mathbf{n}}(Q), Q \rangle \geq c_0 |Q|^2.$$

Proof. (i) From (4.5) and (4.7), the linearized operator $\mathcal{H}_{\mathbf{n}}$ can be written as

$$\mathcal{H}_{\mathbf{n}}(Q) = \psi_1(\mathbf{nn} - \frac{1}{3}\mathbf{I})(\mathbf{nn} : Q) + \psi_2(\mathbf{nn} \cdot Q + Q \cdot \mathbf{nn} - \frac{2}{3}\mathbf{Inn} : Q) + (\psi_3 - \alpha)Q.$$

where ψ_1, ψ_2, ψ_3 are given by (4.6). By (4.4) and definitions of S_2 and S_4 , we have

$$\xi_2 + \xi_3 = \frac{A_2 - A_4}{A_0} = \frac{1}{\alpha}. \quad (4.8)$$

Together with (4.6), we know $\psi_2 + \psi_3 = (\xi_2 + \xi_3)^{-1} = \alpha$. Thus, we get

$$\mathcal{H}_{\mathbf{n}}(Q) = \psi_1(\mathbf{nn} - \frac{1}{3}\mathbf{I})(\mathbf{nn} : Q) + \psi_2(-Q + \mathbf{nn} \cdot Q + Q \cdot \mathbf{nn} - \frac{2}{3}\mathbf{I}(\mathbf{nn} : Q)). \quad (4.9)$$

This yields the assertions in (i) by observing

$$(\mathbf{nn} - \frac{1}{3}\mathbf{I})(\mathbf{nn} : Q) \in \mathbb{Q}_{\mathbf{n}}^{out}, \quad -Q + \mathbf{nn} \cdot Q + Q \cdot \mathbf{nn} - \frac{2}{3}\mathbf{I}(\mathbf{nn} : Q) \in \mathbb{Q}_{\mathbf{n}}^{out}. \quad (4.10)$$

(ii) From the assertion in (i) and (4.10), we have

$$\mathcal{Q}_{\mathbf{n}}^{-1}(Q) = \alpha Q + \mathcal{H}_{\mathbf{n}}(Q).$$

Thus $\mathcal{Q}_{\mathbf{n}}^{-1}(Q) \in \mathbb{Q}_{\mathbf{n}}^{out}$ if and only if $Q \in \mathbb{Q}_{\mathbf{n}}^{out}$. Together with the fact that $\mathcal{Q}_{\mathbf{n}}$ is a bounded operator, we only need to prove that

$$\mathcal{H}_{\mathbf{n}}(\mathcal{Q}_{\mathbf{n}}(B)) : \mathcal{Q}_{\mathbf{n}}(B) \geq c_0 |B|^2,$$

for some positive constant c_0 and any $B \in \mathbb{Q}_{\mathbf{n}}^{out}$. From (4.3), we have

$$\langle B, \mathcal{Q}_{\mathbf{n}}(B) \rangle = \xi_1 |\mathbf{nn} : B|^2 + 2\xi_2 |\mathbf{n} \cdot B|^2 + \xi_3 |B|^2,$$

and

$$\begin{aligned} \langle \mathcal{Q}_{\mathbf{n}}(B), \mathcal{Q}_{\mathbf{n}}(B) \rangle &= \left(\frac{2}{3}(\xi_1 + 2\xi_2)^2 - 2\xi_2^2 + 2\xi_2\xi_3 \right) |\mathbf{nn} : B|^2 \\ &\quad + (2\xi_2^2 + 4\xi_2\xi_3) |\mathbf{n} \cdot B|^2 + \xi_3^2 |B|^2. \end{aligned}$$

Therefore we get

$$\mathcal{H}_{\mathbf{n}}(\mathcal{Q}_{\mathbf{n}}(B)) : \mathcal{Q}_{\mathbf{n}}(B) = (B - \alpha \mathcal{Q}_{\mathbf{n}}(B)) : \mathcal{Q}_{\mathbf{n}}(B) = \beta_1 |\mathbf{nn} : B|^2 + \beta_2 |\mathbf{n} \cdot B|^2 + \beta_3 |B|^2,$$

where the coefficients are given by

$$\begin{aligned} \beta_1 &= \xi_1 - \alpha \left(\frac{2}{3}(\xi_1 + 2\xi_2)^2 - 2\xi_2^2 + 2\xi_2\xi_3 \right), \\ \beta_2 &= 2\xi_2 - \alpha(2\xi_2^2 + 4\xi_2\xi_3), \quad \beta_3 = \xi_3 - \alpha\xi_3^2. \end{aligned}$$

From (4.8), we have

$$\beta_2 + 2\beta_3 = 2(\xi_2 + \xi_3)(1 - \alpha(\xi_2 + \xi_3)) = 0,$$

which implies

$$\begin{aligned}\mathcal{H}_{\mathbf{n}}(\mathcal{Q}_{\mathbf{n}}(B)) : \mathcal{Q}_{\mathbf{n}}(B) &= \beta_1 |\mathbf{nn} : B|^2 + \beta_3 (|B|^2 - 2|B \cdot \mathbf{n}|^2) \\ &= (\beta_1 - 2\beta_3) |\mathbf{nn} : B|^2 + \beta_3 |B|^2,\end{aligned}$$

for $B \in \mathbb{Q}_{\mathbf{n}}^{\text{out}}$. If $\beta_1 > 2\beta_3$, the assertion is apparently true. If $\beta_1 < 2\beta_3$, it is direct to check that for traceless matrix B

$$|B|^2 \geq \frac{3}{2} |B : \mathbf{nn}|^2.$$

Therefore

$$\begin{aligned}\mathcal{H}_{\mathbf{n}}(\mathcal{Q}_{\mathbf{n}}(B)) : \mathcal{Q}_{\mathbf{n}}(B) &= (\beta_1 - 2\beta_3) |\mathbf{nn} : B|^2 + \beta_3 |B|^2 \\ &\geq \frac{2}{3} (\beta_1 - 2\beta_3) |B|^2 + \beta_3 |B|^2 = \frac{1}{3} (2\beta_1 - \beta_3) |B|^2.\end{aligned}$$

Some further tedious calculations give that

$$\begin{aligned}\beta_1 - \frac{\beta_3}{2} &= \xi_1 - \frac{1}{2} \xi_3 - \alpha \left(\frac{2}{3} (\xi_1 + 2\xi_2)^2 - 2\xi_2^2 + 2\xi_2 \xi_3 - \frac{1}{2} \xi_3^2 \right) \\ &= - \frac{8A_0 A_2 + 9A_2^2 - 17A_0 A_4}{4A_0^2} \\ &\quad - \frac{-27A_2^4 + 54A_0 A_2^2 A_4 + A_0^2 (16A_2^2 - 32A_2 A_4 - 11A_4^2)}{8A_0^3 (A_2 - A_4)} \\ &= \frac{9(-3A_2^4 - 2A_0 A_2^2 (A_2 - 4A_4) + A_0^2 (2A_2 - 5A_4) A_4)}{8A_0^3 (A_2 - A_4)} \\ &= \frac{9(A_0 A_4 - A_2^2) (3A_2^2 + 2A_0 A_2 - 5A_0 A_4)}{8A_0^3 (A_2 - A_4)} > 0.\end{aligned}$$

This concludes the proof. \square

We denote by \mathcal{P}^{in} the projection operator from \mathbb{Q} to $\mathbb{Q}_{\mathbf{n}}^{in}$ and by \mathcal{P}^{out} the projection operator from \mathbb{Q} to $\mathbb{Q}_{\mathbf{n}}^{out}$. By direct computation we have

$$|Q - (\mathbf{nn}^\perp + \mathbf{n}^\perp \mathbf{n})|^2 = |Q|^2 - 2|Q \cdot \mathbf{n}|^2 + 2|Q : \mathbf{nn}|^2 + |\mathbf{n}^\perp - (\mathbf{I} - \mathbf{nn}) \cdot Q \cdot \mathbf{n}|^2.$$

Therefore, there holds

$$\begin{aligned}\mathcal{P}^{in}(Q) &= \mathbf{n}[(\mathbf{I} - \mathbf{nn}) \cdot Q \cdot \mathbf{n}] + [(\mathbf{I} - \mathbf{nn}) \cdot Q \cdot \mathbf{n}] \mathbf{n} \\ &= (\mathbf{nn} \cdot Q + Q \cdot \mathbf{nn}) - 2(Q : \mathbf{nn}) \mathbf{nn},\end{aligned}\tag{4.11}$$

$$|\mathcal{P}^{in}(Q)|^2 = 2|Q \cdot \mathbf{n}|^2 - 2|Q : \mathbf{nn}|^2,\tag{4.12}$$

and

$$\mathcal{P}^{out}(Q) = Q - \mathcal{P}^{in}(Q) = Q - (\mathbf{nn} \cdot Q + Q \cdot \mathbf{nn}) + 2(Q : \mathbf{nn}) \mathbf{nn}.\tag{4.13}$$

Another two linear operators will be frequently used in the later are $\mathcal{J}_{\bar{Q}} : \mathbb{R}^{3 \times 3} \mapsto \mathbb{Q}$ and $\mathcal{U}_{\bar{Q}} : \mathbb{Q} \mapsto \mathbb{Q}$, which are defined as

$$\begin{aligned}\mathcal{J}_{\bar{Q}}(A) &:= \frac{1}{2} (\mathcal{M}_{\bar{Q}}(A) + \mathcal{M}_{\bar{Q}}^T(A)) = \frac{1}{6} (A + A^T) + \frac{1}{2} (A^T \cdot \bar{Q} + \bar{Q} \cdot A) - A : M_{\bar{Q}}^{(4)}, \\ \mathcal{U}_{\bar{Q}}(B) &:= M_{\bar{Q}}^{(6)} : B - (\bar{Q} : B) M_{\bar{Q}}^{(4)},\end{aligned}$$

for $A \in \mathbb{R}^{3 \times 3}$ and $B \in \mathbb{Q}$. When $\bar{Q} = S_2(\mathbf{nn} - \frac{1}{3}\mathbf{I})$, and $A \in \mathbb{Q}_{\mathbf{n}}^{\text{out}}$, then by (4.13) we infer that

$$\bar{Q} \cdot A = S_2(\mathbf{nn} - \frac{1}{3}\mathbf{I}) \cdot \mathcal{P}^{\text{out}} A = S_2(A : \mathbf{nn})\mathbf{nn} - \frac{1}{3}S_2\mathcal{P}^{\text{out}} A,$$

which is symmetric. Thus, $\bar{Q} \cdot A = A^T \cdot \bar{Q}$, and then we have

$$\mathcal{M}_{\bar{Q}}(A) = \mathcal{M}_{\bar{Q}}^T(A) = \mathcal{J}_{\bar{Q}}(A). \quad (4.14)$$

A direct consequence of (4.14) and Proposition 6 is that, for $\bar{Q} = S_2(\mathbf{nn} - \frac{1}{3}\mathbf{I})$,

$$\mathcal{M}_{\bar{Q}}(\mathcal{H}_{\bar{Q}}(Q)) = \mathcal{J}_{\bar{Q}}(\mathcal{H}_{\bar{Q}}(Q)), \quad \text{for any } Q \in \mathbb{Q}. \quad (4.15)$$

We denote $\mathcal{J}_{\bar{Q}}$ by $\mathcal{J}_{\mathbf{n}}$ for simplicity when $\bar{Q} = S_2(\mathbf{nn} - \frac{1}{3}\mathbf{I})$. It should be noticed that $\mathcal{J}_{\mathbf{n}}$ is not self-adjoint operator on $\mathbb{R}^{3 \times 3}$ but is self-adjoint on the space \mathbb{Q} .

Direct computation gives that

$$\begin{aligned} \mathcal{Q}_{\mathbf{n}}(\mathbf{nn}^{\perp} + \mathbf{n}^{\perp}\mathbf{n}) &= \left(\frac{2(S_2 - S_4)}{7} + 2\left(\frac{S_4}{35} - \frac{2S_2}{21} + \frac{1}{15}\right) \right) (\mathbf{nn}^{\perp} + \mathbf{n}^{\perp}\mathbf{n}) \in \mathbb{Q}_{\mathbf{n}}^{\text{in}}, \\ \mathcal{J}_{\mathbf{n}}(\mathbf{nn}^{\perp} + \mathbf{n}^{\perp}\mathbf{n}) &= \left(\frac{1}{3} + \frac{1}{6}S_2 - \frac{2(S_2 - S_4)}{7} - 2\left(\frac{S_4}{35} - \frac{2S_2}{21} + \frac{1}{15}\right) \right) (\mathbf{nn}^{\perp} + \mathbf{n}^{\perp}\mathbf{n}) \in \mathbb{Q}_{\mathbf{n}}^{\text{in}}, \end{aligned}$$

which imply

$$\mathcal{Q}_{\mathbf{n}}\mathbb{Q}_{\mathbf{n}}^{\text{in}} \subseteq \mathbb{Q}_{\mathbf{n}}^{\text{in}}, \quad \mathcal{J}_{\mathbf{n}}\mathbb{Q}_{\mathbf{n}}^{\text{in}} \subseteq \mathbb{Q}_{\mathbf{n}}^{\text{in}}.$$

As $\mathcal{Q}_{\mathbf{n}}$ and $\mathcal{J}_{\mathbf{n}}$ are self-adjoint on \mathbb{Q} , we also have

$$\mathcal{Q}_{\mathbf{n}}\mathbb{Q}_{\mathbf{n}}^{\text{out}} \subseteq \mathbb{Q}_{\mathbf{n}}^{\text{out}}, \quad \mathcal{J}_{\mathbf{n}}\mathbb{Q}_{\mathbf{n}}^{\text{out}} \subseteq \mathbb{Q}_{\mathbf{n}}^{\text{out}}.$$

Consequently, we get

$$[\mathcal{Q}_{\mathbf{n}}, \mathcal{P}^{\text{in}}] = [\mathcal{Q}_{\mathbf{n}}, \mathcal{P}^{\text{out}}] = 0, \quad [\mathcal{J}_{\mathbf{n}}, \mathcal{P}^{\text{in}}] = [\mathcal{J}_{\mathbf{n}}, \mathcal{P}^{\text{out}}] = 0. \quad (4.16)$$

5. Rigorous derivation from the Q -tensor theory to the Ericksen-Leslie theory. In this section, by making the so-called Hilbert expansion for the solution of the molecule-based Q -tensor systems (1.27)-(1.29), we present a rigorous derivation from the molecule-based Q -tensor theory to the Ericksen-Leslie theory.

5.1. The Hilbert expansion. Let $(Q^{\varepsilon}, \mathbf{v}^{\varepsilon})$ be a solution of the system (1.34)-(1.36). We perform the following so-called Hilbert expansion:

$$Q^{\varepsilon} = \sum_{k=0}^3 \varepsilon^k Q_k + \varepsilon^3 Q_R \stackrel{\text{def}}{=} \tilde{Q} + \varepsilon^3 Q_R, \quad (5.1)$$

$$\mathbf{v}^{\varepsilon} = \sum_{k=0}^2 \varepsilon^k \mathbf{v}_k + \varepsilon^3 \mathbf{v}_R \stackrel{\text{def}}{=} \tilde{\mathbf{v}} + \varepsilon^3 \mathbf{v}_R, \quad (5.2)$$

where $Q_i (0 \leq i \leq 3)$, $\mathbf{v}_j (0 \leq j \leq 2)$ are independent of ε and will be determined in what follows. (Q_R, \mathbf{v}_R) are called the remainder term which depend upon ε .

Substituting the above expansion to (1.34)-(1.36), and expanding all the terms with respect to ε , we can get several systems of equations to solve $(Q_i, \mathbf{v}_i) (0 \leq i \leq 2)$ and Q_3 by collecting all the terms of the same order with respect to ε . In [27] and [29], the expanding can be performed directly as it involves only polynomials of variables. In contrast, the dependence of B and \mathcal{M}_B on Q is much more complicated here.

First, we make the following formal expansion for Z_{Q^ε} and B_{Q^ε} :

$$B_{Q^\varepsilon} = B_0 + \varepsilon B_1 + \varepsilon^2 B_2 + \varepsilon^3 B_3 + \varepsilon^3 B_R + \varepsilon^4 \mathfrak{R}_B, \quad (5.3)$$

$$Z_{Q^\varepsilon} = Z_0 + \varepsilon Z_1 + \varepsilon^2 Z_2 + \varepsilon^3 Z_3 + \varepsilon^3 Z_R + \varepsilon^4 \mathfrak{R}_Z. \quad (5.4)$$

Here $B_i, Z_i (0 \leq i \leq 3)$ depend on $Q_i (0 \leq i \leq 3)$ only, and B_R, Z_R depends on Q_R and $Q_i (0 \leq i \leq 3)$. Moreover, $B_i, Z_i (0 \leq i \leq 3)$ are independent of ε and B_R, Z_R are the linear functions of Q_R . All the terms with higher order of ε are put in $\varepsilon^4 \mathfrak{R}_B$ and $\varepsilon^4 \mathfrak{R}_Z$. To perform the Hilbert expansion, we have to write B_i, Z_i and B_R, Z_R in terms of Q_i, Q_R explicitly.

By viewing Z_{Q^ε} as a function of B^ε , we have:

$$\begin{aligned} Z_{Q^\varepsilon} &= \int_{\mathbb{S}^2} \exp \left(\mathbf{mm} : \left(\sum_{k=0}^3 \varepsilon^k B_k + \varepsilon^3 B_R + \varepsilon^4 \mathfrak{R}_B \right) \right) d\mathbf{m} \\ &= Z_{Q_0} \left(1 + \varepsilon Q_0 : B_1 + \varepsilon^2 (Q_0 : B_2 + \widehat{Z}_1) + \varepsilon^3 (Q_0 : B_3 + 2\widehat{Z}_2 \right. \\ &\quad \left. + \widehat{Z}_3) + \varepsilon^3 Q_0 : B_R \right) + \varepsilon^4 \mathfrak{R}_Z, \end{aligned} \quad (5.5)$$

where

$$\begin{aligned} \widehat{Z}_1 &= \frac{1}{2Z_{Q_0}} \int_{\mathbb{S}^2} (\mathbf{mm} : B_1)^2 \exp(\mathbf{mm} : B_0) d\mathbf{m}, \\ \widehat{Z}_2 &= \frac{1}{2Z_{Q_0}} \int_{\mathbb{S}^2} (\mathbf{mm} : B_1)(\mathbf{mm} : B_2) \exp(\mathbf{mm} : B_0) d\mathbf{m}, \\ \widehat{Z}_3 &= \frac{1}{6Z_{Q_0}} \int_{\mathbb{S}^2} (\mathbf{mm} : B_1)^3 \exp(\mathbf{mm} : B_0) d\mathbf{m}. \end{aligned}$$

By the expression of Q^ε , we have

$$\begin{aligned} Q^\varepsilon &= \frac{1}{Z_{Q^\varepsilon}} \int_{\mathbb{S}^2} (\mathbf{mm} - \frac{1}{3} \mathbf{I}) \exp \left(\mathbf{mm} : \left(\sum_{k=0}^3 \varepsilon^k B_k + \varepsilon^3 B_R + \varepsilon^4 \mathfrak{R}_B \right) \right) d\mathbf{m} \\ &= \left(Q_0 + \varepsilon (M_{Q_0}^{(4)} - \frac{1}{3} \mathbf{I} Q_0) : B_1 + \varepsilon^2 \left[(M_{Q_0}^{(4)} - \frac{1}{3} \mathbf{I} Q_0) : B_2 + \widehat{Q}_1 \right] \right. \\ &\quad \left. + \varepsilon^3 \left[(M_{Q_0}^{(4)} - \frac{1}{3} \mathbf{I} Q_0) : B_3 + 2\widehat{Q}_2 + \widehat{Q}_3 \right] + \varepsilon^3 \left[(M_{Q_0}^{(4)} - \frac{1}{3} \mathbf{I} Q_0) : B_R \right] + O(\varepsilon^4) \right) \\ &\quad \cdot \left(1 + \varepsilon Q_0 : B_1 + \varepsilon^2 (Q_0 : B_2 + \widehat{Z}_1) \right. \\ &\quad \left. + \varepsilon^3 (Q_0 : B_3 + 2\widehat{Z}_2 + \widehat{Z}_3) + \varepsilon^3 Q_0 : B_R + O(\varepsilon^4) \right)^{-1} \\ &= Q_0 + \varepsilon \left((M_{Q_0}^{(4)} - \frac{1}{3} \mathbf{I} Q_0) : B_1 - (Q_0 : B_1) Q_0 \right) + \varepsilon^2 \left((M_{Q_0}^{(4)} - \frac{1}{3} \mathbf{I} Q_0) : B_2 + \widehat{Q}_1 \right. \\ &\quad \left. - Q_0 (Q_0 : B_2 + \widehat{Z}_1) - \left\{ (M_{Q_0}^{(4)} - \frac{1}{3} \mathbf{I} Q_0) : B_1 \right\} (Q_0 : B_1) + Q_0 (Q_0 : B_1)^2 \right) \\ &\quad + \varepsilon^3 \left((M_{Q_0}^{(4)} - \frac{1}{3} \mathbf{I} Q_0) : B_3 + 2\widehat{Q}_2 + \widehat{Q}_3 - Q_0 (Q_0 : B_3 + 2\widehat{Z}_2 + \widehat{Z}_3) \right. \\ &\quad \left. - (Q_0 : B_2 + \widehat{Z}_1) (M_{Q_0}^{(4)} - \frac{1}{3} \mathbf{I} Q_0) : B_1 - \left[(M_{Q_0}^{(4)} - \frac{1}{3} \mathbf{I} Q_0) : B_2 + \widehat{Q}_1 \right] (Q_0 : B_1) \right. \\ &\quad \left. + 2Q_0 (Q_0 : B_1) (Q_0 : B_2 + \widehat{Z}_1) + (Q_0 : B_1)^2 (M_{Q_0}^{(4)} - \frac{1}{3} \mathbf{I} Q_0) : B_1 - Q_0 (Q_0 : B_1)^3 \right) \\ &\quad \left. + \varepsilon^3 \left((M_{Q_0}^{(4)} - \frac{1}{3} \mathbf{I} Q_0) : B_R - Q_0 (Q_0 : B_R) \right) + O(\varepsilon^4), \end{aligned} \quad (5.6)$$

where

$$\begin{aligned}\widehat{Q}_1 &= \frac{1}{2Z_{Q_0}} \int_{\mathbb{S}^2} (\mathbf{mm} - \frac{1}{3}\mathbf{I})(\mathbf{mm} : B_1)^2 \exp(\mathbf{mm} : B_0) d\mathbf{m}, \\ \widehat{Q}_2 &= \frac{1}{2Z_{Q_0}} \int_{\mathbb{S}^2} (\mathbf{mm} - \frac{1}{3}\mathbf{I})(\mathbf{mm} : B_1)(\mathbf{mm} : B_2) \exp(\mathbf{mm} : B_0) d\mathbf{m}, \\ \widehat{Q}_3 &= \frac{1}{6Z_{Q_0}} \int_{\mathbb{S}^2} (\mathbf{mm} - \frac{1}{3}\mathbf{I})(\mathbf{mm} : B_1)^3 \exp(\mathbf{mm} : B_0) d\mathbf{m}.\end{aligned}$$

Noting the definition of linear operator \mathcal{Q}_{Q_0} and from (5.6) we can deduce that

$$Q_1 = \mathcal{Q}_{Q_0}(B_1), \quad (5.7)$$

$$Q_2 = \mathcal{Q}_{Q_0}(B_2) - (Q_0 : B_1)\mathcal{Q}_{Q_0}(B_1) + \widehat{Q}_1 - \widehat{Z}_1 Q_0, \quad (5.8)$$

$$\begin{aligned}Q_3 &= \mathcal{Q}_{Q_0}(B_3) - (Q_0 : B_1)\mathcal{Q}_{Q_0}(B_2) + ((Q_0 : B_1)^2 - Q_0 : B_2 - \widehat{Z}_1)\mathcal{Q}_{Q_0}(B_1) \\ &\quad + (Q_0 : B_1)(\widehat{Z}_1 Q_0 - \widehat{Q}_1) - (2\widehat{Z}_2 + \widehat{Z}_3)Q_0 + 2\widehat{Q}_2 + \widehat{Q}_3,\end{aligned} \quad (5.9)$$

$$Q_R = \mathcal{Q}_{Q_0}(B_R). \quad (5.10)$$

Thanks to the invertibility of \mathcal{Q}_{Q_0} , we know that B_i can be explicitly given by Q_j ($0 \leq j \leq i$), and B_R is linearly depend on Q_R .

Similarly, we next make the expansion for $M_{Q^\varepsilon}^{(4)}$:

$$M_{Q^\varepsilon}^{(4)} = M_0^{(4)} + \varepsilon M_1^{(4)} + \varepsilon^2 M_2^{(4)} + \varepsilon^3 M_3^{(4)} + \varepsilon^3 M_R^{(4)} + \varepsilon^4 \mathfrak{R}_{M^{(4)}}. \quad (5.11)$$

Then we have

$$\begin{aligned}M_{Q^\varepsilon}^{(4)} &= \frac{1}{Z_{Q^\varepsilon}} \int_{\mathbb{S}^2} \mathbf{mmmm} \exp\left(\mathbf{mm} : \left(\sum_{k=0}^3 \varepsilon^k B_k + \varepsilon^3 B_R + \varepsilon^4 \mathfrak{R}_B\right)\right) d\mathbf{m} \\ &= \left(M_{Q_0}^{(4)} + \varepsilon M_{Q_0}^{(6)} : B_1 + \varepsilon^2 (M_{Q_0}^{(6)} : B_2 + \widehat{M}_1^{(4)}) + \varepsilon^3 (M_{Q_0}^{(6)} : B_3 + 2\widehat{M}_2^{(4)} + \widehat{M}_3^{(4)}) \right. \\ &\quad \left. + \varepsilon^3 (M_{Q_0}^{(6)} : B_R) + O(\varepsilon^4)\right) \cdot \left(1 + \varepsilon Q_0 : B_1 + \varepsilon^2 (Q_0 : B_2 + \widehat{Z}_1) \right. \\ &\quad \left. + \varepsilon^3 (Q_0 : B_3 + 2\widehat{Z}_2 + \widehat{Z}_3) + \varepsilon^3 Q_0 : B_R + O(\varepsilon^4)\right)^{-1} \\ &= M_{Q_0}^{(4)} + \varepsilon \left(M_{Q_0}^{(6)} : B_1 - (Q_0 : B_1)M_{Q_0}^{(4)}\right) \\ &\quad + \varepsilon^2 \left(M_{Q_0}^{(6)} : (B_2 - B_1(Q_0 : B_1)) - M_{Q_0}^{(4)}(Q_0 : B_2 - (Q_0 : B_1)^2 + \widehat{Z}_1) + \widehat{M}_1^{(4)}\right) \\ &\quad + \varepsilon^3 \left(M_{Q_0}^{(6)} : [B_3 + B_2(Q_0 : B_1) + B_1((Q_0 : B_1)^2 - (Q_0 : B_2 + \widehat{Z}_1))\right. \\ &\quad \left. - M_{Q_0}^{(4)}[Q_0 : B_3 + 2\widehat{Z}_2 + \widehat{Z}_3 - 2(Q_0 : B_1)(Q_0 : B_2 + \widehat{Z}_1) + (Q_0 : B_1)^3]\right. \\ &\quad \left. + 2\widehat{M}_2^{(4)} + \widehat{M}_3^{(4)} - \widehat{M}_1(Q_0 : B_1)\right) + \varepsilon^3 \left(M_{Q_0}^{(6)} : B_R - M_{Q_0}^{(4)}(Q_0 : B_R)\right) \\ &\quad + O(\varepsilon^4).\end{aligned} \quad (5.12)$$

Here \widehat{Z}_i are defined after the expansion of Z_{Q^ε} , and $\widehat{M}_i^{(4)}$ are defined as

$$\widehat{M}_1^{(4)} = \frac{1}{2Z_{Q_0}} \int_{\mathbb{S}^2} \mathbf{mmmm}(\mathbf{mm} : B_1)^2 \exp(\mathbf{mm} : B_0) d\mathbf{m},$$

$$\begin{aligned}\widehat{M}_2^{(4)} &= \frac{1}{2Z_{Q_0}} \int_{\mathbb{S}^2} \mathbf{m} \mathbf{m} \mathbf{m} \mathbf{m} (\mathbf{m} \mathbf{m} : B_1) (\mathbf{m} \mathbf{m} : B_2) \exp(\mathbf{m} \mathbf{m} : B_0) d\mathbf{m}, \\ \widehat{M}_3^{(4)} &= \frac{1}{6Z_{Q_0}} \int_{\mathbb{S}^2} \mathbf{m} \mathbf{m} \mathbf{m} \mathbf{m} (\mathbf{m} \mathbf{m} : B_1)^3 \exp(\mathbf{m} \mathbf{m} : B_0) d\mathbf{m}.\end{aligned}$$

Noting the definition of linear operator \mathcal{U}_{Q_0} and from (5.12) we get

$$\begin{aligned}M_1^{(4)} &= \mathcal{U}_{Q_0}(B_1), \\ M_2^{(4)} &= \mathcal{U}_{Q_0}(B_2) - (Q_0 : B_1) \mathcal{U}_{Q_0}(B_1) + \widehat{M}_1^{(4)} - \widehat{Z}_1 M_{Q_0}^{(4)}, \\ M_3^{(4)} &= \mathcal{U}_{Q_0}(B_3) - (Q_0 : B_1) \mathcal{U}_{Q_0}(B_2) + ((Q_0 : B_1)^2 - Q_0 : B_2 - \widehat{Z}_1) \mathcal{U}_{Q_0}(B_1) \\ &\quad + (Q_0 : B_1) (\widehat{Z}_1 M_{Q_0}^{(4)} - \widehat{M}_1^{(4)}) - (2\widehat{Z}_2 + \widehat{Z}_3) M_{Q_0}^{(4)} + 2\widehat{M}_2^{(4)} + \widehat{M}_3^{(4)}, \\ M_R^{(4)} &= \mathcal{U}_{Q_0}(B_R).\end{aligned}$$

Now, we can write down the expansion of the original system (1.34)-(1.36) and collect the terms with same order of ε . Specifically, we have

• **The $O(\varepsilon^{-1})$ system**

$$\mathcal{M}_{Q_0}(B_0 - \alpha Q_0) = 0, \quad (5.13)$$

• **The zero-order term in ε**

$$\frac{\partial Q_0}{\partial t} + \mathbf{v}_0 \cdot \nabla Q_0 = -4\mathcal{J}_{Q_0}(\mathcal{H}_{Q_0}(Q_1) + \mathcal{L}(Q_0)) + 2\mathcal{J}_{Q_0}(\kappa_0^T), \quad (5.14)$$

$$\begin{aligned}\frac{\partial \mathbf{v}_0}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_0 &= -\nabla p_0 + \frac{\gamma}{Re} \Delta \mathbf{v}_0 + \frac{1-\gamma}{2Re} \nabla \cdot (\mathbf{D}_0 : M_{Q_0}^{(4)}) \\ &\quad + \frac{1-\gamma}{Re} \nabla \cdot (2\mathcal{J}_{Q_0}(\mathcal{H}_{Q_0}(Q_1)) + 2\mathcal{M}_{Q_0}(\mathcal{L}(Q_0)) + \sigma^d(Q_0, Q_0)),\end{aligned} \quad (5.15)$$

$$\nabla \cdot \mathbf{v}_0 = 0. \quad (5.16)$$

• **The first-order term in ε**

$$\frac{\partial Q_1}{\partial t} + \mathbf{v}_0 \cdot \nabla Q_1 = -\mathbf{v}_1 \cdot \nabla Q_0 - 4\mathcal{J}_{Q_0}(\mathcal{H}_{Q_0}(Q_2) + \mathcal{L}(Q_1)) + 2\mathcal{J}_{Q_0}(\kappa_1^T) + \mathbf{F}_1, \quad (5.17)$$

$$\begin{aligned}\frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_1 &= -\mathbf{v}_1 \cdot \nabla \mathbf{v}_0 - \nabla p_1 + \frac{\gamma}{Re} \Delta \mathbf{v}_1 \\ &\quad + \frac{1-\gamma}{2Re} \nabla \cdot (\mathbf{D}_0 : M_1^{(4)} + \mathbf{D}_1 : M_{Q_0}^{(4)}) \\ &\quad + \frac{1-\gamma}{Re} \nabla \cdot (2\mathcal{M}_{Q_0}(\mathcal{H}_{Q_0}(Q_2)) + 2\mathcal{M}_{Q_0}(\mathcal{L}(Q_1)) - \mathbf{G}_1 \\ &\quad + \sigma^d(Q_0, Q_1) + \sigma^d(Q_1, Q_0)),\end{aligned} \quad (5.18)$$

$$\nabla \cdot \mathbf{v}_1 = 0. \quad (5.19)$$

• **The second-order term in ε**

$$\begin{aligned}\frac{\partial Q_2}{\partial t} + \mathbf{v}_0 \cdot \nabla Q_2 &= -\mathbf{v}_1 \cdot \nabla Q_1 - \mathbf{v}_2 \cdot \nabla Q_0 - 4\mathcal{J}_{Q_0}(\mathcal{H}_{Q_0}(Q_3) + \mathcal{L}(Q_2)) \\ &\quad + \mathcal{J}_{Q_0}(\kappa_2^T) + \mathbf{F}_2,\end{aligned} \quad (5.20)$$

$$\begin{aligned}
\frac{\partial \mathbf{v}_2}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_2 = & -\mathbf{v}_1 \cdot \nabla \mathbf{v}_1 - \mathbf{v}_2 \cdot \nabla \mathbf{v}_0 - \nabla p_2 + \frac{\gamma}{Re} \Delta \mathbf{v}_2 \\
& + \frac{1-\gamma}{2Re} \nabla \cdot \left(\mathbf{D}_0 : M_2^{(4)} + \mathbf{D}_1 : M_1^{(4)} + \mathbf{D}_2 : M_{Q_0}^{(4)} \right) \\
& + \frac{1-\gamma}{Re} \nabla \cdot \left(2\mathcal{M}_{Q_0}(\mathcal{H}_{Q_0}(Q_3)) + 2\mathcal{M}_{Q_0}(\mathcal{L}(Q_2)) - \mathbf{G}_2 \right. \\
& \left. + \sigma^d(Q_0, Q_2) + \sigma^d(Q_1, Q_1) + \sigma^d(Q_2, Q_0) \right), \tag{5.21}
\end{aligned}$$

$$\nabla \cdot \mathbf{v}_2 = 0. \tag{5.22}$$

Here, $\mathbf{F}_1, \mathbf{F}_2, \mathbf{G}_1, \mathbf{G}_2$ are defined as following:

$$\begin{aligned}
\mathbf{F}_1 = & \bar{\mathbf{F}}_1 + \tilde{\mathbf{F}}_1, \\
\bar{\mathbf{F}}_1 = & 4\alpha \left(Q_1 \cdot Q_1 - (Q_1 + (Q_0 : B_1)Q_0) : M_1^{(4)} - Q_0 : (\widehat{M}_1^{(4)} - \widehat{Z}_1 M_{Q_0}^{(4)}) \right), \\
\tilde{\mathbf{F}}_1 = & Q_1 \cdot (-2\mathcal{L}(Q_0) + \kappa_0^T) + (-2\mathcal{L}(Q_0) + \kappa_0) \cdot Q_1 - 2(-2\mathcal{L}(Q_0) + \mathbf{D}_0) : M_1^{(4)}, \\
\mathbf{G}_1 = & \frac{1}{2} \bar{\mathbf{F}}_1 - 2(Q_1 \cdot \mathcal{L}(Q_0) - \mathcal{L}(Q_0) : M_1^{(4)}),
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{F}_2 = & Q_2 \cdot (2\alpha Q_1 - 2\mathcal{L}(Q_0) + \kappa_0^T) + (2\alpha Q_1 - 2\mathcal{L}(Q_0) + \kappa_0) \cdot Q_2 \\
& + Q_1 \cdot (2\alpha Q_2 - 2\mathcal{L}(Q_1) + \kappa_1^T) + (2\alpha Q_2 - 2\mathcal{L}(Q_1) + \kappa_1) \cdot Q_1 \\
& - 2(2\alpha Q_1 - 2\mathcal{L}(Q_0) + \mathbf{D}_0) : M_2^{(4)} - 2(2\alpha Q_2 - 2\mathcal{L}(Q_1) + \mathbf{D}_1) : M_1^{(4)} \\
& - 4\alpha Q_0 : \left(-(Q_0 : B_1)M_2^{(4)} + ((Q_0 : B_1)^2 - Q_0 : B_2 - \widehat{Z}_1)M_1^{(4)} \right. \\
& \left. + (Q_0 : B_1)(\widehat{Z}_1 M_{Q_0}^{(4)} - \widehat{M}_1^{(4)}) - (2\widehat{Z}_2 + \widehat{Z}_3)M_{Q_0}^{(4)} + 2\widehat{M}_2^{(4)} + \widehat{M}_3^{(4)} \right), \\
\mathbf{G}_2 = & Q_2 \cdot (2\alpha Q_1 - 2\mathcal{L}(Q_0)) + Q_1 \cdot (2\alpha Q_2 - 2\mathcal{L}(Q_1)) \\
& - (2\alpha Q_1 - 2\mathcal{L}(Q_0)) : M_2^{(4)} - (2\alpha Q_2 - 2\mathcal{L}(Q_1)) : M_1^{(4)} \\
& - 2\alpha Q_0 : \left(-(Q_0 : B_1)M_2^{(4)} + ((Q_0 : B_1)^2 - Q_0 : B_2 - \widehat{Z}_1)M_1^{(4)} \right. \\
& \left. + (Q_0 : B_1)(\widehat{Z}_1 M_{Q_0}^{(4)} - \widehat{M}_1^{(4)}) - (2\widehat{Z}_2 + \widehat{Z}_3)M_{Q_0}^{(4)} + 2\widehat{M}_2^{(4)} + \widehat{M}_3^{(4)} \right).
\end{aligned}$$

The equation of $O(\varepsilon^{-1})$ (5.13) is equivalent to $B_0 - \alpha Q_0 = 0$. Thanks to Proposition 4, Q_0 takes the form

$$Q_0(t, \mathbf{x}) = S_2(\mathbf{n}(t, \mathbf{x})\mathbf{n}(t, \mathbf{x}) - \frac{1}{3}\mathbf{I}), \tag{5.23}$$

for some $\mathbf{n}(t, \mathbf{x}) \in \mathbb{S}^2$.

The evolution of $\mathbf{n}(t, \mathbf{x})$ is determined by the $O(1)$ system (5.14)-(5.16). At first glance, this system is not closed since it involves Q_1 which is unknown. However, if we project (5.14) into the subspace $\mathbb{Q}_{\mathbf{n}}^{\text{in}} = \text{Ker } \mathcal{H}_{\mathbf{n}}$, then Q_1 is vanished in (5.14) by Proposition 6. In addition, if we project (5.14) into the subspace $\mathbb{Q}_{\mathbf{n}}^{\text{out}} = (\text{Ker } \mathcal{H}_{\mathbf{n}})^{\perp}$, then we can solve $\mathcal{H}_{\mathbf{n}}(Q_1)$ in terms of (Q_0, \mathbf{v}_0) . Thus Q_1 can also be eliminated in (5.15). Actually, the following proposition shows that the system (5.14)-(5.16) implies $(\mathbf{n}, \mathbf{v}_0)$ satisfies the Ericksen-Leslie system with coefficients depending on the molecule parameters. One can see the detailed proof in [12].

Proposition 7. *If (\mathbf{v}_0, Q_0) is a strong solution of the system (5.14)–(5.16), then $(\mathbf{n}, \mathbf{v}_0)$ is necessary a solution of the Ericksen-Leslie system (1.3)–(1.5), where the coefficients are given by (1.37)–(1.39).*

In the next subsections, we will show how to solve $Q_i (1 \leq i \leq 3)$ and $\mathbf{v}_j (1 \leq j \leq 2)$ from (5.17–5.22). The whole procedure is very similar to the one used in [27] and [29].

5.2. Existence of the Hilbert expansion. Assume that $(\mathbf{v}_0, \mathbf{n})$ is a solution of the systems (1.3)–(1.5) on $[0, T]$ such that

$$\mathbf{v}_0 \in C([0, T]; H^k), \quad \nabla \mathbf{n} \in C([0, T]; H^k)$$

for $k \geq 20$. Since $Q_0 = S_2(\mathbf{n}(t, \mathbf{x})\mathbf{n}(t, \mathbf{x}) - \frac{1}{3}\mathbf{I})$, we have $Q_0 \in C([0, T], H^{k+1})$.

Let $Q_1 = Q_1^\top + Q_1^\perp$ with $Q_1^\top \in \mathbb{Q}_{\mathbf{n}}^{in}$ and $Q_1^\perp \in \mathbb{Q}_{\mathbf{n}}^{out}$. Notice that we can solve Q_1^\perp by the equation (5.14) and have $Q_1^\perp \in C([0, T]; H^{k-1})$. In order to solve (\mathbf{v}_1, Q_1^\top) , we need to derive a closed system for (\mathbf{v}_1, Q_1^\top) from (5.17)–(5.19). We will also show that this system is linear and have a closed energy estimate, which implies the solution (\mathbf{v}_1, Q_1^\top) will not blow up in $[0, T]$.

In what follows, we denote by $L(Q_1^\top, \mathbf{v}_1)$ the terms which only depend on (Q_1^\top, \mathbf{v}_1) (not their derivatives) linearly with the coefficients belonging to $C([0, T]; H^{k-1})$. We also use $R \in C([0, T]; H^{k-3})$ to denote the terms depending only on \mathbf{n}, \mathbf{v}_0 and Q_1^\perp .

Lemma 5.1. *It holds that*

$$\begin{aligned} \mathcal{P}^{out} \left(\frac{\partial Q_1}{\partial t} + \mathbf{v}_0 \cdot \nabla Q_1 \right) &= L(Q_1^\top) + R, \\ \mathcal{P}^{in} \left(\frac{\partial Q_1}{\partial t} + \mathbf{v}_0 \cdot \nabla Q_1 \right) &= \frac{\partial Q_1^\top}{\partial t} + \mathbf{v}_0 \cdot \nabla Q_1^\top + L(Q_1^\top) + R. \end{aligned}$$

Proof. The proof can be found in [29]. \square

For any $Q \in \mathbb{Q}$, we set

$$\begin{aligned} \bar{\mathbf{F}}_1(Q) &= 4\alpha \left(Q^2 - [Q + Q_0(\mathcal{Q}_{\mathbf{n}}^{-1}(Q) : Q_0)] : \mathcal{U}_{\mathbf{n}}(\mathcal{Q}_{\mathbf{n}}^{-1}(Q)) \right. \\ &\quad \left. - Q_0 : (\widehat{M}_1(Q) - \widehat{Z}_1(Q)M_{Q_0}^{(4)}) \right), \end{aligned}$$

where $\widehat{M}_1(Q)$ and $\widehat{Z}_1(Q)$ are nonlinear functions with respect to Q ,

$$\begin{aligned} \widehat{M}_1(Q) &= \frac{1}{2Z_{Q_0}} \int_{\mathbb{S}^2} \mathbf{m} \mathbf{m} \mathbf{m} \mathbf{m} (\mathbf{m} \mathbf{m} : \mathcal{Q}_{\mathbf{n}}^{-1}(Q))^2 \exp(\mathbf{m} \mathbf{m} : B_0) d\mathbf{m}, \\ \widehat{Z}_1(Q) &= \frac{1}{2Z_{Q_0}} \int_{\mathbb{S}^2} (\mathbf{m} \mathbf{m} : \mathcal{Q}_{\mathbf{n}}^{-1}(Q))^2 \exp(\mathbf{m} \mathbf{m} : B_0) d\mathbf{m}. \end{aligned}$$

Therefore, note that $Q_1 = Q_1^\top + Q_1^\perp$, we have

$$\bar{\mathbf{F}}_1 = \bar{\mathbf{F}}_1(Q_1) = \bar{\mathbf{F}}_1(Q_1^\top) + L(Q_1^\top),$$

where the definition of $L(\cdot)$ is as the above. The next lemma tells us that when we take the projection \mathcal{P}^{in} on $\bar{\mathbf{F}}_1$, the terms which are nonlinear with respect to Q_1^\top will vanish.

Lemma 5.2. $\bar{\mathbf{F}}_1(Q_1^\top) \in \mathbb{Q}_{\mathbf{n}}^{out}$, that is, $\mathcal{P}^{in} \bar{\mathbf{F}}_1 = L(Q_1^\top)$.

Proof. Let $Q_1^\top = \mathbf{n}\tilde{\mathbf{n}} + \tilde{\mathbf{n}}\mathbf{n}$ where $\tilde{\mathbf{n}} \perp \mathbf{n}$. It suffices to prove that $\bar{\mathbf{F}}_1(Q_1^\top) : (\mathbf{n}\mathbf{p} + \mathbf{p}\mathbf{n}) = 0$ for any $\mathbf{p} \perp \mathbf{n}$.

Due to the definition of $\mathcal{Q}_\mathbf{n}^{-1}$ we know $\mathcal{Q}_\mathbf{n}^{-1}(Q_1^\top) = (\psi_2 + \psi_3)Q_1^\top$, where ψ_2, ψ_3 are coefficients defined in (4.6). Thus we have

$$Q_0 : \mathcal{Q}_\mathbf{n}^{-1}(Q_1^\top) = S_2(\psi_2 + \psi_3)(\mathbf{n}\mathbf{n} - \frac{1}{3}\mathbf{I}) : (\mathbf{n}\tilde{\mathbf{n}} + \tilde{\mathbf{n}}\mathbf{n}) = 0.$$

Direct calculation yields that

$$\begin{aligned} & \left(Q_1^\top : \mathcal{U}_\mathbf{n}(\mathcal{Q}_\mathbf{n}^{-1}(Q_1^\top)) \right) : (\mathbf{n}\mathbf{p} + \mathbf{p}\mathbf{n}) \\ &= \frac{8(\psi_2 + \psi_3)}{Z_{Q_0}} \int_{\mathbb{S}^2} (\mathbf{m} \cdot \mathbf{n})^3 (\mathbf{m} \cdot \tilde{\mathbf{n}})^2 (\mathbf{m} \cdot \mathbf{p}) \exp(\eta(\mathbf{m} \cdot \mathbf{n})^2) d\mathbf{m}, \\ & (Q_0 : \widehat{M}_1^{(4)}(Q_1^\top)) : (\mathbf{n}\mathbf{p} + \mathbf{p}\mathbf{n}) \\ &= \frac{4S_2(\psi_2 + \psi_3)^2}{Z_{Q_0}} \int_{\mathbb{S}^2} ((\mathbf{m} \cdot \mathbf{n})^2 - \frac{1}{3}) (\mathbf{m} \cdot \mathbf{n})^3 (\mathbf{m} \cdot \tilde{\mathbf{n}})^2 (\mathbf{m} \cdot \mathbf{p}) \exp(\eta(\mathbf{m} \cdot \mathbf{n})^2) d\mathbf{m}, \\ & \widehat{Z}_1(Q_1^\top)(Q_0 : M_{Q_0}^{(4)}) : (\mathbf{n}\mathbf{p} + \mathbf{p}\mathbf{n}) \\ &= \frac{2S_2\widehat{Z}_1(Q_1^\top)}{Z_{Q_0}} \int_{\mathbb{S}^2} ((\mathbf{m} \cdot \mathbf{n})^2 - \frac{1}{3}) (\mathbf{m} \cdot \mathbf{n})(\mathbf{m} \cdot \mathbf{p}) \exp(\eta(\mathbf{m} \cdot \mathbf{n})^2) d\mathbf{m}. \end{aligned}$$

By the coordinate invariance, we may assume $\mathbf{n} = (0, 0, 1)^T$ and $\tilde{\mathbf{n}} = (a_1, b_1, 0)^T$, $\mathbf{p} = (a_2, b_2, 0)^T$. Let $\mathbf{m} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)^T$, then

$$\begin{aligned} & \left(Q_1^\top : \mathcal{U}_\mathbf{n}(\mathcal{Q}_\mathbf{n}^{-1}(Q_1^\top)) \right) : (\mathbf{n}\mathbf{p} + \mathbf{p}\mathbf{n}) \\ &= \frac{8(\psi_2 + \psi_3)}{Z_{Q_0}} \int_0^{2\pi} \int_0^\pi \cos^3 \theta \sin^4 \theta (a_1 \cos \varphi + b_1 \sin \varphi)^2 \\ & \quad \cdot (a_2 \cos \varphi + b_2 \sin \varphi) e^{\eta \cos^2 \theta} d\theta d\varphi \\ &= 0. \end{aligned}$$

Similarly, we have

$$(Q_0 : \widehat{M}_1^{(4)}(Q_1^\top)) : (\mathbf{n}\mathbf{p} + \mathbf{p}\mathbf{n}), \quad \widehat{Z}_1(Q_1^\top)(Q_0 : M_{Q_0}^{(4)}) : (\mathbf{n}\mathbf{p} + \mathbf{p}\mathbf{n}) = 0.$$

This completes the proof of Lemma 5.2. \square

We are now in a position to derive the systems of (\mathbf{v}_1, Q_1^\top) . We denote

$$\begin{aligned} \mathbf{A}_1 &= \mathcal{P}^{in}(\mathcal{J}_\mathbf{n}(\mathcal{L}(Q_1^\top))), \quad \mathbf{A}_2 = \mathcal{P}^{out}(\mathcal{J}_\mathbf{n}(\mathcal{L}(Q_1^\top))), \\ \mathbf{B}_1 &= \mathcal{P}^{in}(\mathcal{J}_{Q_0}(\nabla \mathbf{v}_1)), \quad \mathbf{B}_2 = \mathcal{P}^{out}(\mathcal{J}_{Q_0}(\nabla \mathbf{v}_1)). \end{aligned}$$

Taking the projection \mathcal{P}^{in} on both sides of (5.17), note that $\mathcal{H}_\mathbf{n}(Q_2) \in \mathbb{Q}_\mathbf{n}^{out}$ and $\mathcal{J}_\mathbf{n}(\mathcal{L}(Q_1)) = \mathcal{J}_\mathbf{n}(\mathcal{L}(Q_1^\top)) + R$, from Lemma 5.1 and Lemma 5.2 we get

$$\frac{\partial Q_1^\top}{\partial t} + \mathbf{v}_0 \cdot \nabla Q_1^\top = -4\mathbf{A}_1 + 2\mathbf{B}_1 + L(Q_1^\top) + L(\mathbf{v}_1) + R. \quad (5.24)$$

Here we have absorbing $\mathcal{P}^{in}(\mathbf{v}_1 \cdot \nabla Q_0)$ into $L(\mathbf{v}_1)$. Taking the projection \mathcal{P}^{out} on both sides of (5.17), we have

$$-4\mathcal{J}_\mathbf{n}(\mathcal{H}_\mathbf{n}(Q_2)) - 4\mathbf{A}_2 + 2\mathbf{B}_2 + \bar{\mathbf{F}}_1(Q_1^\top) + L(Q_1^\top) + L(\mathbf{v}_1) + R = 0,$$

which implies that

$$-2\mathcal{J}_\mathbf{n}(\mathcal{H}_\mathbf{n}(Q_2)) + \mathbf{G}_1 = 2\mathbf{A}_2 - \mathbf{B}_2 + L(Q_1^\top) + L(\mathbf{v}_1) + R. \quad (5.25)$$

Substituting (5.25) to (5.18) and together with (5.24), we obtain the following closed system for (\mathbf{v}_1, Q_1^\top)

$$\frac{\partial Q_1^\top}{\partial t} + \mathbf{v}_0 \cdot \nabla Q_1^\top = -4\mathbf{A}_1 + \mathbf{B}_1 + L(Q_1^\top) + L(\mathbf{v}_1) + R, \quad (5.26)$$

$$\begin{aligned} \frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_1 = & -\nabla p_1 + \frac{\gamma}{Re} \Delta \mathbf{v}_1 + \frac{1-\gamma}{2Re} \nabla \cdot (\mathbf{D}_1 : M_{Q_0}^{(4)}) \\ & - \frac{1-\gamma}{Re} \left(\nabla \cdot (2\mathbf{A}_2 - \mathbf{B}_2 - 2\mathcal{M}_{Q_0}(\mathcal{L}(Q_1^\top)) + L(Q_1^\top) + R) \right. \\ & \left. - \nabla \cdot (\sigma^d(Q_0, Q_1^\top) + \sigma^d(Q_1^\top, Q_0)) \right) + L(\mathbf{v}_1), \end{aligned} \quad (5.27)$$

$$\nabla \cdot \mathbf{v}_1 = 0. \quad (5.28)$$

Apparently, (5.26)-(5.28) is a linear system of (\mathbf{v}_1, Q_1^\top) . To prove its solvability, we give a priori estimate for the energy

$$E_k \stackrel{\text{def}}{=} \sum_{|\ell|=0}^{k-4} \left(\langle \partial^\ell \mathbf{v}_1, \partial^\ell \mathbf{v}_1 \rangle + \frac{1-\gamma}{2Re} \langle \partial^\ell Q_1^\top, \mathcal{L}(\partial^\ell Q_1^\top) \rangle \right) + \langle Q_1^\top, Q_1^\top \rangle.$$

We will prove that there exists a positive constant C such that

$$\frac{d}{dt} E_k \leq C(E_k + \|R(t)\|_{H^{k-3}}), \quad (5.29)$$

which ensure that the systems (5.26)-(5.28) have a unique solution (\mathbf{v}_1, Q_1^\top) on $[0, T]$ satisfying

$$\mathbf{v}_1 \in C([0, T]; H^{k-4}), \quad Q_1^\top \in C([0, T]; H^{k-3}). \quad (5.30)$$

Without loss of generality, we only prove (5.29) in the case of $\ell = 0$ and the proof is similar for the general case. When $\ell = 0$, the corresponding energy is given by

$$E_1 = \langle \mathbf{v}_1, \mathbf{v}_1 \rangle + \langle Q_1^\top, Q_1^\top \rangle + \frac{1-\gamma}{2Re} \langle Q_1^\top, \mathcal{L}(Q_1^\top) \rangle.$$

First, we get by (5.26) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \langle Q_1^\top, Q_1^\top \rangle &= \langle -4\mathcal{J}_n(\mathcal{L}(Q_1^\top)) + \mathcal{J}_n(\nabla \mathbf{v}_1), Q_1^\top \rangle + \langle L(\mathbf{v}_1) + L(Q_1^\top) + R, Q_1^\top \rangle \\ &\leq \delta \|\nabla \mathbf{v}_1\|_{L^2}^2 + C_\delta \|Q_1^\top\|_{H^1}^2 + C(\|\mathbf{v}_1\|_{L^2}^2 + \|R\|_{L^2}^2). \end{aligned} \quad (5.31)$$

Meanwhile, we can obtain from (5.26) and (5.27) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\frac{2Re}{1-\gamma} \langle \mathbf{v}_1, \mathbf{v}_1 \rangle + 2 \langle Q_1^\top, \mathcal{L}(Q_1^\top) \rangle \right) &= \frac{2Re}{1-\gamma} \langle \partial_t \mathbf{v}_1, \mathbf{v}_1 \rangle + 2 \langle \partial_t Q_1^\top, \mathcal{L}(Q_1^\top) \rangle \\ &= -\frac{2\gamma}{1-\gamma} \|\nabla \mathbf{v}_1\|_{L^2}^2 - \langle \mathbf{D}_1 : M_{Q_0}^{(4)}, \mathbf{D}_1 \rangle + \underbrace{2 \langle 2\mathbf{A}_2 - \mathbf{B}_2 - 2\mathcal{M}_{Q_0}(\mathcal{L}(Q_1^\top)), \nabla \mathbf{v}_1 \rangle}_{I_1} \\ &\quad + \underbrace{2 \langle L(\mathbf{v}_1) + L(Q_1^\top) + R - \frac{Re}{1-\gamma} (\sigma^d(Q_0, Q_1^\top) + \sigma^d(Q_1^\top, Q_0)), \nabla \mathbf{v}_1 \rangle}_{I_2} \\ &\quad - \underbrace{2 \langle \mathbf{v}_0 \cdot \nabla Q_1^\top, \mathcal{L}(Q_1^\top) \rangle}_{I_3} + \underbrace{2 \langle -4\mathbf{A}_1 + \mathbf{B}_1, \mathcal{L}(Q_1^\top) \rangle}_{I_4} \\ &\quad + \underbrace{2 \langle L(\mathbf{v}_1) + L(Q_1^\top) + R, \mathcal{L}(Q_1^\top) \rangle}_{I_5}. \end{aligned}$$

For I_2, I_3 and I_5 , we have

$$I_3 = -2 \int_{\mathbb{R}^3} \left(L_1 \partial_j \mathbf{v}_{0i} \partial_i Q_{1i'j'}^\top \partial_j Q_{1i'j'}^\top + 2L_2 (\partial_l \mathbf{v}_{0j} \partial_j Q_{1kl}^\top \partial_m Q_{1km}^\top + \partial_k \mathbf{v}_{0j} \partial_j Q_{1kl}^\top \partial_m Q_{1lm}^\top) \right) d\mathbf{x} \leq C \|Q_1^\top\|_{H^1}^2, \quad (5.32)$$

$$I_2 + I_5 \leq \delta \|\nabla \mathbf{v}_1\|_{L^2}^2 + C_\delta (\|\mathbf{v}_1\|_{L^2}^2 + \|Q_1^\top\|_{H^1}^2 + \|R\|_{H^1}^2). \quad (5.33)$$

Now we turn to estimate $I_1 + I_4$. Recalling the fact that for any $Q \in \mathbb{Q}_n^{in}(\mathbb{Q}_n^{out})$, $\mathcal{J}_n(Q)$ and $\mathcal{M}_{Q_0}(Q)$ belong to $\mathbb{Q}_n^{in}(\mathbb{Q}_n^{out})$, we have:

$$\begin{aligned} \langle \mathcal{P}^{out}(\mathcal{J}_n(\nabla \mathbf{v}_1)), \nabla \mathbf{v}_1 \rangle &= \langle \mathcal{P}^{out}(\mathcal{J}_n(\nabla \mathbf{v}_1)), \mathbf{D}_1 \rangle = \langle \mathcal{M}_n(\nabla \mathbf{v}_1), \mathcal{P}^{out}(\mathbf{D}_1) \rangle \\ &= \langle \nabla \mathbf{v}_1, \mathcal{M}_n(\mathcal{P}^{out}(\mathbf{D}_1)) \rangle = \langle \nabla \mathbf{v}_1, \mathcal{J}_n(\mathcal{P}^{out}(\mathbf{D}_1)) \rangle \\ &= \langle \mathcal{P}^{out}(\mathbf{D}_1), \mathcal{J}_n(\mathcal{P}^{out}(\mathbf{D}_1)) \rangle \geq 0, \end{aligned} \quad (5.34)$$

where we have repeatedly used the symmetry of $\mathcal{J}_n(\cdot)$ and the self-adjointness of \mathcal{M}_n (note that $\mathcal{M}_n(\cdot)$ is not symmetric and \mathcal{J}_n is not self-adjoint). Similarly, it holds that

$$\langle \mathcal{J}_n(\mathcal{L}(Q_1^\top)), \mathcal{P}^{in}(\mathcal{L}(Q_1^\top)) \rangle \geq 0. \quad (5.35)$$

On the other hand, thanks to (4.14) and (4.16), we have

$$\begin{aligned} &\langle \mathcal{P}^{out}(\mathcal{J}_n(\mathcal{L}(Q_1^\top))), \nabla \mathbf{v}_1 \rangle + \langle \mathcal{P}^{in}(\mathcal{J}_n(\nabla \mathbf{v}_1)), \mathcal{L}(Q_1^\top) \rangle \\ &= \langle \mathcal{J}_n(\mathcal{P}^{out}(\mathcal{L}(Q_1^\top))), \nabla \mathbf{v}_1 \rangle + \langle \mathcal{J}_n(\nabla \mathbf{v}_1), \mathcal{P}^{in}(\mathcal{L}(Q_1^\top)) \rangle \\ &= \langle \mathcal{M}_n(\mathcal{P}^{out}(\mathcal{L}(Q_1^\top))), \nabla \mathbf{v}_1 \rangle + \langle \mathcal{M}_n(\nabla \mathbf{v}_1), \mathcal{P}^{in}(\mathcal{L}(Q_1^\top)) \rangle \\ &= \langle \mathcal{M}_{Q_0}(\mathcal{L}(Q_1^\top)), \nabla \mathbf{v}_1 \rangle. \end{aligned} \quad (5.36)$$

Combining (5.34)-(5.36), we get

$$I_1 + I_4 \leq 0. \quad (5.37)$$

Therefore, we obtain the following energy inequality

$$\frac{d}{dt} E_1 \leq C(E_1 + \|R\|_{H^1}^2),$$

which indicates the existence of (\mathbf{v}_1, Q_1^\top) .

Again, we write $Q_2 = Q_2^\top + Q_2^\perp$ with $Q_2^\top \in \mathbb{Q}_n^{in}$ and $Q_2^\perp \in \mathbb{Q}_n^{out}$. By (5.25) we can solve Q_2^\perp as

$$Q_2^\perp = \frac{1}{2} \mathcal{H}_n^{-1} \mathcal{J}_n^{-1} (-2\mathbf{A}_2 + \frac{1}{2}\mathbf{B}_2 - \mathbf{G}_1 - L(\mathbf{v}_1) - L(Q_1^\top) - R) \in C([0, T]; H^{k-5}). \quad (5.38)$$

Then, (\mathbf{v}_2, Q_2^\top) can be solved in a similar way as (\mathbf{v}_1, Q_1^\top) . Q_3 can be solved similarly as in (5.38) (unique up to a term in \mathbb{Q}_n^{in}). We omit the details and leave them to the interest readers.

To summarize, we have proved:

Proposition 8. *Let $(\mathbf{v}_0, \mathbf{n})$ be a solution of (1.3)-(1.5) on $[0, T]$ and satisfy*

$$\mathbf{v}_0 \in C([0, T]; H^k), \quad \nabla \mathbf{n} \in C([0, T]; H^k) \quad \text{for } k \geq 20.$$

There exists the solution $(\mathbf{v}_i, Q_i)(i = 0, 1, 2)$ and $Q_3 \in \mathbb{Q}_{\mathbf{n}}^{out}$ of the system (5.17)-(5.22) satisfying

$$\mathbf{v}_i \in C([0, T]; H^{k-4i}), \quad Q_i \in C([0, T]; H^{k+1-4i})(i = 0, 1, 2), \quad Q_3 \in C([0, T]; H^{k-11}).$$

5.3. The system for the remainder. In this subsection, we focus on the derivation of systems of the remainder and uniform estimates for the remainder. Throughout this subsection, we assume that $\mathbf{v}_i \in C([0, T]; H^{k-4i})$ for $i = 0, 1, 2$ and $Q_i \in C([0, T]; H^{k+1-4i})$ for $i = 0, 1, 2, 3$. We denote by C a constant depending on $\sum_{i=0}^2 \sup_{t \in [0, T]} \|\mathbf{v}_i(t)\|_{H^{k-4i}}$ and $\sum_{i=0}^3 \sup_{t \in [0, T]} \|Q_i(t)\|_{H^{k+1-4i}}$, and independent of ε .
Let

$$E = \|Q_R\|_{H^1} + \varepsilon \|\Delta Q_R\|_{L^2} + \varepsilon^2 \|\nabla \Delta Q_R\|_{L^2} + \|\mathbf{v}_R\|_{L^2} + \varepsilon \|\nabla \mathbf{v}_R\|_{L^2} + \varepsilon^2 \|\Delta \mathbf{v}_R\|_{L^2},$$

$$F = \varepsilon \|\nabla \mathcal{L}(Q_R)\|_{L^2} + \varepsilon^2 \|\Delta \mathcal{L}(Q_R)\|_{L^2} + \varepsilon^2 \|\Delta \nabla \mathbf{v}_R\|_{L^2}.$$

By Sobolev embedding inequality, for $k = 0, 1, 2$, we have

$$\varepsilon^k \|Q_R\|_{H^k} + \varepsilon^k \|\mathbf{v}_R\|_{H^k} \leq E, \quad \varepsilon \|Q_R\|_{L^\infty} + \varepsilon^2 \|\mathbf{v}_R\|_{L^\infty} \leq CE, \quad (5.39)$$

$$\varepsilon^{k+1} \|\mathcal{L}(Q_R)\|_{H^k} + \varepsilon^3 \|\nabla \mathbf{v}_R\|_{L^\infty} \leq C(E + \varepsilon F), \quad (5.40)$$

for some constant C . To simplify the formulation, we introduce a notation \mathfrak{R} to denote all the terms (called *good terms*) which can be controlled by

$$\|\mathfrak{R}\|_{L^2} + \varepsilon \|\nabla \mathfrak{R}\|_{L^2} + \varepsilon^2 \|\Delta \mathfrak{R}\|_{L^2} \leq C(\varepsilon E)(1 + E + \varepsilon F) + \varepsilon f(E), \quad (5.41)$$

where $C(\cdot)$ and $f(\cdot): \mathbb{R}^+ \cup \{0\} \mapsto \mathbb{R}^+ \cup \{0\}$ are increasing functions. They may depend on $\|Q_i\|$ and the parameters of the system, but are independent of ε . The main feature of the righthand side is that it is almost controlled by $C(1 + E)$ when $\varepsilon \rightarrow 0$. Therefore, we can deduce a closed energy estimate uniformly in ε , see Proposition 9. Since $\|Q_0 - Q^*\|_{H^k}, \|Q_i\|_{H^k}(k \leq 3, 1 \leq i \leq 3)$ are all bounded by a constant independent on ε , we have that

$$\|Q^\varepsilon - Q^*\|_{H^k} \leq C + \varepsilon^3 \|Q_R\|_{H^k} \leq C(\varepsilon E), \quad \|\mathbf{v}^\varepsilon\|_{H^k} \leq C(\varepsilon E).$$

We explain the motivation to introduce this definition. To control the remainder term, first we have to write down the evolution equation for Q_R and \mathbf{v}_R . In other words, we have to calculate

$$\frac{1}{\varepsilon^3} \left(\frac{\partial}{\partial t} Q^\varepsilon - \frac{\partial}{\partial t} Q_0 - \varepsilon \frac{\partial}{\partial t} Q_1 - \varepsilon^2 \frac{\partial}{\partial t} Q_2 - \varepsilon^3 \frac{\partial}{\partial t} Q_3 \right).$$

The system for $(Q^\varepsilon, \mathbf{v}^\varepsilon)$ can be written in the following abstract form:

$$\frac{\partial}{\partial t} Q^\varepsilon = \frac{1}{\varepsilon} \mathbf{F}(Q^\varepsilon) + \mathbf{G}(Q^\varepsilon, \mathbf{v}^\varepsilon), \quad (5.42)$$

$$\frac{\partial}{\partial t} \mathbf{v}^\varepsilon = P_{\text{div}} \nabla \cdot \left(\frac{1}{\varepsilon} \mathbf{H}(Q^\varepsilon) + \mathbf{J}(Q^\varepsilon, \mathbf{v}^\varepsilon) \right), \quad (5.43)$$

where P_{div} is projection operator which projects a vector field to its solenoidal part, and

$$\begin{aligned}\mathbf{F}(Q) &= -6Q + 2\alpha(\mathcal{M}_Q(Q) + \mathcal{M}_Q^T(Q)), \quad \mathbf{H}(Q) = \frac{1-\gamma}{Re}(3Q - 2\alpha\mathcal{M}_Q(Q)), \\ \mathbf{G}(Q, \mathbf{v}) &= -2(\mathcal{M}_Q(\mathcal{L}(Q)) + \mathcal{M}_Q^T(\mathcal{L}(Q))) + (\mathcal{M}_Q(\nabla \mathbf{v}) + \mathcal{M}_Q^T(\nabla \mathbf{v})) - \mathbf{v} \cdot \nabla Q \\ &\triangleq \mathbf{G}_1(Q) + \mathbf{G}_2(Q, \mathbf{v}) + \mathbf{G}_3(Q, \mathbf{v}), \\ \mathbf{J}(Q, \mathbf{v}) &= -\frac{2(1-\gamma)}{Re}\mathcal{M}_Q(\mathcal{L}(Q)) + \frac{1-\gamma}{Re}\sigma^d(Q, Q) \\ &\quad + \frac{1-\gamma}{2Re}\mathbf{D} : M_Q^{(4)} - \mathbf{v} \otimes \mathbf{v} + \frac{2\gamma}{Re}\mathbf{D} \\ &\triangleq \mathbf{J}_1(Q) + \mathbf{J}_2(Q) + \mathbf{J}_3(Q, \mathbf{v}) + \mathbf{J}_4(\mathbf{v}) + \frac{2\gamma}{Re}\mathbf{D}.\end{aligned}$$

Then we have

$$\begin{aligned}\frac{\partial}{\partial t}Q_R &= \frac{1}{\varepsilon^4}(\mathbf{F}(Q^\varepsilon) - \mathbf{F}(\tilde{Q})) + \frac{1}{\varepsilon^3}(\mathbf{G}(Q^\varepsilon, \mathbf{v}^\varepsilon) - \mathbf{G}(\tilde{Q}, \tilde{\mathbf{v}})) \\ &\quad + \frac{1}{\varepsilon^3}\left(\frac{1}{\varepsilon}\mathbf{F}(\tilde{Q}) + \mathbf{G}(\tilde{Q}, \tilde{\mathbf{v}}) - \frac{\partial}{\partial t}\tilde{Q}\right), \\ \frac{\partial}{\partial t}\mathbf{v}_R &= P_{\text{div}}\left(\frac{1}{\varepsilon^4}(\mathbf{H}(Q^\varepsilon) - \mathbf{H}(\tilde{Q})) + \frac{1}{\varepsilon^3}(\mathbf{J}(Q^\varepsilon, \mathbf{v}^\varepsilon) - \mathbf{J}(\tilde{Q}, \tilde{\mathbf{v}}))\right) \\ &\quad + \frac{1}{\varepsilon^3}\left(P_{\text{div}}\left(\frac{1}{\varepsilon}\mathbf{H}(\tilde{Q}) + \mathbf{J}(\tilde{Q}, \tilde{\mathbf{v}})\right) - \frac{\partial}{\partial t}\tilde{\mathbf{v}}\right).\end{aligned}$$

By the choices of $Q_i (0 \leq i \leq 3)$, $\mathbf{v}_j (0 \leq j \leq 2)$, we know that

$$\left\|\frac{1}{\varepsilon^3}\left(\frac{1}{\varepsilon}\mathbf{F}(\tilde{Q}) + \mathbf{G}(\tilde{Q}, \tilde{\mathbf{v}}) - \frac{\partial}{\partial t}\tilde{Q}\right)\right\|_{H^2}, \quad \left\|\frac{1}{\varepsilon^3}(\mathbf{J}(Q^\varepsilon, \mathbf{v}^\varepsilon) - \mathbf{J}(\tilde{Q}, \tilde{\mathbf{v}}))\right\|_{H^2} \quad (5.44)$$

are bounded by a constant uniformly in ε , then they are good terms.

Lemma 5.3. *For the difference terms arising from \mathbf{G} and \mathbf{J} , we have*

$$\mathbf{G}_1(Q^\varepsilon) - \mathbf{G}_1(\tilde{Q}) = -2\varepsilon^3(\mathcal{M}_{Q_0}(\mathcal{L}(Q_R)) + \mathcal{M}_{Q_0}^T(\mathcal{L}(Q_R))) + \varepsilon^3\mathfrak{R}, \quad (5.45)$$

$$\mathbf{G}_2(Q^\varepsilon, \mathbf{v}^\varepsilon) - \mathbf{G}_2(\tilde{Q}, \tilde{\mathbf{v}}) = \varepsilon^3(\mathcal{M}_{Q_0}(\nabla \mathbf{v}_R) + \mathcal{M}_{Q_0}^T(\nabla \mathbf{v}_R)) + \varepsilon^3\mathfrak{R}, \quad (5.46)$$

$$\mathbf{G}_3(Q^\varepsilon, \mathbf{v}^\varepsilon) - \mathbf{G}_3(\tilde{Q}, \tilde{\mathbf{v}}) = \varepsilon^3\mathfrak{R}, \quad (5.47)$$

$$\mathbf{J}_1(Q^\varepsilon) - \mathbf{J}_1(\tilde{Q}) = -\varepsilon^3\frac{2(1-\gamma)}{Re}\mathcal{M}_{Q_0}(\mathcal{L}(Q_R)) + \varepsilon^3\mathfrak{R}, \quad (5.48)$$

$$\mathbf{J}_2(Q^\varepsilon) - \mathbf{J}_2(\tilde{Q}) = \varepsilon^3\mathfrak{R}, \quad (5.49)$$

$$\mathbf{J}_3(Q^\varepsilon, \mathbf{v}^\varepsilon) - \mathbf{J}_3(\tilde{Q}, \tilde{\mathbf{v}}) = \varepsilon^3\frac{1-\gamma}{2Re}\mathbf{D}_R : M_{Q_0}^{(4)} + \varepsilon^3\mathfrak{R}, \quad (5.50)$$

$$\mathbf{J}_4(\mathbf{v}^\varepsilon) - \mathbf{J}_4(\tilde{\mathbf{v}}) = \varepsilon^3\mathfrak{R}. \quad (5.51)$$

Proof. First, by Lemma 2.5, for $0 \leq k \leq 2$, we have

$$\begin{aligned}&\|\mathcal{M}_{Q^\varepsilon}(\mathcal{L}(Q^\varepsilon)) - \mathcal{M}_{\tilde{Q}}(\mathcal{L}(Q^\varepsilon))\|_{H^k} \\ &\leq \|\mathcal{M}_{Q^\varepsilon}(\mathcal{L}(\tilde{Q})) - \mathcal{M}_{\tilde{Q}}(\mathcal{L}(\tilde{Q}))\|_{H^k} + \varepsilon^3\|\mathcal{M}_{Q^\varepsilon}(\mathcal{L}(Q_R)) - \mathcal{M}_{\tilde{Q}}(\mathcal{L}(Q_R))\|_{H^k} \\ &\leq C(\|Q^\varepsilon\|_{L^\infty}, \|\tilde{Q}\|_{L^\infty}, \|Q^\varepsilon - Q^*\|_{H^k}, \|\tilde{Q} - Q^*\|_{H^k})\|\varepsilon^3Q_R\|_{H^k}\|\mathcal{L}(\tilde{Q})\|_{H^{k+2}}\end{aligned}$$

$$\begin{aligned}
& + \varepsilon^3 C(\|Q^\varepsilon\|_{L^\infty}, \|\tilde{Q}\|_{L^\infty}) \|\mathcal{L}(Q_R)\|_{H^k} \|\varepsilon^3 Q_R\|_{L^\infty} \\
& + \varepsilon^3 C(\|Q^\varepsilon\|_{L^\infty}, \|\tilde{Q}\|_{L^\infty}, \|Q^\varepsilon - Q^*\|_{H^k}, \|\tilde{Q} - Q^*\|_{H^k}) \|\mathcal{L}(Q_R)\|_{L^\infty} \|\varepsilon^3 Q_R\|_{H^k}, \\
& \varepsilon^3 \|\mathcal{M}_{\tilde{Q}}(\mathcal{L}(Q_R)) - \mathcal{M}_{Q_0}(\mathcal{L}(Q_R))\|_{H^k} \\
& \leq \varepsilon^3 C(\|Q_0\|_{L^\infty}, \|\tilde{Q}\|_{L^\infty}, \|Q_0 - Q^*\|_{H^{k+2}}, \|\tilde{Q} - Q^*\|_{H^{k+2}}) \\
& \quad \cdot \|\tilde{Q} - Q_0\|_{H^{k+2}} \|\mathcal{L}(Q_R)\|_{H^k}.
\end{aligned}$$

Using $\|\tilde{Q} - Q_0\|_{H^{k+2}} = \varepsilon\|Q_1\|_{H^{k+2}} + \varepsilon^2\|Q_2\|_{H^{k+2}} + \varepsilon^3\|Q_3\|_{H^{k+2}} \leq CE$, we have

$$\begin{aligned}
& \varepsilon^k \|\mathcal{M}_{Q^\varepsilon}(\mathcal{L}(Q^\varepsilon)) - \mathcal{M}_{\tilde{Q}}(\mathcal{L}(Q^\varepsilon))\|_{H^k} \leq C(\varepsilon E) \varepsilon^3 E + C(\varepsilon E) \varepsilon^4 E(E + F), \\
& \varepsilon^{k+3} \|\mathcal{M}_{\tilde{Q}}(\mathcal{L}(Q_R)) - \mathcal{M}_{Q_0}(\mathcal{L}(Q_R))\|_{H^k} \leq \varepsilon^3 C(E + \varepsilon F).
\end{aligned}$$

Thus we obtain

$$\mathcal{M}_{Q^\varepsilon}(\mathcal{L}(Q^\varepsilon)) = \varepsilon^3 \mathcal{M}_{Q_0}(\mathcal{L}(Q_R)) + \varepsilon^3 \mathfrak{R}.$$

This implies (5.48). (5.45) and (5.46) can be proved in the same way. Moreover, $M_{Q_1}^{(4)} - M_{Q_2}^{(4)}$ shares the same estimate with $\mathcal{M}_{Q_1} - \mathcal{M}_{Q_2}$, so (5.50) is also true.

For (5.49), we have

$$\begin{aligned}
\varepsilon^k \|\sigma^d(Q^\varepsilon, Q^\varepsilon) - \sigma^d(\tilde{Q}, \tilde{Q})\|_{H^k} &= \varepsilon^{3+k} \|\sigma^d(Q^\varepsilon, Q_R) + \sigma^d(Q_R, \tilde{Q})\|_{H^k} \\
&\leq C\varepsilon^3 \|\varepsilon^k \nabla Q_R\|_{H^k} (1 + \|\varepsilon^3 \nabla Q_R\|_{L^\infty}) \\
&\leq \varepsilon^3 C(1 + \varepsilon E)E.
\end{aligned}$$

In the same way, (5.47) and (5.51) can be deduced. \square

Lemma 5.4. *For the difference terms arising from \mathbf{H} and \mathbf{F} , we have*

$$\mathbf{H}(Q^\varepsilon) - \mathbf{H}(\tilde{Q}) = 2\varepsilon^3 \frac{1-\gamma}{Re} \mathcal{M}_{Q_0}(\mathcal{H}_{Q_0}(Q_R)) + \varepsilon^4 \mathfrak{R}, \quad (5.52)$$

$$\mathbf{F}(Q^\varepsilon) - \mathbf{F}(\tilde{Q}) = 2\varepsilon^3 \left(\mathcal{M}_{Q_0}(\mathcal{H}_{Q_0}(Q_R)) + \mathcal{M}_{Q_0}^T(\mathcal{H}_{Q_0}(Q_R)) \right) + \varepsilon^4 \mathfrak{R}. \quad (5.53)$$

Proof. First, we have

$$\begin{aligned}
& 3Q^\varepsilon - 2\alpha \mathcal{M}_{Q^\varepsilon}(Q^\varepsilon) - (3\tilde{Q} - 2\alpha \mathcal{M}_{\tilde{Q}}(\tilde{Q})) \\
&= 2\mathcal{M}_{Q^\varepsilon}(B_{Q^\varepsilon} - \alpha Q^\varepsilon) - 2\mathcal{M}_{\tilde{Q}}(B_{\tilde{Q}} - \alpha \tilde{Q}) \\
&= 2\mathcal{M}_{Q^\varepsilon}(B_{Q^\varepsilon} - \alpha Q^\varepsilon) - 2\mathcal{M}_{\tilde{Q}}(B_{Q^\varepsilon} - \alpha Q^\varepsilon) \\
&\quad + 2\mathcal{M}_{\tilde{Q}}(B_{Q^\varepsilon} - \alpha Q^\varepsilon) - 2\mathcal{M}_{\tilde{Q}}(B_{\tilde{Q}} - \alpha \tilde{Q}).
\end{aligned}$$

Using Talylor expansion for $B_Q = B(Q)$, we get

$$\begin{aligned}
\|B_{Q^\varepsilon} - B_{\tilde{Q}} - \varepsilon^3 \mathcal{Q}_{\tilde{Q}}^{-1}(Q_R)\|_{H^k} &= \left\| \varepsilon^6 \int_0^1 s(Q_R : \nabla Q)^2 B_{(\tilde{Q} + s\varepsilon^3 Q_R)} ds \right\|_{H^k} \\
&\leq \varepsilon^6 C(\varepsilon^3 \|Q_R\|_{H^2}) \|Q_R\|_{H^k} \|Q_R\|_{H^2} \leq \varepsilon^5 C(\varepsilon E) E \|Q_R\|_{H^k}. \quad (5.54)
\end{aligned}$$

Therefore, $B_{Q^\varepsilon} - B_{\tilde{Q}} - \varepsilon^3 \mathcal{Q}_{\tilde{Q}}^{-1}(Q_R) = \varepsilon^4 \mathfrak{R}$, which implies

$$\begin{aligned}
\mathcal{M}_{\tilde{Q}}(B_{Q^\varepsilon} - \alpha Q^\varepsilon) - \mathcal{M}_{\tilde{Q}}(B_{\tilde{Q}} - \alpha \tilde{Q}) &= \varepsilon^3 \mathcal{M}_{\tilde{Q}}(\mathcal{Q}_{\tilde{Q}}^{-1}(Q_R) - \alpha Q_R) + \varepsilon^4 \mathfrak{R} \\
&= \varepsilon^3 \mathcal{M}_{Q_0}(\mathcal{Q}_{Q_0}^{-1}(Q_R) - \alpha Q_R) + \varepsilon^4 \mathfrak{R}.
\end{aligned}$$

By Lemma 2.5, we can obtain

$$\begin{aligned} & \|\mathcal{M}_{Q^\varepsilon}(B_{Q^\varepsilon} - \alpha Q^\varepsilon) - \mathcal{M}_{\tilde{Q}}(B_{Q^\varepsilon} - \alpha Q^\varepsilon)\|_{H^k} \\ & \leq C(\|\varepsilon^3 Q_R\|_{H^2}) \|\varepsilon^3 Q_R\|_{H^k} \|B_{Q^\varepsilon} - B_{Q_0} - \alpha(Q^\varepsilon - Q_0)\|_{H^2} \\ & \leq C(\|\varepsilon^3 Q_R\|_{H^2}) \varepsilon^4 \|Q_R\|_{H^k} (1 + \|\varepsilon^2 Q_R\|_{H^2}) \leq \varepsilon^4 C(\varepsilon E) \|Q_R\|_{H^k} (1 + \varepsilon E). \end{aligned}$$

Thus

$$\mathcal{M}_{Q^\varepsilon}(B_{Q^\varepsilon} - \alpha Q^\varepsilon) - \mathcal{M}_{\tilde{Q}}(B_{Q^\varepsilon} - \alpha Q^\varepsilon) \in \mathfrak{R}. \quad (5.55)$$

Combining (5.54) and (5.55) and recalling the definition of $\mathcal{H}_{Q_0}(Q_R) := \mathcal{Q}_{Q_0}^{-1}(Q_R) - \alpha Q_R$, we get (5.52) and (5.53). \square

Combining Lemma 5.3 with Lemma 5.4, we finally arrive at

$$\frac{\partial Q_R}{\partial t} = -4\mathcal{J}_n\left(\frac{1}{\varepsilon}\mathcal{H}_n(Q_R) + \mathcal{L}(Q_R)\right) + 2\mathcal{J}_n(\nabla \mathbf{v}_R) + \mathfrak{R}, \quad (5.56)$$

$$\begin{aligned} \frac{\partial \mathbf{v}_R}{\partial t} &= -\nabla p_R + \frac{\gamma}{Re}\Delta \mathbf{v}_R + \frac{1-\gamma}{2Re}\nabla \cdot (\mathbf{D}_R : M_{\tilde{Q}}^{(4)}) \\ &\quad + \frac{1-\gamma}{Re}\nabla \cdot \left(2\mathcal{M}_{Q_0}\left(\frac{1}{\varepsilon}\mathcal{H}_n(Q_R) + \mathcal{L}(Q_R)\right)\right) + \nabla \cdot \mathfrak{R} + \mathfrak{R}, \end{aligned} \quad (5.57)$$

$$\nabla \cdot \mathbf{v}_R = 0. \quad (5.58)$$

5.4. Uniform estimates for the remainder. In order to obtain the uniform energy estimates, we introduce

$$\mathcal{H}_n^\varepsilon(Q_R) = \mathcal{H}_n(Q_R) + \varepsilon \mathcal{L}(Q_R),$$

and the following energy functional

$$\begin{aligned} \mathfrak{E}(t) &= \frac{1}{2} \int \left(|\mathbf{v}_R|^2 + \mathcal{J}_n^{-1}(Q_R) : Q_R + \frac{1-\gamma}{\varepsilon Re} \mathcal{H}_n^\varepsilon(Q_R) : Q_R \right) + \varepsilon^2 \left(|\nabla \mathbf{v}_R|^2 \right. \\ &\quad \left. + \frac{1-\gamma}{\varepsilon Re} \mathcal{H}_n^\varepsilon(\nabla Q_R) : \nabla Q_R \right) + \varepsilon^4 \left(|\Delta \mathbf{v}_R|^2 + \frac{1-\gamma}{\varepsilon Re} \mathcal{H}_n^\varepsilon(\Delta Q_R) : \Delta Q_R \right) dx, \\ \mathfrak{F}(t) &= \int \left(\frac{\gamma}{Re} |\nabla \mathbf{v}_R|^2 + \frac{4(1-\gamma)}{\varepsilon^2 Re} \mathcal{J}_n(\mathcal{H}_n^\varepsilon(Q_R)) : \mathcal{H}_n^\varepsilon(Q_R) \right) \\ &\quad + \varepsilon^2 \left(\frac{\gamma}{Re} |\Delta \mathbf{v}_R|^2 + \frac{4(1-\gamma)}{\varepsilon^2 Re} \mathcal{J}_n(\mathcal{H}_n^\varepsilon(\nabla Q_R)) : \mathcal{H}_n^\varepsilon(\nabla Q_R) \right) \\ &\quad + \varepsilon^4 \left(\frac{\gamma}{Re} |\nabla \Delta \mathbf{v}_R|^2 + \frac{4(1-\gamma)}{\varepsilon^2 Re} \mathcal{J}_n(\mathcal{H}_n^\varepsilon(\Delta Q_R)) : \mathcal{H}_n^\varepsilon(\Delta Q_R) \right) dx. \end{aligned}$$

Lemma 5.5. *There holds*

$$\|Q_R\|_{H^1} + \|(\varepsilon \nabla^2 Q_R, \varepsilon^2 \nabla^3 Q_R)\|_{L^2} + \|(\mathbf{v}_R, \varepsilon \nabla \mathbf{v}_R, \varepsilon^2 \nabla^2 \mathbf{v}_R)\|_{L^2} \leq C \mathfrak{E}^{\frac{1}{2}}, \quad (5.59)$$

$$\left\| \left(\frac{1}{\varepsilon} \mathcal{H}_n^\varepsilon(Q_R), \nabla \mathcal{H}_n^\varepsilon(Q_R), \varepsilon \Delta \mathcal{H}_n^\varepsilon(Q_R) \right) \right\|_{L^2} \leq C (\mathfrak{E} + \mathfrak{F})^{\frac{1}{2}}, \quad (5.60)$$

$$\|(\varepsilon \nabla \mathcal{L}(Q_R), \varepsilon^2 \Delta \mathcal{L}(Q_R))\|_{L^2} + \|(\nabla \mathbf{v}_R, \varepsilon \nabla^2 \mathbf{v}_R, \varepsilon^2 \nabla^3 \mathbf{v}_R)\|_{L^2} \leq C (\mathfrak{E} + \mathfrak{F})^{\frac{1}{2}}. \quad (5.61)$$

Proof. The first estimate follows from the non-negativity of \mathcal{H}_n . The second estimate can be deduced from the strict positivity of \mathcal{J}_n and the following estimates for communicators

$$\begin{aligned} \|\partial_i \mathcal{H}_n^\varepsilon(Q_R) - \mathcal{H}_n^\varepsilon(\partial_i Q_R)\|_{L^2} &\leq C \|Q_R\|_{L^2}, \\ \|\Delta \mathcal{H}_n^\varepsilon(Q_R) - \mathcal{H}_n^\varepsilon(\Delta Q_R)\|_{L^2} &\leq C \|Q_R\|_{H^1}. \end{aligned}$$

For the last one, we have

$$\begin{aligned}\|\varepsilon \partial_i \mathcal{L}(Q_R)\|_{L^2} &= \|\mathcal{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R) - \mathcal{H}_{\mathbf{n}}(\partial_i Q_R)\|_{L^2} \\ &\leq \|\mathcal{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R)\|_{L^2} + C\|\partial_i Q_R\|_{L^2} \leq C(\mathfrak{E} + \mathfrak{F})^{\frac{1}{2}}, \\ \|\varepsilon^2 \Delta \mathcal{L}(Q_R)\|_{L^2} &\leq \|\varepsilon \mathcal{H}_{\mathbf{n}}^\varepsilon(\Delta Q_R)\|_{L^2} + C\|\varepsilon \Delta Q_R\|_{L^2} \leq C(\mathfrak{E} + \mathfrak{F})^{\frac{1}{2}}.\end{aligned}$$

The estimates for \mathbf{v}_R is straightforward to prove. The proof is completed. \square

Corollary 1. $E \leq C\mathfrak{E}^{1/2}$, $F \leq C(\mathfrak{E} + \mathfrak{F})^{1/2}$.

Now, we state the a priori estimate for the remainder (Q_R, \mathbf{v}_R) .

Proposition 9. *There exist two functions C and f depending on Q_i, \mathbf{v}_j and the parameters of the system (except ε), such that if (\mathbf{v}_R, Q_R) be a strong solution of the system (5.56)–(5.58) on $[0, T]$, then for any $t \in [0, T]$, it holds that*

$$\frac{d}{dt} \mathfrak{E}(t) + \mathfrak{F}(t) \leq C(\varepsilon \mathfrak{E})(1 + \mathfrak{E}) + \varepsilon f(\mathfrak{E}) + C(\varepsilon \mathfrak{E})\varepsilon \mathfrak{F}.$$

Proof. First, for $B \in \mathbb{Q}$, and $A \in \mathbb{R}^{3 \times 3}$, we have:

$$\langle \mathcal{M}_{Q_0}(\mathcal{H}_{\mathbf{n}}^\varepsilon(B)), A \rangle = \langle \mathcal{H}_{\mathbf{n}}^\varepsilon(B), \mathcal{M}_{Q_0}(A) \rangle = \langle \mathcal{J}_{\mathbf{n}}(A), \mathcal{H}_{\mathbf{n}}^\varepsilon(B) \rangle. \quad (5.62)$$

This relation will be repeatedly used in the proof.

Step 1. L^2 -estimate. Using the equation (5.56) and Lemma 5.5, we have

$$\begin{aligned}\left\langle \frac{\partial Q_R}{\partial t}, \mathcal{J}_{\mathbf{n}}^{-1}(Q_R) \right\rangle + \frac{4}{\varepsilon} \langle \mathcal{H}_{\mathbf{n}}^\varepsilon(Q_R), Q_R \rangle &= \langle 2\nabla \mathbf{v}_R, Q_R \rangle + \langle \mathfrak{R}, \mathcal{J}_{\mathbf{n}}^{-1}(Q_R) \rangle \\ &\leq C\|Q_R\|_{L^2} (\|\nabla \mathbf{v}_R\|_{L^2} + \|\mathfrak{R}\|_{L^2}) \leq \delta_0 \mathfrak{F} + C_{\delta_0} \mathfrak{E} + C\|\mathfrak{R}\|_{L^2}^2.\end{aligned} \quad (5.63)$$

We can also obtain

$$\begin{aligned}&\frac{Re}{1-\gamma} \left\langle \frac{\partial \mathbf{v}_R}{\partial t}, \mathbf{v}_R \right\rangle + \left\langle \frac{\partial Q_R}{\partial t}, \frac{1}{\varepsilon} \mathcal{H}_{\mathbf{n}}^\varepsilon(Q_R) \right\rangle \\ &= -\frac{\gamma}{1-\gamma} \|\nabla \mathbf{v}_R\|_{L^2}^2 - \frac{1}{2} \langle \mathbf{D}_R : M_{\tilde{Q}}^{(4)}, \mathbf{D}_R \rangle - \frac{2}{\varepsilon} \langle \mathcal{M}_{Q_0}(\mathcal{H}_{\mathbf{n}}^\varepsilon(Q_R)), \nabla \mathbf{v}_R \rangle \\ &\quad + \frac{Re}{1-\gamma} \langle \nabla \cdot \mathfrak{R} + \mathfrak{R}, \mathbf{v}_R \rangle \\ &\quad - \frac{4}{\varepsilon^2} \langle \mathcal{J}_{\mathbf{n}}(\mathcal{H}_{\mathbf{n}}^\varepsilon(Q_R)), \mathcal{H}_{\mathbf{n}}^\varepsilon(Q_R) \rangle + \langle 2\mathcal{J}_{\mathbf{n}}(\nabla \mathbf{v}_R), \frac{1}{\varepsilon} \mathcal{H}_{\mathbf{n}}^\varepsilon(Q_R) \rangle + \langle \mathfrak{R}, \frac{1}{\varepsilon} \mathcal{H}_{\mathbf{n}}^\varepsilon(Q_R) \rangle.\end{aligned}$$

Using (5.62) with $(A, B) = (\nabla \mathbf{v}_R, Q_R)$, and the fact that $\langle \mathbf{D}_R : M_{\tilde{Q}}^{(4)}, \mathbf{D}_R \rangle \geq 0$, we have

$$\begin{aligned}&\frac{Re}{1-\gamma} \left\langle \frac{\partial \mathbf{v}_R}{\partial t}, \mathbf{v}_R \right\rangle + \left\langle \frac{\partial Q_R}{\partial t}, \frac{1}{\varepsilon} \mathcal{H}_{\mathbf{n}}^\varepsilon(Q_R) \right\rangle \\ &\quad + \frac{\gamma}{1-\gamma} \|\nabla \mathbf{v}_R\|_{L^2}^2 + \frac{4}{\varepsilon^2} \langle \mathcal{J}_{\mathbf{n}}(\mathcal{H}_{\mathbf{n}}^\varepsilon(Q_R)), \mathcal{H}_{\mathbf{n}}^\varepsilon(Q_R) \rangle \\ &\leq \frac{Re}{1-\gamma} \langle \nabla \cdot \mathfrak{R} + \mathfrak{R}, \mathbf{v}_R \rangle + \langle \mathfrak{R}, \frac{1}{\varepsilon} \mathcal{H}_{\mathbf{n}}^\varepsilon(Q_R) \rangle \leq \delta_0 \mathfrak{F} + C\mathfrak{E} + C\|\mathfrak{R}\|_{L^2}^2.\end{aligned} \quad (5.64)$$

Step 2. H^1 -estimate. Using (5.56)-(5.58), we have

$$\begin{aligned}
& \frac{Re}{1-\gamma} \left\langle \frac{\partial}{\partial t} \partial_i \mathbf{v}_R, \partial_i \mathbf{v}_R \right\rangle + \left\langle \frac{\partial}{\partial t} \partial_i Q_R, \frac{1}{\varepsilon} \mathcal{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R) \right\rangle \\
&= -\frac{\gamma}{1-\gamma} \|\nabla \partial_i \mathbf{v}_R\|_{L^2}^2 - \frac{1}{2} \langle \partial_i \mathbf{D}_R : M_{\tilde{Q}}^{(4)}, \partial_i \mathbf{D}_R \rangle - \frac{2}{\varepsilon} \langle \mathcal{M}_{Q_0}(\mathcal{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R)), \nabla \partial_i \mathbf{v}_R \rangle \\
&\quad - \frac{1}{2} \langle \mathbf{D}_R : \partial_i M_{\tilde{Q}}^{(4)}, \partial_i \mathbf{D}_R \rangle + \frac{2}{\varepsilon} \langle [\mathcal{M}_{Q_0} \mathcal{H}_{\mathbf{n}}^\varepsilon, \partial_i] Q_R, \nabla \partial_i \mathbf{v}_R \rangle \\
&\quad - \frac{4}{\varepsilon^2} \langle \mathcal{J}_{\mathbf{n}}(\mathcal{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R)), \mathcal{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R) \rangle + \langle 2\mathcal{J}_{\mathbf{n}}(\nabla \partial_i \mathbf{v}_R), \frac{1}{\varepsilon} \mathcal{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R) \rangle \\
&\quad - \frac{4}{\varepsilon^2} \langle [\mathcal{J}_{\mathbf{n}} \mathcal{H}_{\mathbf{n}}^\varepsilon, \partial_i] Q_R, \mathcal{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R) \rangle + \langle 2[\mathcal{J}_{\mathbf{n}}, \partial_i] \nabla \mathbf{v}_R, \frac{1}{\varepsilon} \mathcal{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R) \rangle \\
&\quad + \frac{Re}{1-\gamma} \langle \nabla \cdot \partial_i \mathfrak{R} + \partial_i \mathfrak{R}, \partial_i \mathbf{v}_R \rangle + \langle \partial_i \mathfrak{R}, \frac{1}{\varepsilon} \mathcal{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R) \rangle.
\end{aligned}$$

It is straightforward to obtain the estimates:

$$\begin{aligned}
& |\langle \mathbf{D}_R : \partial_i M_{\tilde{Q}}^{(4)}, \partial_i \mathbf{D}_R \rangle| \leq C \|\mathbf{D}_R\|_{L^2} \|\partial_i \mathbf{D}_R\|_{L^2} \leq \varepsilon^{-2} (\delta_0 \mathfrak{F} + C\mathfrak{E}), \\
& \varepsilon^{-1} \langle [\mathcal{M}_{Q_0} \mathcal{H}_{\mathbf{n}}^\varepsilon, \partial_i] Q_R, \nabla \partial_i \mathbf{v}_R \rangle = \varepsilon^{-1} \langle [\mathcal{M}_{Q_0} \mathcal{H}_{\mathbf{n}}, \partial_i] Q_R + \varepsilon [\mathcal{M}_{Q_0} \mathcal{L}, \partial_i] Q_R, \nabla \partial_i \mathbf{v}_R \rangle \\
& \leq \varepsilon^{-2} C (\|Q_R\|_{L^2} + \varepsilon \|Q_R\|_{H^2}) \|\varepsilon \nabla \partial_i \mathbf{v}_R\|_{L^2} \\
& \leq \varepsilon^{-2} (\delta_0 \mathfrak{F} + C\mathfrak{E}), \\
& -\varepsilon^{-2} \langle [\mathcal{J}_{\mathbf{n}} \mathcal{H}_{\mathbf{n}}^\varepsilon, \partial_i] Q_R, \mathcal{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R) \rangle \leq \varepsilon^{-2} C (\|Q_R\|_{L^2} + \varepsilon \|Q_R\|_{H^2}) \|\mathcal{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R)\|_{L^2} \\
& \leq \varepsilon^{-2} (\delta_0 \mathfrak{F} + C\mathfrak{E}), \\
& \varepsilon^{-1} \langle [\mathcal{J}_{\mathbf{n}}, \partial_i] \nabla \mathbf{v}_R, \mathcal{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R) \rangle \leq \varepsilon^{-1} \|\nabla \mathbf{v}_R\|_{L^2} \|\mathcal{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R)\|_{L^2} \leq \varepsilon^{-2} (\delta_0 \mathfrak{F} + C\mathfrak{E}).
\end{aligned}$$

Therefore, by the cancelation relation (5.62) with taking $(A, B) = (\nabla \partial_i \mathbf{v}_R, \partial_i Q_R)$, we have

$$\begin{aligned}
& \frac{\varepsilon^2 Re}{1-\gamma} \left\langle \frac{\partial}{\partial t} \partial_i \mathbf{v}_R, \partial_i \mathbf{v}_R \right\rangle + \varepsilon \left\langle \frac{\partial}{\partial t} \partial_i Q_R, \mathcal{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R) \right\rangle \\
& + \frac{\varepsilon^2 \gamma}{1-\gamma} \|\nabla \partial_i \mathbf{v}_R\|_{L^2}^2 + 4 \langle \mathcal{J}_{\mathbf{n}}(\mathcal{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R)), \mathcal{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R) \rangle \\
& \leq \delta_0 \mathfrak{F} + C\mathfrak{E} + C \|\varepsilon \partial_i \mathfrak{R}\|_{L^2}^2.
\end{aligned} \tag{5.65}$$

Step 3. H^2 -estimate. Using (5.56)-(5.58), we get

$$\begin{aligned}
& \frac{Re}{1-\gamma} \left\langle \frac{\partial}{\partial t} \Delta \mathbf{v}_R, \Delta \mathbf{v}_R \right\rangle + \left\langle \frac{\partial}{\partial t} \Delta Q_R, \frac{1}{\varepsilon} \mathcal{H}_{\mathbf{n}}^\varepsilon(\Delta Q_R) \right\rangle \\
&= -\frac{\gamma}{1-\gamma} \|\nabla \Delta \mathbf{v}_R\|_{L^2}^2 - \frac{1}{2} \langle \Delta \mathbf{D}_R : M_{\tilde{Q}}^{(4)}, \Delta \mathbf{D}_R \rangle - \frac{2}{\varepsilon} \langle \mathcal{M}_{Q_0}(\mathcal{H}_{\mathbf{n}}^\varepsilon(\Delta Q_R)), \nabla \Delta \mathbf{v}_R \rangle \\
&\quad + \frac{1}{2} \langle [\Delta, M_{\tilde{Q}}^{(4)}] : \mathbf{D}_R, \Delta \mathbf{D}_R \rangle + \frac{2}{\varepsilon} \langle [\mathcal{M}_{Q_0} \mathcal{H}_{\mathbf{n}}^\varepsilon, \Delta] Q_R, \nabla \Delta \mathbf{v}_R \rangle \\
&\quad - \frac{4}{\varepsilon^2} \langle \mathcal{J}_{\mathbf{n}}(\mathcal{H}_{\mathbf{n}}^\varepsilon(\Delta Q_R)), \mathcal{H}_{\mathbf{n}}^\varepsilon(\Delta Q_R) \rangle + \langle 2\mathcal{J}_{\mathbf{n}}(\nabla \Delta \mathbf{v}_R), \frac{1}{\varepsilon} \mathcal{H}_{\mathbf{n}}^\varepsilon(\Delta Q_R) \rangle \\
&\quad - \frac{4}{\varepsilon^2} \langle [\mathcal{J}_{\mathbf{n}} \mathcal{H}_{\mathbf{n}}^\varepsilon, \Delta] Q_R, \mathcal{H}_{\mathbf{n}}^\varepsilon(\Delta Q_R) \rangle + \langle 2[\mathcal{J}_{\mathbf{n}}, \Delta] \nabla \mathbf{v}_R, \frac{1}{\varepsilon} \mathcal{H}_{\mathbf{n}}^\varepsilon(\Delta Q_R) \rangle \\
&\quad + \frac{Re}{1-\gamma} \langle \nabla \cdot \Delta \mathfrak{R} + \Delta \mathfrak{R}, \Delta \mathbf{v}_R \rangle + \langle \Delta \mathfrak{R}, \frac{1}{\varepsilon} \mathcal{H}_{\mathbf{n}}^\varepsilon(\Delta Q_R) \rangle.
\end{aligned}$$

Similar to Step 2, we can obtain

$$\begin{aligned}
\langle [\Delta, M_Q^{(4)}] : \mathbf{D}_R, \Delta \mathbf{D}_R \rangle &\leq C\varepsilon^{-4} \|\varepsilon^2 \mathbf{D}_R\|_{H^1} \|\varepsilon^2 \Delta \mathbf{D}_R\|_{L^2} \leq \varepsilon^{-4} (\delta_0 \mathfrak{F} + C\mathfrak{E}), \\
\varepsilon^{-1} \langle [\mathcal{M}_{Q_0} \mathcal{H}_{\mathbf{n}}^\varepsilon, \Delta] Q_R, \nabla \Delta \mathbf{v}_R \rangle &= \varepsilon^{-1} \langle [\mathcal{M}_{Q_0} \mathcal{H}_{\mathbf{n}}, \Delta] Q_R + \varepsilon [\mathcal{M}_{Q_0} \mathcal{L}, \Delta] Q_R, \nabla \Delta \mathbf{v}_R \rangle \\
&\leq \varepsilon^{-4} C (\|\varepsilon Q_R\|_{H^1} + \|\varepsilon^2 \mathcal{L}(Q_R)\|_{H^1}) \|\varepsilon^2 \nabla \Delta \mathbf{v}_R\|_{L^2} + \varepsilon^{-4} (\delta_0 \mathfrak{F} + C\mathfrak{E}), \\
-\varepsilon^{-2} \langle [\mathcal{J}_{\mathbf{n}} \mathcal{H}_{\mathbf{n}}^\varepsilon, \Delta] Q_R, \mathcal{H}_{\mathbf{n}}^\varepsilon(\Delta Q_R) \rangle &\leq \varepsilon^{-4} C (\|\varepsilon Q_R\|_{H^1} + \|\varepsilon^2 \mathcal{L}(Q_R)\|_{H^1}) \|\varepsilon \mathcal{H}_{\mathbf{n}}^\varepsilon(\Delta Q_R)\|_{L^2} \leq \varepsilon^{-4} (\delta_0 \mathfrak{F} + C\mathfrak{E}), \\
\varepsilon^{-1} \langle [\mathcal{J}_{\mathbf{n}}, \Delta] \nabla \mathbf{v}_R, \mathcal{H}_{\mathbf{n}}^\varepsilon(\Delta Q_R) \rangle &\leq \varepsilon^{-4} \|\varepsilon^2 \nabla \mathbf{v}_R\|_{H^1} \|\varepsilon \mathcal{H}_{\mathbf{n}}^\varepsilon(\Delta Q_R)\|_{L^2} \leq \varepsilon^{-4} (\delta_0 \mathfrak{F} + C\mathfrak{E}).
\end{aligned}$$

Using (5.62) again, we have

$$\begin{aligned}
&\frac{\varepsilon^4 Re}{1-\gamma} \left\langle \frac{\partial}{\partial t} \Delta \mathbf{v}_R, \Delta \mathbf{v}_R \right\rangle + \varepsilon^3 \left\langle \frac{\partial}{\partial t} \Delta Q_R, \mathcal{H}_{\mathbf{n}}^\varepsilon(\Delta Q_R) \right\rangle \\
&+ \frac{\varepsilon^4 \gamma}{1-\gamma} \|\nabla \Delta \mathbf{v}_R\|_{L^2}^2 + 4\varepsilon^2 \langle \mathcal{J}_{\mathbf{n}}(\mathcal{H}_{\mathbf{n}}^\varepsilon(\Delta Q_R)), \mathcal{H}_{\mathbf{n}}^\varepsilon(\Delta Q_R) \rangle \\
&\leq \delta_0 \mathfrak{F} + C\mathfrak{E} + C \|\varepsilon^2 \Delta \mathfrak{R}\|_{L^2}^2.
\end{aligned} \tag{5.66}$$

Step 4. The completion of energy estimate. Recalling (4.9) we get

$$\mathcal{H}_{\mathbf{n}}^\varepsilon(Q) = \psi_1(\mathbf{nn} - \frac{1}{3}\mathbf{I})(\mathbf{nn} : Q) + \psi_2(-Q + \mathbf{nn} \cdot Q + Q \cdot \mathbf{nn} - \frac{2}{3}\mathbf{Inn} : Q) + \varepsilon \mathcal{L}(Q).$$

With $Q_R : \mathbf{I} = \text{tr } Q_R = 0$, it yields

$$\begin{aligned}
&\frac{1}{\varepsilon} \frac{d}{dt} \langle Q_R, \mathcal{H}_{\mathbf{n}}^\varepsilon(Q_R) \rangle \\
&= \frac{2}{\varepsilon} \left\langle \frac{\partial}{\partial t} Q_R, \mathcal{H}_{\mathbf{n}}^\varepsilon(Q_R) \right\rangle + \frac{1}{\varepsilon} \left\langle Q_R, \psi_1(\mathbf{nn} : Q_R) \partial_t(\mathbf{nn}) + \psi_1(\partial_t(\mathbf{nn}) : Q_R) \mathbf{nn} \right. \\
&\quad \left. + \psi_2(\partial_t(\mathbf{nn}) \cdot Q_R + Q_R \cdot \partial_t(\mathbf{nn})) \right\rangle \\
&= \frac{2}{\varepsilon} \left\langle \frac{\partial}{\partial t} Q_R, \mathcal{H}_{\mathbf{n}}^\varepsilon(Q_R) \right\rangle + \frac{2}{\varepsilon} \left\langle Q_R, \psi_1(\partial_t(\mathbf{nn}) : Q_R) \mathbf{nn} + \psi_2 \partial_t(\mathbf{nn}) \cdot Q_R \right\rangle.
\end{aligned}$$

Lemma 5.6 tell us that

$$\begin{aligned}
&\frac{2}{\varepsilon} \left\langle Q_R, \psi_1(\partial_t(\mathbf{nn}) : Q_R) \mathbf{nn} + \psi_2 \partial_t(\mathbf{nn}) \cdot Q_R \right\rangle \\
&\leq \delta \left\| \frac{1}{\varepsilon} \mathcal{H}_{\mathbf{n}}^\varepsilon(Q_R) \right\|_{L^2}^2 + C_\delta \left(\frac{1}{\varepsilon} \langle \mathcal{H}_{\mathbf{n}}^\varepsilon(Q_R), Q_R \rangle + \|Q_R\|_{L^2}^2 \right).
\end{aligned}$$

Thus we have

$$\frac{1}{2\varepsilon} \frac{d}{dt} \langle Q_R, \mathcal{H}_{\mathbf{n}}^\varepsilon(Q_R) \rangle \leq \frac{1}{\varepsilon} \left\langle \frac{\partial}{\partial t} Q_R, \mathcal{H}_{\mathbf{n}}^\varepsilon(Q_R) \right\rangle + \delta \mathfrak{F} + C\mathfrak{E}.$$

Similarly, the following inequalities hold

$$\begin{aligned}
\frac{\varepsilon}{2} \frac{d}{dt} \langle \partial_i Q_R, \mathcal{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R) \rangle &\leq \varepsilon \left\langle \frac{\partial}{\partial t} \partial_i Q_R, \mathcal{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R) \right\rangle + \delta \mathfrak{F} + C\mathfrak{E}, \\
\frac{\varepsilon^3}{2} \frac{d}{dt} \langle \Delta Q_R, \mathcal{H}_{\mathbf{n}}^\varepsilon(\Delta Q_R) \rangle &\leq \varepsilon^3 \left\langle \frac{\partial}{\partial t} \Delta Q_R, \mathcal{H}_{\mathbf{n}}^\varepsilon(\Delta Q_R) \right\rangle + \delta \mathfrak{F} + C\mathfrak{E}.
\end{aligned}$$

Together with (5.63)-(5.66), we arrive at

$$\frac{1}{2} \frac{d}{dt} \mathfrak{E}(t) + \mathfrak{F}(t) \leq \delta \mathfrak{F} + C_\delta \mathfrak{E} + \|\mathfrak{R}\|_{L^2}^2 + \|\varepsilon \nabla \mathfrak{R}\|_{L^2}^2 + \|\varepsilon^2 \Delta \mathfrak{R}\|_{L^2}^2.$$

Recalling that \mathfrak{R} denotes *good terms* with

$$\|\mathfrak{R}\|_{L^2} + \varepsilon \|\nabla \mathfrak{R}\|_{L^2} + \varepsilon^2 \|\Delta \mathfrak{R}\|_{L^2} \leq C(\varepsilon E)(1 + E + \varepsilon F) + \varepsilon f(E),$$

and Corollary 1, we have

$$\frac{1}{2} \frac{d}{dt} \mathfrak{E}(t) + \mathfrak{F}(t) \leq \delta \mathfrak{F} + C_\delta \mathfrak{E} + C(\varepsilon^2 \mathfrak{E})(1 + \mathfrak{E} + \varepsilon^2 \mathfrak{F}) + \varepsilon^2 f(\mathfrak{E}).$$

Taking δ enough small leads to the Proposition 9. \square

Lemma 5.6. *For any $\delta > 0$, there exists a constant $C = C(\delta, \|\nabla_{t,x} \mathbf{n}\|_{L^\infty}, \|\nabla \mathbf{n}_t\|_{L^\infty})$ such that for any $Q \in \mathbb{R}_{sym,0}^{3 \times 3}$, it holds that*

$$\begin{aligned} \frac{1}{\varepsilon} \langle \partial_t(\mathbf{nn}) \cdot Q, Q \rangle &\leq \delta \left\| \frac{1}{\varepsilon} \mathcal{H}_n^\varepsilon(Q) \right\|_{L^2}^2 + C_\delta \left(\frac{1}{\varepsilon} \langle \mathcal{H}_n^\varepsilon(Q), Q \rangle + \|Q\|_{L^2}^2 \right), \\ \frac{1}{\varepsilon} \langle Q : \partial_t(\mathbf{nn}), Q : \mathbf{nn} \rangle &\leq \delta \left\| \frac{1}{\varepsilon} \mathcal{H}_n^\varepsilon(Q) \right\|_{L^2}^2 + C_\delta \left(\frac{1}{\varepsilon} \langle \mathcal{H}_n^\varepsilon(Q), Q \rangle + \|Q\|_{L^2}^2 \right). \end{aligned}$$

The proof of Lemma 5.6 can be found in [29].

5.5. Proof of Theorem 1.2. Given the initial data $(\mathbf{v}_0^\varepsilon, Q_0^\varepsilon) \in H^2 \times H^3$, thanks to Theorem 1.1 there exists a maximal time $T_\varepsilon > 0$ and a unique solution $(\mathbf{v}^\varepsilon, Q^\varepsilon)$ of the system (1.34)-(1.36) such that

$$\mathbf{v}^\varepsilon \in C([0, T_\varepsilon]; H^2) \cap L^2(0, T_\varepsilon; H^3), \quad Q^\varepsilon \in C([0, T_\varepsilon]; H^3).$$

Now we prove that $T_\varepsilon \geq T$. Suppose it is not. By Proposition 8, the solution has the expansion

$$\begin{aligned} \mathbf{v}^\varepsilon &= \mathbf{v}_0 + \varepsilon \mathbf{v}_1 + \varepsilon^2 \mathbf{v}_2 + \varepsilon^3 \mathbf{v}_R^\varepsilon, \\ Q^\varepsilon &= Q_0 + \varepsilon Q_1 + \varepsilon^2 Q_2 + \varepsilon^3 Q_3 + \varepsilon^3 Q_R^\varepsilon. \end{aligned}$$

For the remainder $(\mathbf{v}_R^\varepsilon, Q_R^\varepsilon)$, we infer from Proposition 9 that

$$\frac{d}{dt} \mathfrak{E}(t) + \mathfrak{F}(t) \leq C(\varepsilon \mathfrak{E})(1 + \mathfrak{E}) + \varepsilon f(\mathfrak{E}) + C(\varepsilon \mathfrak{E}) \varepsilon \mathfrak{F},$$

for any $t \in [0, T_\varepsilon]$. Thanks to the assumptions of Theorem 1.2, we know that

$$\mathfrak{E}(0) \leq C_1 \left(\|\mathbf{v}_{I,R}^\varepsilon\|_{H^2} + \|Q_{I,R}^\varepsilon\|_{H^3} + \varepsilon^{-1} \|\mathcal{P}^{out}(Q_{I,R}^\varepsilon)\|_{L^2} \right) \leq C_1 E_0.$$

Let $E_1 = (2 + C_1 E_0) e^T - 2 > \mathfrak{E}(0)$, and

$$T_1 = \sup\{t \in [0, T_\varepsilon] : \mathfrak{E}(t) \leq E_1\}.$$

Thus, if we take ε_0 small enough such that

$$C(\varepsilon_0 E_1) \leq 1, \quad \varepsilon_0 f(E_1) \leq 1, \quad \varepsilon_0 \leq 1/2,$$

then for $t \leq T_1$, it holds that

$$\frac{d}{dt} \mathfrak{E}(t) \leq 2 + \mathfrak{E}(t). \tag{5.67}$$

If $T_\varepsilon < T$, Gronwall's inequality gives that for $t \leq T_1$,

$$\mathfrak{E}(t) \leq e^t (2 + C_1 E_0) - 2 < E_1,$$

which implies $T_1 = T_\varepsilon$ and at time T_ε , $(\mathbf{v}_\varepsilon, Q_\varepsilon) \in H^2 \times H^3$, which is contradict with our assumption. Thus $T \leq T_\varepsilon$, and $\mathfrak{E}(t) \leq E_1$ for $t \in [0, T]$. Then Theorem 1.2 follows.

6. Appendix.

6.1. Some basic estimates in Sobolev spaces. The following product estimates and commutator estimates are well-known, see [25] for example, and frequently used in this paper.

Lemma 6.1. *Let $s \geq 0$. Then for any multi-index $\alpha, \beta, \gamma, \delta$, there holds*

$$\begin{aligned} \|\partial^\alpha f \partial^\beta g\|_{H^s} &\leq C(\|f\|_{L^\infty} \|g\|_{H^{s+|\alpha|+|\beta|}} + \|g\|_{L^\infty} \|f\|_{H^{s+|\alpha|+|\beta|}}); \\ \|\partial^\alpha f \partial^\beta g\|_{H^s} &\leq C\|f\|_{H^{s+|\alpha|+|\gamma|}} \|g\|_{H^{s+|\beta|+|\delta|}}, \quad \text{if } s + |\gamma| + |\delta| \geq 2. \end{aligned}$$

In particular, we have

$$\begin{aligned} \|fg\|_{H^s} &\leq C(\|f\|_{L^\infty} \|g\|_{H^s} + \|g\|_{L^\infty} \|f\|_{H^s}); \\ \|fg\|_{H^s} &\leq C\|f\|_{H^s} \|g\|_{H^s}, \quad \text{if } s \geq 2; \\ \|fg\|_{H^k} &\leq C \min\{\|f\|_{H^k} \|g\|_{H^2}, \|f\|_{H^2} \|g\|_{H^k}\}, \quad \text{if } 0 \leq k \leq 2. \end{aligned}$$

Lemma 6.2. *Let $s \geq 0$ and $F(\cdot) \in C^\infty(\mathbb{R}^d)$ with $F(0) = 0$. Then*

$$\|F(f)\|_{H^s} \leq C(\|f\|_{L^\infty}) \|f\|_{H^s}.$$

Lemma 6.3. *Let a be a multiple index. There holds*

$$\|[\partial^a, g]f\|_{L^2} \leq C(\|\nabla g\|_{L^\infty} \|f\|_{H^{|a|-1}} + \|\nabla g\|_{H^{|a|-1}} \|f\|_{L^\infty}).$$

In particular, if $|a| \geq 2$, it holds

$$\begin{aligned} \|[\partial^a, g]f\|_{L^2} &\leq C\|g\|_{H^{|a|+1}} \|f\|_{H^{|a|-1}}, \\ \|[\partial^{a+1}, g]f\|_{L^2} &\leq C\|g\|_{H^{|a|+1}} \|f\|_{H^{|a|}}. \end{aligned}$$

Lemma 6.4. *Let Ω be a convex domain in \mathbb{R}^d and $k \geq 0$ be an integer. $F(\cdot) \in C^\infty(\Omega)$ and $k' = \max\{k, 2\}$. Then*

$$\|F(u) - F(v)\|_{H^k} \leq C(\|u\|_{L^\infty}, \|v\|_{L^\infty})(1 + \|u\|_{H^{k'}} + \|v\|_{H^{k'}})\|u - v\|_{H^k}.$$

Proof. We may assume that $F'(0) = 0$, since if not, we can consider $G(u) = F(u) - u \cdot F'(0)$. By the fact that

$$F(u) - F(v) = (u - v) \cdot \int_0^1 F'(v + t(u - v)) dt,$$

we have

$$\begin{aligned} \|F(u) - F(v)\|_{L^2} &\leq \|u - v\|_{L^2} \sup_{t \in [0,1]} \|F'(v + t(u - v))\|_{L^\infty} \\ &\leq C(\|u\|_{L^\infty}, \|v\|_{L^\infty}) \|u - v\|_{L^2}, \\ \|\nabla(F(u) - F(v))\|_{L^2} &\leq \|\nabla(u - v)\|_{L^2} \sup_{t \in [0,1]} \|F'(v + t(u - v))\|_{L^\infty} \\ &\quad + \|u - v\|_{H^1} \sup_{t \in [0,1]} \|\nabla(F'(v + t(u - v)))\|_{H^1} \\ &\leq C(\|u\|_{L^\infty}, \|v\|_{L^\infty})(\|u\|_{H^2} + \|v\|_{H^2}) \|u - v\|_{H^1}, \end{aligned}$$

and for $k \geq 2$,

$$\begin{aligned} \|F(u) - F(v)\|_{H^k} &\leq C \left(\|u - v\|_{L^\infty} \sup_{t \in [0,1]} \|F'(v + t(u - v))\|_{H^k} \right. \\ &\quad \left. + \|u - v\|_{H^k} \sup_{t \in [0,1]} \|F'(v + t(u - v))\|_{L^\infty} \right) \\ &\leq C(\|u\|_{L^\infty}, \|v\|_{L^\infty})(1 + \|u\|_{H^k} + \|v\|_{H^k})\|u - v\|_{H^k}. \end{aligned}$$

Here, we have used the following estimate which is induced by Lemma 6.2:

$$\begin{aligned} \|F'(v + t(u - v))\|_{H^k} &\leq C(\|v + t(u - v)\|_{L^\infty})\|v + t(u - v)\|_{H^k} \\ &\leq C(\|u\|_{L^\infty}, \|v\|_{L^\infty})(\|u\|_{H^k} + \|v\|_{H^k}). \end{aligned}$$

This concludes the proof. \square

Acknowledgments. The authors would like to thank Professor Zhifei Zhang for his helpful discussion. W. Wang is supported by China Postdoctoral Science Foundation under Grant 2013M540010 and 2014T70008. P. Zhang is partly supported by NSF of China under Grant 21274005, 11421110001 and 114211101.

REFERENCES

- [1] H. Abels, G. Dolzmann and Y. Liu, [Well-posedness of a fully-coupled Navier-Stokes/ \$Q\$ -tensor system with inhomogeneous boundary data](#), *SIAM J. Math. Anal.*, **46** (2014), 3050–3077.
- [2] H. Abels, G. Dolzmann and Y. Liu, Strong solutions for the Beris-Edwards model for nematic liquid crystals with homogeneous Dirichlet boundary conditions, preprint, [arXiv:1312.5988](#).
- [3] H. Bahouri, J.-Y. Chemin and R. Danchin, [Fourier Analysis and Nonlinear Partial Differential Equations](#), Fundamental Principles of Mathematical Sciences, 343, Springer, Heidelberg, 2011.
- [4] J. M. Ball and A. Majumdar, [Nematic liquid crystals: From Maier-Saupe to a continuum theory](#), *Mol. Cryst. Liq. Cryst.*, **525** (2010), 1–11.
- [5] A. N. Beris and B. J. Edwards, *Thermodynamics of Flowing Systems with Internal Microstructure*, Oxford Engrg. Sci. Ser., 36, Oxford University Press, Oxford, New York, 1994.
- [6] P. G. De Gennes, *The Physics of Liquid Crystals*, Clarendon Press, Oxford, 1974.
- [7] W. E and P. Zhang, [A molecular kinetic theory of inhomogeneous liquid crystal flow and the small Deborah number limit](#), *Methods Appl. Anal.*, **13** (2006), 181–198.
- [8] J. Ericksen, [Conservation laws for liquid crystals](#), *Trans. Soc. Rheol.*, **5** (1961), 23–34.
- [9] I. Fatkullin and V. Slastikov, [Critical points of the Onsager functional on a sphere](#), *Nonlinearity*, **18** (2005), 2565–2580.
- [10] J. Feng, C. V. Chaubal and L. G. Leal, [Closure approximations for the Doi theory: Which to use in simulating complex flows of liquid-crystalline polymers?](#), *J. Rheol.*, **42** (1998), 1095–1119.
- [11] J. Feng, G. Sgalari and L. G. Leal, [A theory for flowing nematic polymers with orientational distortion](#), *J. Rheol.*, **44** (2000), 1085–1101.
- [12] J. Han, Y. Luo, W. Wang, P. Zhang and Z. Zhang, [From microscopic theory to macroscopic theory: A systematic study on modeling for liquid crystals](#), *Arch. Ration. Mech. Anal.*, **215** (2015), 741–809.
- [13] J. Huang and S. Ding, [Global well-posedness for a coupled incompressible Navier-Stokes and \$Q\$ -tensor system](#), preprint, [arXiv:1405.1863](#).
- [14] J. Huang, F. H. Lin and C. Wang, [Regularity and existence of global solutions to the Ericksen-Leslie system in \$\mathbb{R}^2\$](#) , *Commun. Math. Phys.*, **331** (2014), 805–850.
- [15] N. Kuzuu and M. Doi, [Constitutive equation for nematic liquid crystals under weak velocity gradient derived from a molecular kinetic equation](#), *J. Phys. Soc. Japan*, 52 (1983), 3486–3494.
- [16] F. M. Leslie, [Some constitutive equations for liquid crystals](#), *Arch. Ration. Mech. Anal.*, **28** (1968), 265–283.
- [17] F.-H. Lin and C. Liu, [Existence of solutions for the Ericksen-Leslie system](#), *Arch. Ration. Mech. Anal.*, **154** (2000), 135–156.

- [18] H. Liu, H. Zhang and P. Zhang, [Axial symmetry and classification of stationary solutions of Doi-Onsager equation on the sphere with Maier-Saupe potential](#), *Comm. Math. Sci.*, **3** (2005), 201–218.
- [19] A. Majumdar and A. Zarnescu, [Landau-De Gennes theory of nematic liquid crystals: The Oseen-Frank limit and beyond](#), *Arch. Ration. Mech. Anal.*, **196** (2010), 227–280.
- [20] N. J. Mottram and C. J. P. Newton, Introduction to Q -tensor theory, preprint, [arXiv:1409.3542](#).
- [21] M. Paicu and A. Zarnescu, [Energy dissipation and regularity for a coupled Navier-Stokes and \$Q\$ -tensor system](#), *Arch. Ration. Mech. Anal.*, **203** (2012), 45–67.
- [22] M. Paicu and A. Zarnescu, [Global existence and regularity for the full coupled Navier-Stokes and \$Q\$ -tensor system](#), *SIAM J. Math. Anal.*, **43** (2011), 2009–2049.
- [23] O. Parodi, [Stress tensor for a nematic liquid crystal](#), *J. Phys. France*, **31** (1970), 581–584.
- [24] T. Qian and P. Sheng, [Generalized hydrodynamic equations for nematic liquid crystals](#), *Phys. Rev. E*, **58** (1998), 7475–7485.
- [25] H. Triebel, [Theory of Function Spaces](#), Monographs in Mathematics, Birkhäuser Verlag, Basel, Boston, 1983.
- [26] M. Wang and W. Wang, [Global existence of weak solution for the 2-D Ericksen-Leslie system](#), *Calc. Var. Partial Differ. Equ.*, **51** (2014), 915–962.
- [27] W. Wang, P. Zhang and Z. Zhang, [The small Deborah number limit of the Doi-Onsager equation to the Ericksen-Leslie equation](#), *Comm. Pure Appl. Math.*, **68** (2015), 1326–1398.
- [28] W. Wang, P. Zhang and Z. Zhang, [Well-posedness of the Ericksen-Leslie system](#), *Arch. Ration. Mech. Anal.*, **210** (2013), 837–855.
- [29] W. Wang, P. Zhang and Z. Zhang, [Rigorous derivation from Landau-de Gennes theory to Ericksen-Leslie theory](#), *SIAM J. Math. Anal.*, **47** (2015), 127–158.

Received October 2014; revised January 2015.

E-mail address: srli@pku.edu.cn

E-mail address: wangw07@zju.edu.cn, wangw07@pku.edu.cn

E-mail address: pzhang@pku.edu.cn