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RIGOROUS JUSTIFICATION OF THE UNIAXIAL LIMIT FROM THE QIAN–SHENG INERTIAL Q-TENSOR THEORY TO THE ERICKSEN–LESLIE THEORY*

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Abstract. In this paper, we rigorously justify the connection between the Qian–Sheng inertial Q-tensor model and the full Ericksen–Leslie model for the liquid crystal flow. By using the Hilbert expansion method, we prove that when the elastic coefficients tend to zero (also called the uniaxial limit), the solution to the Qian–Sheng inertial model will converge to the solution to the full inertial Ericksen–Leslie system.

Key words. liquid crystals, Ericksen–Leslie model, Qian–Sheng model, Q-tensor theory

AMS subject classifications. Primary, 35Q35; Secondary, 35Q30, 76D05

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1. Introduction.

Liquid crystals are a state of matter with physical properties between liquid and solid, in which molecules tend to align in a preferred direction. In the nematic liquid phase, the molecules exhibit long-range orientational order but no positional order. In physics, different order parameters are introduced to characterize the anisotropic behavior of liquid crystals, which lead to different models. There are three kinds of widely accepted theories to model nematic liquid crystal flows: *the Ericksen–Leslie theory*, *the Landau–de Gennes theory*, and *the Doi–Onsager theory*. The first two are macroscopic theories based on continuum mechanics, while the latter is a microscopic kinetic theory derived from the viewpoint of statistical mechanics. As they are derived from different considerations and are widely used in liquid crystal studies, to explore the connection between different theories is an important problem. In this paper, we aim to study the rigorous connection between the Ericksen–Leslie model and the Qian–Sheng model—a representative model in the Landau–de Gennes framework.

Before introducing the Ericksen–Leslie model and the Qian–Sheng model, we list some notation and conventions. Throughout this paper, the Einstein summation convention is utilized. The space of symmetric traceless tensors is defined as

$$\mathbb{S}_0^3 \stackrel{\text{def}}{=} \left\{ Q \in \mathbb{R}^{3\times 3}:\ Q_{ij} = Q_{ji},\ Q_{ii} = 0 \right\},$$

which is endowed with the inner product $Q_1 : Q_2 = Q_{1ij}Q_{2ij}$. The set $\mathbb{S}_0^3$ is a five-dimensional subspace of $\mathbb{R}^{3\times 3}$. The matrix norm on $\mathbb{S}_0^3$ is defined as $|Q| \stackrel{\text{def}}{=} \sqrt{\text{Tr}Q^2} = \sqrt{Q_{ij}Q_{ij}}$. For two tensors $A, B \in \mathbb{S}_0^3$ we denote $(A \cdot B)_{ij} = A_{ik}B_{kj}$ and $A : B =$

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$A_{ij}B_{ij}$, and their commutator $[A,B] = A\cdot B - B\cdot A$. For any $Q_1, Q_2 \in L^2(\mathbb{R}^3)^{3\times 3}$, the corresponding inner product is defined by

$$\langle Q_1, Q_2\rangle \overset{\text{def}}{=} \int_{\mathbb{R}^3} Q_{1ij}(\mathbf{x}) : Q_{2ij}(\mathbf{x})\mathrm{d}\mathbf{x}.$$

We use $f_{,i}$ to denote $\partial_i f$ and $\mathbf{I}$ to denote the 3×3 identity tensor. In addition, the superscripted dot denotes the material derivative, i.e., $\dot{f} = (\partial_t + \mathbf{v}\cdot\nabla)f$, where the fluid velocity $\mathbf{v}$ can be understood from the context.

In this paper, we denote by $\mathbf{n}_1 \otimes \mathbf{n}_2$ the tensor product of two vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ and usually omit the symbol $\otimes$ for simplicity.

1.1. Ericksen–Leslie theory. The hydrodynamic theory of nematic liquid crystals was initiated in the seminal work of Ericksen [9] and Leslie [20] in the 1960s. In this theory, the local state of molecular alignments is described by a unit vector $\mathbf{n} \in \mathbb{S}^2$, called the director. The corresponding total free energy, called the Oseen–Frank energy, is given by

$$\begin{aligned} E_F(\mathbf{n},\nabla\mathbf{n}) = \frac{k_1}{2}(\nabla\cdot\mathbf{n})^2 + \frac{k_2}{2}(\mathbf{n}\cdot(\nabla\times\mathbf{n}))^2 + \frac{k_3}{2}|\mathbf{n}\times(\nabla\times\mathbf{n})|^2 \\ + \frac{k_2+k_4}{2}\big(\mathrm{tr}(\nabla\mathbf{n})^2 - (\nabla\cdot\mathbf{n})^2\big), \end{aligned} \tag{1.1}$$

where $k_1, \dots, k_4$ are constants depending on the material and the temperature.

The full inertial Ericksen–Leslie system can be given as follows:

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \nabla\cdot\sigma, \tag{1.2}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{1.3}$$

$$\mathbf{n}\times\big(I\ddot{\mathbf{n}} - \mathbf{h} + \gamma_1\mathbf{N} + \gamma_2\mathbf{D}\cdot\mathbf{n}\big) = 0, \tag{1.4}$$

where $\mathbf{v}$ is the fluid velocity, p is the pressure penalizing the incompressible condition (1.3) of $\mathbf{v}$, and I is *the moment of inertial density* usually considered as a small parameter. The inertial term $\ddot{\mathbf{n}}$ is the material derivative of $\dot{\mathbf{n}}$. Equations (1.2) and (1.4) reflect the conservation laws of linear momentum and angular momentum, respectively. The stress tensor σ consists of the viscous (Leslie) stress σ^L and the elastic (Ericksen) stress σ^E, i.e., $\sigma = \sigma^L + \sigma^E$, which can be described by the following phenomenological constitutive relations:

$$\sigma^L = \alpha_1(\mathbf{nn}:\mathbf{D})\mathbf{nn} + \alpha_2\mathbf{nN} + \alpha_3\mathbf{Nn} + \alpha_4\mathbf{D} + \alpha_5\mathbf{nn}\cdot\mathbf{D} + \alpha_6\mathbf{D}\cdot\mathbf{nn}, \tag{1.5}$$

$$\sigma^E = -\frac{\partial E_F}{\partial(\nabla\mathbf{n})}\cdot(\nabla\mathbf{n})^T, \tag{1.6}$$

where

$$\mathbf{D} = \frac{1}{2}\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right), \quad \mathbf{\Omega} = \frac{1}{2}\left(\nabla\mathbf{v} - (\nabla\mathbf{v})^T\right), \quad \mathbf{N} = \dot{\mathbf{n}} - \mathbf{\Omega}\cdot\mathbf{n}.$$

Additionally, the molecular field $\mathbf{h}$ is given by

$$\mathbf{h} = -\frac{\delta E_F}{\delta\mathbf{n}} = -\frac{\partial E_F}{\partial\mathbf{n}} + \nabla\cdot\frac{\partial E_F}{\partial(\nabla\mathbf{n})}.$$

The six constants $\alpha_1, \dots, \alpha_6$ in (1.5) are called the Leslie viscosity coefficients. They and the coefficients γ_1, γ_2 together satisfy the following relations:

$$\alpha_2 + \alpha_3 = \alpha_6 - \alpha_5, \tag{1.7}$$

$$\gamma_1 = \alpha_3 - \alpha_2, \quad \gamma_2 = \alpha_6 - \alpha_5. \tag{1.8}$$

The equality (1.7) is referred to as *Parodi's relation* derived from the Onsager reciprocal relation of irreversible thermodynamics. The relations (1.7)–(1.8) will guarantee that the full Ericksen–Leslie system (1.2)–(1.4) fulfills the energy dissipation law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left(\frac{1}{2}|\mathbf{v}|^2+\frac{I}{2}|\dot{\mathbf{n}}|^2+E_F\right)\mathrm{d}\mathbf{x}=-\int_{\mathbb{R}^3}\left(\left(\alpha_1+\frac{\gamma_2^2}{\gamma_1}\right)(\mathbf{D}:\mathbf{nn})^2+\alpha_4|\mathbf{D}|^2\right.$$
$$\left.+\left(\alpha_5+\alpha_6-\frac{\gamma_2^2}{\gamma_1}\right)|\mathbf{D}\cdot\mathbf{n}|^2+\frac{1}{\gamma_1}\big|\mathbf{n}\times(\mathbf{h}-I\ddot{\mathbf{n}})\big|^2\right)\mathrm{d}\mathbf{x}. \tag{1.9}$$

It is worth emphasizing that the inertial term I in (1.4) is responsible for the hyperbolic feature of the equation describing the molecular orientation. If the inertial term is neglected, then the system (1.2)–(1.4) is immediately transformed into its noninertial counterpart which is a parabolic-type system.

Concerning the noninertial version of the Ericksen–Leslie theory, the well-posedness results are addressed [22, 23, 33] and the references therein. In particular, under a natural physical condition on the Leslie coefficients, Wang, Zhang, and Zhang [33] proved the well-posedness of the system, and the global existence of a weak solution in the two-dimensional case was shown in [14, 32]. Lin and Wang [24] proved the global existence of a weak solution for the three-dimensional case with the initial director field lying in the upper hemisphere. For more related works on the noninertial Ericksen–Leslie system, for instance, see [23, 36, 8] and the references therein.

On the other hand, there are also some analytical works devoted to the original inertial Ericksen–Leslie system. Very recently, Jiang and Luo [16] established the well-posedness for the full inertial Ericksen–Leslie system in the context of classical solutions. Cai and Wang [4] studied the global well-posedness of classical solutions to the inertial Ericksen–Leslie model with positive γ_1.

1.2. Landau–de Gennes theory. Landau–de Gennes theory [6] is capable of providing a rather comprehensive description of the local behavior of the medium, since it accounts for more complex phenomena of liquid crystals, such as line defects and biaxial configurations. This theory employs a symmetric and traceless tensorial order parameter $Q(\mathbf{x})$ to characterize the alignment behavior of molecular orientations. Physically, $Q(\mathbf{x})$ could be understood as the second order traceless moment of f:

$$Q(\mathbf{x})=\int_{\mathbb{S}^2}\left(\mathbf{mm}-\frac{1}{3}\mathbf{I}\right)f(\mathbf{x},\mathbf{m})\mathrm{d}\mathbf{m},$$

where $f(\mathbf{x},\mathbf{m})$ represents the microscopic distribution of molecules with the orientation parallel to $\mathbf{m}$ at material point $\mathbf{x}$. The tensor $Q(\mathbf{x})$ is said to be *isotropic* if all its eigenvalues are zero, *uniaxial* if it has two equal nonzero eigenvalues, and *biaxial* if its three eigenvalues are distinct.

In the absence of boundary constraint and external field, the Landau–de Gennes free energy is given as follows:

$$\mathcal{F}(Q,\nabla Q)=\int_{\mathbb{R}^3}\left\{-\frac{a}{2}\mathrm{Tr}(Q^2)-\frac{b}{3}\mathrm{Tr}(Q^3)+\frac{c}{4}(\mathrm{Tr}(Q^2))^2\right.$$
$$\left.+\frac{1}{2}\Big(L_1|\nabla Q|^2+L_2Q_{ij,j}Q_{ik,k}+L_3Q_{ij,k}Q_{ik,j}\Big)\right\}\mathrm{d}\mathbf{x}$$
$$\stackrel{\mathrm{def}}{=}\int_{\mathbb{R}^3}\Big(f_b(Q)+f_e(\nabla Q)\Big)\mathrm{d}\mathbf{x}, \tag{1.10}$$

where a, b, c are nonnegative parameters which may depend on the material and temperature, and $L_i (i = 1, 2, 3)$ are material-dependent elastic constants. f_b is the bulk energy density describing the isotropic-nematic phase transition, while the elastic energy density f_e penalizes spatial nonhomogeneities. The interested reader is referred to [6, 28] for detailed introductions.

Up to now, some dynamic Q-tensor theories have been established to model nematic liquid crystal flows, which are either derived from the molecular kinetic theory by closure approximations such as [11, 12] or directly obtained by a variational method such as the Beris–Edwards model [3] and the Qian–Sheng model [31]. The well-posedness results of the Beris–Edwards system on whole space and bounded domain can be found in [30, 29, 13] and [1, 2, 25], respectively. For the inertial Qian–Sheng model, De Anna and Zarnescu [5] studied the local well-posedness for bounded initial data and global well-posedness under the assumptions of the small initial data. For the nonviscous version of the Qian–Sheng model, Feireisl et al. [10] proved a global existence of the dissipative solution which is inspired by that of incompressible Euler equations.

The Qian–Sheng model [31] is a hydrodynamical model which reads as

$$J\ddot{Q} + \mu_1\dot{Q} = \mathbf{H} - \frac{\mu_2}{2}\mathbf{D} + \mu_1[\mathbf{\Omega}, Q], \tag{1.11}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \nabla\cdot\big(\sigma + \sigma^d\big), \tag{1.12}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{1.13}$$

where $\dot{Q} = (\partial_t + \mathbf{v}\cdot\nabla)Q$, $\ddot{Q} = (\partial_t + \mathbf{v}\cdot\nabla)\dot{Q}$, and the viscous stress σ, the distortion stress σ^d, and the molecular field $\mathbf{H}$ are respectively given by

$$\begin{aligned}\sigma &= \beta_1 Q(Q:\mathbf{D}) + \beta_4\mathbf{D} + \beta_5\mathbf{D}\cdot Q + \beta_6 Q\cdot\mathbf{D} + \beta_7(\mathbf{D}\cdot Q^2 + Q^2\cdot\mathbf{D})\\ &\quad + \frac{\mu_2}{2}(\dot{Q} - [\mathbf{\Omega}, Q]) + \mu_1\big[Q, (\dot{Q} - [\mathbf{\Omega}, Q])\big],\end{aligned} \tag{1.14}$$

$$\sigma^d_{ij} = -\frac{\partial\mathcal{F}}{\partial Q_{kl,j}}\partial_i Q_{kl},$$

$$\mathbf{H}_{ij} = -\left(\frac{\delta\mathcal{F}(Q,\nabla Q)}{\delta Q}\right)_{ij} = -\frac{\partial\mathcal{F}}{\partial Q_{ij}} + \partial_k\left(\frac{\partial\mathcal{F}}{\partial Q_{ij,k}}\right). \tag{1.15}$$

Moreover, in (1.11), J stands for the *inertial density* which is usually small. The viscosity coefficients $\beta_1, \beta_4, \beta_5, \beta_6, \beta_7, \mu_1$, and μ_2 in (1.14)–(1.15) can be linked by the following relation:

$$\beta_6 - \beta_5 = \mu_2. \tag{1.16}$$

The system (1.11)–(1.13) possesses an energy dissipation law; see (A.1) in the appendix. Here we remark that, comparing with the original Qian–Sheng model in [31], we add a new viscosity term $\beta_7(D_{ik}Q_{kl}Q_{lj} + Q_{ik}Q_{kl}D_{lj})$ in (1.14) to ensure that the energy of the system will always dissipate without assuming any relation between β_5 and β_6. Indeed, if $\beta_7 = 0$, we have to assume $\beta_5 + \beta_6 = 0$; otherwise the energy may not dissipate (see Lemma A.1). However, the condition $\beta_5 + \beta_6 = 0$ is so strong that it cannot be satisfied by many liquid crystal materials. Therefore, we introduce the β_7 term and assume that

$$
(1.17)\qquad \begin{cases} \beta_1, \beta_4, \mu_1 > 0,\ \beta_4 - \dfrac{\mu_2^2}{4\mu_1} > 0,\ \beta_7 \geq 0; \\ (\beta_5+\beta_6)^2 < 8\beta_7\left(\beta_4 - \dfrac{\mu_2^2}{4\mu_1}\right) \text{ if } \beta_7 \neq 0; \beta_5+\beta_6 = 0 \text{ if } \beta_7 = 0. \end{cases}
$$

The detailed discussion of the dissipative relation is postponed to Lemma A.1 in the appendix.

1.3. Motivations and main results. The intricate connection between different dynamical theories for liquid crystals is not only of significance in mathematical literature but also directly related to many physical properties. The fundamental subject, generally involving the singular limit problem, has drawn a lot of attention in the physics and applied mathematics communities. In this respect, the formal asymptotic expansions were first constructed by Kuzzu and Doi [19] to derive the homogeneous noninertial Ericksen–Leslie system from the Doi–Onsager system and to determine the Leslie coefficients, under the small Deborah number limit. E and Zhang [7] extended the Kuzzu–Doi derivation and obtained the inhomogeneous noninertial Ericksen–Leslie system. In particular, the Ericksen stress is derived from a body force. Their formal derivation was rigorously justified by Wang, Zhang, and Zhang [35] under the small Deborah number limit. Based on the same spirit, Li, Wang, and Zhang [21] provided a strict derivation from the molecular-based Q-tensor system, obtained from the molecular kinetic theory by the Bingham closure, to the noninertial Ericksen–Leslie system. Similar rigorous results were initiated by Wang, Zhang, and Zhang in [34] concerning the Beris–Edwards system in a Landau–de Gennes framework. A unified formulation for liquid crystal modeling was put forward by Han et al. in [12] to establish relations between microscopic theories and macroscopic theories. There are also some interesting works which have explored the relations between different dynamical theories for liquid crystals in the framework of weak solutions; see [26].

Recently, to better understand the limit of zero inertia for the full Ericksen–Leslie model, Jiang et al. [18] first studied a limit connecting a scaled wave map with heat flow into the unit sphere $\mathbb{S}^2$. Later on, Jiang and others [15, 17] investigated the zero inertial limit from the full inertial Ericksen–Leslie model to the noninertial one.

The main goal of this paper is to rigorously justify the connection between the inertial Qian–Sheng model and the full inertial Ericksen–Leslie model, in a sense of smooth solutions. Our methods and results can be seen as an extension of the work initiated by Wang, Zhang, and Zhang in [34], who proved the uniaxial limit for the noninertial Beris–Edwards model. The main framework of our proof follows the strategy in [34], that is, constructing approximated solutions (near the inertial Ericksen–Leslie solutions) by using the Hilbert expansion for the inertial Qian–Sheng model, and then deriving the uniform estimates for the difference between true solutions and approximations. Some new and essential obstacles appear due to the nonlinear hyperbolic structure of the Q-tensor equation. Roughly speaking, in this case, the dissipation part of the energy is not strong enough to estimate the singular terms, and in addition, the nonlinear inertial term will bring some extra terms with high order derivatives. To overcome these difficulties, the energy has to be delicately modified such that these singular or high order terms can be absorbed. A more detailed discussion for the main challenges and the novelty of our work is presented at the end of Theorem 1.1.

In contrast to the constants a, b, c, the elastic coefficients $L_i (i = 1, 2, 3)$ in (1.10) are usually regarded as being small, so we consider the following rescaled energy functional with a small parameter ε:

$$\mathcal{F}_\varepsilon(Q,\nabla Q)=\int_{\mathbb{R}^3}\left(\frac{1}{\varepsilon}f_b(Q)+f_e(\nabla Q)\right)\mathrm{d}\mathbf{x}, \tag{1.18}$$

and $a,b,c,L_i(1\le i\le 3)\sim O(1)$. We assume that the elastic coefficients L_i satisfy

$$L_1>0,\ L_1+L_2+L_3>0,$$

which will ensure that the elastic energy is strictly positive (see Lemma 2.5 in [34]), i.e., there exists some constant $L_0=L_0(L_1,L_2,L_3)>0$ such that

$$\int_{\mathbb{R}^3} f_e(\nabla Q)\mathrm{d}\mathbf{x}\ge L_0\|\nabla Q\|_{L^2}. \tag{1.19}$$

Then the Qian–Sheng model with a small parameter ε can be written as

$$J\ddot{Q}^\varepsilon+\mu_1\dot{Q}^\varepsilon=\mathbf{H}^\varepsilon-\frac{\mu_2}{2}\mathbf{D}^\varepsilon+\mu_1[\mathbf{\Omega}^\varepsilon,Q^\varepsilon], \tag{1.20}$$

$$\frac{\partial\mathbf{v}^\varepsilon}{\partial t}+\mathbf{v}^\varepsilon\cdot\nabla\mathbf{v}^\varepsilon=-\nabla p^\varepsilon+\nabla\cdot\left(\sigma_\varepsilon+\sigma_\varepsilon^d\right), \tag{1.21}$$

$$\nabla\cdot\mathbf{v}^\varepsilon=0, \tag{1.22}$$

where $\dot{Q}^\varepsilon=(\partial_t+\mathbf{v}^\varepsilon\cdot\nabla)Q^\varepsilon,\ \ddot{Q}^\varepsilon=(\partial_t+\mathbf{v}^\varepsilon\cdot\nabla)\dot{Q}^\varepsilon$, and

$$\begin{aligned}
&\mathbf{D}^\varepsilon=\frac{1}{2}\left(\nabla\mathbf{v}^\varepsilon+(\nabla\mathbf{v}^\varepsilon)^T\right),\ \mathbf{\Omega}^\varepsilon=\frac{1}{2}\left(\nabla\mathbf{v}^\varepsilon-(\nabla\mathbf{v}^\varepsilon)^T\right),\\
&\sigma_\varepsilon=\beta_1Q^\varepsilon(Q^\varepsilon:\mathbf{D}^\varepsilon)+\beta_4\mathbf{D}^\varepsilon+\beta_5\mathbf{D}^\varepsilon\cdot Q^\varepsilon+\beta_6Q^\varepsilon\cdot\mathbf{D}^\varepsilon+\beta_7(\mathbf{D}^\varepsilon\cdot Q^{\varepsilon 2}+Q^{\varepsilon 2}\cdot\mathbf{D}^\varepsilon)\\
&\qquad+\frac{\mu_2}{2}(\dot{Q}^\varepsilon-[\mathbf{\Omega}^\varepsilon,Q^\varepsilon])+\mu_1\left[Q^\varepsilon,(\dot{Q}^\varepsilon-[\mathbf{\Omega}^\varepsilon,Q^\varepsilon])\right],\\
&(\sigma^d_\varepsilon)_{ji}=-\frac{\partial\mathcal{F}_\varepsilon}{\partial Q^\varepsilon_{kl,j}}Q^\varepsilon_{kl,i}\stackrel{\text{def}}{=}\sigma^d(Q^\varepsilon,Q^\varepsilon).
\end{aligned}$$

The tensor $\sigma^d(Q,\overline{Q})$ is denoted as

$$\sigma^d_{ji}(Q,\overline{Q})=-\left(L_1Q_{kl,j}\overline{Q}_{kl,i}+L_2Q_{km,m}\overline{Q}_{kj,i}+L_3Q_{kj,l}\overline{Q}_{kl,i}\right).$$

The molecular field $\mathbf{H}^\varepsilon$ is given by

$$\mathbf{H}^\varepsilon(Q)=-\frac{1}{\varepsilon}\frac{\partial f_b}{\partial Q}+\partial_i\left(\frac{\partial f_e}{\partial Q_{,i}}\right)\stackrel{\text{def}}{=}-\frac{1}{\varepsilon}\mathscr{T}(Q)-\mathscr{L}(Q),$$

where two operators $\mathscr{T}$ and $\mathscr{L}$ are respectively defined by

$$\begin{aligned}
\mathscr{T}(Q)&=-aQ-bQ^2+c|Q|^2Q+\frac{1}{3}b|Q|^2\mathbf{I},\\
(\mathscr{L}(Q))_{kl}&=-\left(L_1\Delta Q_{kl}+\frac{1}{2}(L_2+L_3)\left(Q_{km,ml}+Q_{lm,mk}-\frac{2}{3}\delta_{kl}Q_{ij,ij}\right)\right).
\end{aligned}$$

For a given director field $\mathbf{n}(t,\mathbf{x})$ and $s=\frac{b+\sqrt{b^2+24ac}}{4c}$, we define

$$\begin{aligned}
&\mathscr{P}^{out}(Q)=Q-(\mathbf{nn}\cdot Q+Q\cdot\mathbf{nn})+2(Q:\mathbf{nn})\mathbf{nn},\\
&\mathscr{H}_\mathbf{n}(Q)=bs\left(Q-(\mathbf{nn}\cdot Q+Q\cdot\mathbf{nn})+\frac{2}{3}(Q:\mathbf{nn})\mathbf{I}\right)+2cs^2(Q:\mathbf{nn})\left(\mathbf{nn}-\frac{1}{3}\mathbf{I}\right),
\end{aligned}$$

which will be explained in subsection 2.1. We also take the viscosity coefficients in the full inertial Ericksen–Leslie model as

$$\begin{cases} \alpha_1 = \beta_1 s^2, \quad \alpha_2 = \dfrac{1}{2}\mu_2 s - \mu_1 s^2, \\ \alpha_3 = \dfrac{1}{2}\mu_2 s + \mu_1 s^2, \quad \alpha_4 = \beta_4 - \dfrac{1}{3}(\beta_5 + \beta_6)s + \dfrac{2}{9}\beta_7 s^2, \\ \alpha_5 = \beta_5 s + \dfrac{1}{3}\beta_7 s^2, \quad \alpha_6 = \beta_6 s + \dfrac{1}{3}\beta_7 s^2, \end{cases} \tag{1.23}$$

and the coefficients γ_1, γ_2 and the inertial coefficient I are

$$\gamma_1 = 2\mu_1 s^2, \quad \gamma_2 = \mu_2 s, \quad I = 2Js^2. \tag{1.24}$$

In addition, the elastic constants in the Oseen–Frank energy are given by

$$k_1 = k_3 = 2(L_1 + L_2 + L_3)s^2, \quad k_2 = 2L_1 s^2, \quad k_4 = L_3 s^2. \tag{1.25}$$

Throughout this paper, we assume that the viscosity coefficient μ_1 is large enough compared with the inertial coefficient J, i.e., $\mu_1 \gg J$, and the condition (1.17) holds, and the elastic constants $L_i (i = 1, 2, 3)$ satisfy $L_1 > 0,\ L_1 + L_2 + L_3 > 0$. The main result of this paper is stated as follows.

THEOREM 1.1. *Let* $(\mathbf{n}(t, \mathbf{x}), \mathbf{v}(t, \mathbf{x}))$ *be a smooth solution of the full inertial Ericksen–Leslie model* (1.2)–(1.4) *on* $[0, T]$ *with the coefficients given by* (1.23)–(1.25), *which satisfies*

$$(\mathbf{v}, \partial_t \mathbf{n}, \nabla \mathbf{n}) \in L^\infty([0, T]; H^k) \quad \textit{for} \quad k \geq 20.$$

Let $Q_0(t, \mathbf{x}) = s\big(\mathbf{n}(t, \mathbf{x})\mathbf{n}(t, \mathbf{x}) - \frac{1}{3}\mathbf{I}\big)$, *and the functions* $(Q_1, Q_2, Q_3, \mathbf{v}_1, \mathbf{v}_2)$ *are determined by Proposition* 2.4. *Suppose that the initial data* $(Q^\varepsilon(0, \mathbf{x}), \partial_t Q^\varepsilon(0, \mathbf{x}), \mathbf{v}^\varepsilon(0, \mathbf{x}))$ *takes the form*

$$Q^\varepsilon(0, \mathbf{x}) = \sum_{k=0}^{3} \varepsilon^k Q_k(0, \mathbf{x}) + \varepsilon^3 Q^\varepsilon_{R,0}(\mathbf{x}), \quad \mathbf{v}^\varepsilon(0, \mathbf{x}) = \sum_{k=0}^{2} \varepsilon^k \mathbf{v}_k(0, \mathbf{x}) + \varepsilon^3 \mathbf{v}^\varepsilon_{R,0}(\mathbf{x}),$$

$$\partial_t Q^\varepsilon(0, \mathbf{x}) = \sum_{k=0}^{3} \varepsilon^k \partial_t Q_k(0, \mathbf{x}) + \varepsilon^3 \partial_t Q^\varepsilon_{R,0}(\mathbf{x}),$$

where $(Q^\varepsilon_{R,0}(\mathbf{x}), \partial_t Q^\varepsilon_{R,0}(\mathbf{x}), \mathbf{v}^\varepsilon_{R,0}(\mathbf{x}))$ *fulfills*

$$\|\mathbf{v}^\varepsilon_{R,0}\|_{H^2} + \|Q^\varepsilon_{R,0}\|_{H^3} + \|\partial_t Q^\varepsilon_{R,0}\|_{H^2} + \varepsilon^{-1}\|\mathscr{P}^{out}(Q^\varepsilon_{R,0})\|_{L^2} \leq E_0.$$

Then there exists $\varepsilon_0 > 0$ *and* $E_1 > 0$ *such that for all* $\varepsilon < \varepsilon_0$, *the inertial Qian–Sheng model* (1.20)–(1.22) *has a unique solution* $(Q^\varepsilon(t, \mathbf{x}), \mathbf{v}^\varepsilon(t, \mathbf{x}))$ *on* $[0, T]$ *that has the Hilbert expansion*

$$Q^\varepsilon(t, \mathbf{x}) = \sum_{k=0}^{3} \varepsilon^k Q_k(t, \mathbf{x}) + \varepsilon^3 Q^\varepsilon_R(t, \mathbf{x}), \quad \mathbf{v}^\varepsilon(t, \mathbf{x}) = \sum_{k=0}^{2} \varepsilon^k \mathbf{v}_k(t, \mathbf{x}) + \varepsilon^3 \mathbf{v}^\varepsilon_R(t, \mathbf{x}),$$

where, for any $t \in [0, T]$, $(Q^\varepsilon_R, \mathbf{v}^\varepsilon_R)$ *satisfies*

$$\mathfrak{E}\left(Q^\varepsilon_R(t), \mathbf{v}^\varepsilon_R(t)\right) \leq E_1.$$

Here $\mathfrak{E}$ is defined by

$$\mathfrak{E}(Q,\mathbf{v}) \overset{def}{=} \int_{\mathbb{R}^3} \left(|\mathbf{v}|^2 + |Q|^2 + |\dot{Q}|^2 + \frac{1}{\varepsilon}\mathscr{H}^\varepsilon_{\mathbf{n}}(Q) : Q \right) + \varepsilon^2 \left(|\nabla\mathbf{v}|^2 + |\partial_i \dot{Q}|^2 + \frac{1}{\varepsilon}\mathscr{H}^\varepsilon_{\mathbf{n}}(\partial_i Q) : \partial_i Q \right) + \varepsilon^4 \left(|\Delta\mathbf{v}|^2 + |\Delta\dot{Q}|^2 + \frac{1}{\varepsilon}\mathscr{H}^\varepsilon_{\mathbf{n}}(\Delta Q) : \Delta Q \right), \tag{1.26}$$

and $\dot{Q} = (\partial_t + \tilde{\mathbf{v}}\cdot\nabla)Q$, $\tilde{\mathbf{v}} = \sum_{k=0}^2 \varepsilon^k \mathbf{v}_k$, $\mathscr{H}^\varepsilon_{\mathbf{n}}(Q) = \mathscr{H}_{\mathbf{n}}(Q) + \varepsilon\mathscr{L}(Q)$ *in* (1.26) *and the constant* E_1 *is independent of* ε.

Let us say a few words on the rough idea of proving the main result. We first make a formal expansion for the solution $(Q^\varepsilon, \mathbf{v}^\varepsilon)$:

$$\begin{aligned} Q^\varepsilon(t,\mathbf{x}) &= Q_0(t,\mathbf{x}) + \varepsilon Q_1(t,\mathbf{x}) + \varepsilon^2 Q_2(t,\mathbf{x}) + \varepsilon^3 Q_3(t,\mathbf{x}) + \varepsilon^3 Q_R(t,\mathbf{x}),\\ \mathbf{v}^\varepsilon(t,\mathbf{x}) &= \mathbf{v}_0(t,\mathbf{x}) + \varepsilon\mathbf{v}_1(t,\mathbf{x}) + \varepsilon^2\mathbf{v}_2(t,\mathbf{x}) + \varepsilon^3\mathbf{v}_R(t,\mathbf{x}). \end{aligned}$$

If we plug the above expansion into the inertial Qian–Sheng system (1.20)–(1.22), then we obtain a hierarchy of equations in subsection 2.2. The $O(\varepsilon^{-1})$ equation gives $\mathscr{T}(Q_0) = 0$, which implies by Proposition 2.1 that

$$Q_0 = s\left(\mathbf{nn} - \frac{1}{3}\mathbf{I}\right)$$

for some $\mathbf{n} \in \mathbb{S}^2$ and $s = \frac{b+\sqrt{b^2+24ac}}{4c}$. For the $O(1)$ system, we can obtain that $(\mathbf{v}_0, \mathbf{n})$ is exactly a solution of the full inertial Ericksen–Leslie system with the coefficients given by (1.23)–(1.25). Moreover, it can be shown that the existence of $(Q_i, \mathbf{v}_i)$ with $i \geq 1$ for $O(\varepsilon^i)$ can be guaranteed by the fact that $(Q_i, \mathbf{v}_i)$ satisfies a linear dissipative system; see Proposition 2.4.

The core part to rigorously justify the uniaxial limit is to prove the uniform (in ε) bounds for the remainders $(Q_R, \mathbf{v}_R)$. For this end, we write the equation for $(Q_R, \mathbf{v}_R)$ which roughly reads as

$$J\ddot{Q}_R + \mu_1\dot{Q}_R = -\frac{1}{\varepsilon}\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R) - \frac{\mu_2}{2}\mathbf{D}_R + \mu_1[\mathbf{\Omega}_R, Q_0] + \widetilde{\mathbf{F}}_R + \cdots, \tag{1.27}$$

$$\frac{\partial\mathbf{v}_R}{\partial t} + \tilde{\mathbf{v}}\cdot\nabla\mathbf{v}_R = -\nabla p_R + \nabla\cdot\left(\frac{\mu_2}{2}(\dot{Q}_R - [\mathbf{\Omega}_R, Q_0]) + \mu_1\big[Q_0, (\dot{Q}_R - [\mathbf{\Omega}_R, Q_0])\big]\right) + \cdots. \tag{1.28}$$

The main difficult terms are the term $\frac{1}{\varepsilon}\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R)$ which is singular in ε and the term $\widetilde{\mathbf{F}}_R$ (see (3.2) for the precise definition) which includes $\partial_t\mathbf{v}_R$ and second order derivatives of Q_R which will cause the problem of loss of derivatives in energy estimates.

To control the singular term $\frac{1}{\varepsilon}\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R)$, a natural method is to multiply the equation of Q_R with $\dot{Q}_R$, which will bring a term $\langle\frac{1}{\varepsilon}\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R), Q_R\rangle$ into the energy. However, the operator $\mathscr{H}^\varepsilon_{\mathbf{n}}$ is dependent of t, and its time derivative will bring some difficult terms such as

$$\frac{1}{\varepsilon}\langle\dot{\overline{\mathbf{nn}}}\cdot Q_R, Q_R\rangle, \quad \frac{1}{\varepsilon}\langle(Q_R : \dot{\overline{\mathbf{nn}}})\mathbf{nn}, Q_R\rangle. \tag{1.29}$$

The dissipative energy offers us the control of $\|\dot{Q}_R - [\mathbf{\Omega}_R, Q_0] - \frac{\mu_2}{\mu_1}\mathbf{D}_R\|^2_{L^2}$, which is equivalent to $\frac{1}{\varepsilon^2}\|\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R)\|^2_{L^2}$ in the noninertial case $J = 0$ due to (1.27) and then can be used to control the singular terms (1.29); see [34, Lemma 4.1]. However, it does not work in the inertial case since the equivalence does not hold.

To overcome this difficulty, we choose a delicate modified energy $\widetilde{\mathfrak{E}}$ (see (3.22)). Roughly speaking, we add the term

$$\left\langle \mathscr{H}_{\mathbf{n}}^{-1}\mathcal{G}\left(Q_R^{\top}\right), \dot{Q}_R\right\rangle$$

(see (3.22) for the precise definition) into the energy, such that its time derivative could cancel the above singular terms by using the structure of the system. A key related estimate is given in Lemma 3.4.

To estimate the terms with high order derivatives in $\widetilde{\mathbf{F}}_R$, we introduce the energy terms related to $\mathbf{v}_R\cdot\nabla Q^{\varepsilon}$; see (3.13)–(3.14). Then some hard terms can be absorbed into the time derivatives of these new introduced energy terms, and others are eliminated by using the symmetric structure of the system. These estimates are summarized in Lemmas 3.5–3.6.

Furthermore, we can show that the energy functional $\widetilde{\mathfrak{E}}$ is positive and $\widetilde{\mathfrak{E}}\sim\mathfrak{E}$ if $\mu_1\gg J$, and thus accomplish the main steps of the proof for Theorem 1.1 in principle.

2. The Hilbert expansion.

This section is devoted to deriving the Hilbert expansion for the inertial Qian–Sheng system (1.20)–(1.22). In particular, we will show that the $O(1)$ system is just the full Ericksen–Leslie system. The existence of the $O(\varepsilon^k)(k\geq 1)$ system in the Hilbert expansion will also be proved.

We first give some preliminary results about critical points and the linearized operator.

2.1. Critical points and the linearized operator.

A tensor Q_0 is called a critical point of $F_b(Q)$ if $\mathscr{T}(Q_0):=\frac{\partial F_b}{\partial Q}\big|_{Q=Q_0}=0$. The following characterization of critical points can be seen from [27, 34].

Proposition 2.1. *$\mathscr{T}(Q)=0$ if and only if $Q=s(\mathbf{nn}-\frac{1}{3}\mathbf{I})$ for some $\mathbf{n}\in\mathbb{S}^2$, where $s=0$ or a solution of $2cs^2-bs+3a=0$, that is,*

$$s_1=\frac{b+\sqrt{b^2+24ac}}{4c}\quad or\quad s_2=\frac{b-\sqrt{b^2+24ac}}{4c}.$$

Moreover, the critical point $Q_0=s(\mathbf{nn}-\frac{1}{3}\mathbf{I})$ is stable if $s=s_1$.

Given a critical point $Q_0=s(\mathbf{nn}-\frac{1}{3}\mathbf{I})$, the linearized operator $\mathscr{H}_{Q_0}$ of $\mathscr{T}(Q)$ around Q_0 is given by

$$\mathscr{H}_{Q_0}(Q)=aQ-b\left(Q_0\cdot Q+Q\cdot Q_0\right)+c|Q_0|^2Q+2(Q_0:Q)\left(cQ_0+\frac{b}{3}\mathbf{I}\right).$$

Then a direct calculation yields

$$\mathscr{H}_{Q_0}(Q)=bs\left(Q-(\mathbf{nn}\cdot Q+Q\cdot\mathbf{nn})+\frac{2}{3}(Q:\mathbf{nn})\mathbf{I}\right)+2cs^2(Q:\mathbf{nn})\left(\mathbf{nn}-\frac{1}{3}\mathbf{I}\right)$$
$$\stackrel{\text{def}}{=}\mathscr{H}_{\mathbf{n}}(Q).\tag{2.1}$$

The kernel space of the linearized operator $\mathscr{H}_{\mathbf{n}}$, being a two-dimensional subspace of $\mathbb{S}_0^3$, can be defined as

$$\mathrm{Ker}\mathscr{H}_{\mathbf{n}}\stackrel{\text{def}}{=}\left\{\mathbf{nn}^{\perp}+\mathbf{n}^{\perp}\mathbf{n}\in\mathbb{S}_0^3:\mathbf{n}^{\perp}\in\mathbb{V}_{\mathbf{n}}\right\}$$

for any given $\mathbf{n} \in \mathbb{S}^2$, where $\mathbb{V}_{\mathbf{n}} \stackrel{\text{def}}{=} \{\mathbf{n}^\perp \in \mathbb{R}^3 : \mathbf{n}^\perp \cdot \mathbf{n} = 0\}$. Let $\mathscr{P}^{in}$ be the projection operator from $\mathbb{S}_0^3$ to $\mathrm{Ker}\mathscr{H}_{\mathbf{n}}$ and $\mathscr{P}^{out}$ the projection operator from $\mathbb{S}_0^3$ to $(\mathrm{Ker}\mathscr{H}_{\mathbf{n}})^\perp$. Using the simple fact that

$$|Q - (\mathbf{n}\mathbf{n}^\perp + \mathbf{n}^\perp\mathbf{n})|^2 = |Q|^2 - 2|Q\cdot\mathbf{n}|^2 + 2|Q:\mathbf{n}\mathbf{n}|^2 + |\mathbf{n}^\perp - (\mathbf{I} - \mathbf{n}\mathbf{n})\cdot Q\cdot\mathbf{n}|^2,$$

the projection operators $\mathscr{P}^{in}$ and $\mathscr{P}^{out}$ are expressed as, respectively,

$$\begin{aligned}\mathscr{P}^{in}(Q) &= \mathbf{n}[(\mathbf{I}-\mathbf{n}\mathbf{n})\cdot Q\cdot\mathbf{n}] + [(\mathbf{I}-\mathbf{n}\mathbf{n})\cdot Q\cdot\mathbf{n}]\mathbf{n}\\ &= (\mathbf{n}\mathbf{n}\cdot Q + Q\cdot\mathbf{n}\mathbf{n}) - 2(Q:\mathbf{n}\mathbf{n})\mathbf{n}\mathbf{n},\end{aligned}\tag{2.2}$$

$$\begin{aligned}\mathscr{P}^{out}(Q) &= Q - \mathscr{P}^{in}(Q)\\ &= Q - (\mathbf{n}\mathbf{n}\cdot Q + Q\cdot\mathbf{n}\mathbf{n}) + 2(Q:\mathbf{n}\mathbf{n})\mathbf{n}\mathbf{n}.\end{aligned}\tag{2.3}$$

The important properties of the linearized operator $\mathscr{H}_{\mathbf{n}}$ can be found in [34].

PROPOSITION 2.2. (i) *For any* $\mathbf{n} \in \mathbb{S}^2$, *it holds that* $\mathscr{H}_{\mathbf{n}}\mathrm{Ker}\mathscr{H}_{\mathbf{n}} = 0$, *i.e.*, $\mathscr{H}_{\mathbf{n}}(Q) \in (\mathrm{Ker}\mathscr{H}_{\mathbf{n}})^\perp$.

(ii) *There exists a constant* $C_0 = c_0(a,b,c) > 0$ *such that for any* $Q \in (\mathrm{Ker}\mathscr{H}_{\mathbf{n}})^\perp$,

$$\mathscr{H}_{\mathbf{n}}(Q) : Q \geq c_0|Q|^2.$$

(iii) $\mathscr{H}_{\mathbf{n}}$ *is a* 1-1 *map on* $(\mathrm{Ker}\mathscr{H}_{\mathbf{n}})^\perp$ *and its inverse* $\mathscr{H}_{\mathbf{n}}^{-1}$ *is given by*

$$\begin{aligned}\mathscr{H}_{\mathbf{n}}^{-1}(Q) &= \frac{1}{bs}\Big(Q - (\mathbf{n}\mathbf{n}\cdot Q + Q\cdot\mathbf{n}\mathbf{n}) + \frac{2}{3}(Q:\mathbf{n}\mathbf{n})\mathbf{I}\Big)\\ &\quad + \frac{4b+2cs}{bs(4cs-b)}(Q:\mathbf{n}\mathbf{n})\left(\mathbf{n}\mathbf{n} - \frac{1}{3}\mathbf{I}\right).\end{aligned}\tag{2.4}$$

2.2. The Hilbert expansion. Let $(Q^\varepsilon, \mathbf{v}^\varepsilon)$ be a solution of the system (1.20)–(1.22). We perform the following Hilbert expansion:

$$Q^\varepsilon = \sum_{k=0}^{3}\varepsilon^k Q_k + \varepsilon^3 Q_R \stackrel{\text{def}}{=} \widetilde{Q} + \varepsilon^3 Q_R,\tag{2.5}$$

$$\mathbf{v}^\varepsilon = \sum_{k=0}^{2}\varepsilon^k \mathbf{v}_k + \varepsilon^3 \mathbf{v}_R \stackrel{\text{def}}{=} \tilde{\mathbf{v}} + \varepsilon^3\mathbf{v}_R,\tag{2.6}$$

where $Q_i(0\leq i\leq 3), \mathbf{v}_j(0\leq j\leq 2)$ do not depend on ε, while $(Q_R, \mathbf{v}_R)$ is called the remainder term which depend upon ε.

As shown in (2.9)–(2.18) below, inserting the Hilbert expansion (2.5)–(2.6) into the system (1.20)–(1.22) and equating like powers of ε leads to a hierarchy of equations. We will prove that $(Q_i, \mathbf{v}_i)(0\leq i\leq 2)$ and Q_3 can be determined in this way: Q_0 must be a critical point of $\mathscr{T}(Q)$, and the system of $(Q_0, \mathbf{v}_0)$ could be reduced to the full inertial Ericksen–Leslie system, while $(Q_i, \mathbf{v}_i)(1\leq i\leq 2)$ and Q_3 solve the linear equations obtained by using the projection operators.

For $Q_i \in \mathbb{R}^{3\times 3}(i=1,2,3)$, we introduce the following definitions:

$$\mathscr{B}(Q_1, Q_2) \stackrel{\text{def}}{=} Q_1\cdot Q_2 + Q_2^T\cdot Q_1^T - \frac{2}{3}(Q_1:Q_2)\mathbf{I},$$

$$\mathscr{C}(Q_1,Q_2,Q_3) \stackrel{\text{def}}{=} Q_1(Q_2:Q_3) + Q_2(Q_1:Q_3) + Q_3(Q_1:Q_2).$$

Let $\widehat{Q}^\varepsilon = Q_1 + \varepsilon Q_2 + \varepsilon^2 Q_3$, just as the polynomial expansion technique adopted in [34], we get the expansion of $\mathscr{T}(Q^\varepsilon)$ in ε as follows:

$$\mathscr{T}(Q^\varepsilon) = \mathscr{T}(Q_0) + \varepsilon \mathscr{H}_{\mathbf{n}}(Q_1) + \varepsilon^2(\mathscr{H}_{\mathbf{n}}(Q_2) + \mathbf{B}_1) + \varepsilon^3(\mathscr{H}_{\mathbf{n}}(Q_3) + \mathbf{B}_2) + \varepsilon^3 \mathscr{H}_{\mathbf{n}}(Q_R) + \varepsilon^4 \mathscr{T}_R^\varepsilon, \tag{2.7}$$

where $\mathbf{B}_1, \mathbf{B}_2$, and $\mathbf{B}^\varepsilon$, independent of Q_R, are respectively

$$\begin{aligned}
\mathbf{B}_1 &= -\frac{b}{2}\mathscr{B}(Q_1, Q_1) + c\mathscr{C}(Q_0, Q_1, Q_1),\\
\mathbf{B}_2 &= -b\mathscr{B}(Q_1, Q_2) + 2c\mathscr{C}(Q_0, Q_1, Q_2),\\
\mathbf{B}^\varepsilon &= -\frac{b}{2}\sum_{\substack{i+j\geq 4\\ 1\leq i,j\leq 3}} \varepsilon^{i+j-4}\mathscr{B}(Q_i, Q_j)\\
&\quad + \frac{c}{3}\sum_{\substack{i+j+k\geq 4\\ \text{at least two of } i,j,k \text{ are not zero}}} \varepsilon^{i+j+k-4}\mathscr{C}(Q_i, Q_j, Q_k),
\end{aligned}$$

and the fourth order term $\mathscr{T}_R^\varepsilon$ in ε is given by

$$\begin{aligned}
\mathscr{T}_R^\varepsilon &= \mathbf{B}^\varepsilon - b\mathscr{B}(\widehat{Q}^\varepsilon, Q_R) + c\mathscr{C}(Q_R, \widehat{Q}^\varepsilon, Q_0) + \frac{c}{2}\varepsilon\mathscr{C}(Q_R, \widehat{Q}^\varepsilon, \widehat{Q}^\varepsilon)\\
&\quad - \frac{b}{2}\varepsilon^2\mathscr{B}(Q_R, Q_R) + c\varepsilon^2\mathscr{C}(Q_R, Q_R, Q_0 + \varepsilon\widehat{Q}^\varepsilon) + c\varepsilon^5\mathscr{C}(Q_R, Q_R, Q_R).
\end{aligned}\tag{2.8}$$

For the sake of brevity, we also denote

$$\begin{aligned}
\mathbf{H}_0 &= \mathscr{H}_{\mathbf{n}}(Q_1) + \mathscr{L}(Q_0),\\
\mathbf{H}_1 &= \mathscr{H}_{\mathbf{n}}(Q_2) + \mathscr{L}(Q_1) + \mathbf{B}_1,\\
\mathbf{H}_2 &= \mathscr{H}_{\mathbf{n}}(Q_3) + \mathscr{L}(Q_2) + \mathbf{B}_2.
\end{aligned}$$

We are now in a position to write the expansion of the original system (1.20)–(1.22) and collect the terms (independent of Q_R) with the same order of ε. Specifically, we have the following:

- The $O(\varepsilon^{-1})$ system:

$$\mathscr{T}(Q_0) = 0. \tag{2.9}$$

- Zero order term in ε:

$$J\ddot{Q}_0 + \mu_1\dot{Q}_0 = -\mathbf{H}_0 - \frac{\mu_2}{2}\mathbf{D}_0 + \mu_1[\mathbf{\Omega}_0, Q_0], \tag{2.10}$$

$$\begin{aligned}
\frac{\partial \mathbf{v}_0}{\partial t} + \mathbf{v}_0\cdot\nabla\mathbf{v}_0 &= -\nabla p_0 + \nabla\cdot\Big(\beta_1 Q_0(Q_0:\mathbf{D}_0) + \beta_4\mathbf{D}_0 + \beta_5\mathbf{D}_0\cdot Q_0\\
&\quad + \beta_6 Q_0\cdot\mathbf{D}_0 + \beta_7(\mathbf{D}_0\cdot Q_0^2 + Q_0^2\cdot\mathbf{D}_0)\\
&\quad + \frac{\mu_2}{2}\mathcal{N}_0 + \mu_1[Q_0, \mathcal{N}_0] + \sigma^d(Q_0, Q_0)\Big),
\end{aligned}\tag{2.11}$$

$$\nabla\cdot\mathbf{v}_0 = 0, \tag{2.12}$$

where

$$\ddot{Q}_0 = (\partial_t + \mathbf{v}_0\cdot\nabla)\dot{Q}_0, \quad \dot{Q}_0 = (\partial_t + \mathbf{v}_0\cdot\nabla)Q_0, \quad \mathcal{N}_0 = \dot{Q}_0 - [\mathbf{\Omega}_0, Q_0].$$

- First order term in ε:

$$\begin{aligned}
J\ddot{Q}_1 + \mu_1\dot{Q}_1 &= -\mathbf{H}_1 - \frac{\mu_2}{2}\mathbf{D}_1 + \mu_1\big([\mathbf{\Omega}_1, Q_0] + [\mathbf{\Omega}_0, Q_1] - \mathbf{v}_1\cdot\nabla Q_0\big)\\
&\quad - J\Big(2\mathbf{v}_1\cdot\nabla(\partial_t Q_0) + \partial_t\mathbf{v}_1\cdot\nabla Q_0 + (\mathbf{v}_1\cdot\nabla)\mathbf{v}_0\cdot\nabla Q_0\\
&\quad + (\mathbf{v}_0\cdot\nabla)\mathbf{v}_1\cdot\nabla Q_0 + (\mathbf{v}_1\mathbf{v}_0:\nabla^2)Q_0 + (\mathbf{v}_0\mathbf{v}_1:\nabla^2)Q_0\Big),
\end{aligned}\tag{2.13}$$

$$\begin{aligned}\frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_0\cdot\nabla\mathbf{v}_1 = {} & -\mathbf{v}_1\cdot\nabla\mathbf{v}_0 - \nabla p_1 + \nabla\cdot\Big(\beta_1\big(Q_0(Q_0:\mathbf{D}_1)\\ & + Q_0(Q_1:\mathbf{D}_0) + Q_1(Q_0:\mathbf{D}_0)\big) + \beta_4\mathbf{D}_1\\ & + \beta_5(\mathbf{D}_0\cdot Q_1 + \mathbf{D}_1\cdot Q_0) + \beta_6(Q_0\cdot\mathbf{D}_1 + Q_1\cdot\mathbf{D}_0)\\ & + \beta_7\big(\mathbf{D}_1\cdot Q_0^2 + Q_0^2\cdot\mathbf{D}_1 + \mathbf{D}_0\cdot Q_1\cdot Q_0 + \mathbf{D}_0\cdot Q_0\cdot Q_1\\ & + Q_1\cdot Q_0\cdot\mathbf{D}_0 + Q_0\cdot Q_1\cdot\mathbf{D}_0\big) + \frac{\mu_2}{2}\overline{\mathcal{N}}_1\\ & + \mu_1([Q_1,\mathcal{N}_0] + [Q_0,\overline{\mathcal{N}}_1]) + \sigma^d(Q_1,Q_0) + \sigma^d(Q_0,Q_1)\Big),\end{aligned} \tag{2.14}$$

$$\nabla\cdot\mathbf{v}_1 = 0, \tag{2.15}$$

where

$$\begin{aligned}&\ddot{Q}_1 = (\partial_t + \mathbf{v}_0\cdot\nabla)\dot{Q}_1, \quad \dot{Q}_1 = (\partial_t + \mathbf{v}_0\cdot\nabla)Q_1,\\ &\mathcal{N}_1 = \dot{Q}_1 - [\mathbf{\Omega}_0, Q_1], \quad \overline{\mathcal{N}}_1 = \mathcal{N}_1 + \mathbf{v}_1\cdot\nabla Q_0 - [\mathbf{\Omega}_1, Q_0].\end{aligned}$$

• Second order term in ε:

$$\begin{aligned}J\ddot{Q}_2 + \mu_1\dot{Q}_2 = {} & -\mathbf{H}_2 - \frac{\mu_2}{2}\mathbf{D}_2 + \mu_1\Big([\mathbf{\Omega}_2, Q_0] + [\mathbf{\Omega}_0, Q_2] + [\mathbf{\Omega}_1, Q_1]\\ & - \mathbf{v}_2\cdot\nabla\mathbf{v}_0 - \mathbf{v}_1\cdot\nabla Q_1\Big)\\ & - J\Big(2\mathbf{v}_1\cdot\nabla(\partial_t Q_1) + 2\mathbf{v}_2\cdot\nabla(\partial_t Q_0) + \partial_t\mathbf{v}_1\cdot\nabla Q_1\\ & + \partial_t\mathbf{v}_2\cdot\nabla Q_0 + (\mathbf{v}_1\cdot\nabla)\mathbf{v}_1\cdot\nabla Q_0 + (\mathbf{v}_0\cdot\nabla)\mathbf{v}_1\cdot\nabla Q_1\\ & + (\mathbf{v}_0\mathbf{v}_1:\nabla^2)Q_1 + (\mathbf{v}_1\mathbf{v}_0:\nabla^2)Q_1 + (\mathbf{v}_1\mathbf{v}_1:\nabla^2)Q_0\\ & + (\mathbf{v}_2\mathbf{v}_0:\nabla^2)Q_0 + (\mathbf{v}_0\mathbf{v}_2:\nabla^2)Q_0\Big),\end{aligned} \tag{2.16}$$

$$\begin{aligned}\frac{\partial \mathbf{v}_2}{\partial t} + \mathbf{v}_0\cdot\nabla\mathbf{v}_2 = {} & -\mathbf{v}_2\cdot\nabla\mathbf{v}_0 - \mathbf{v}_1\cdot\nabla\mathbf{v}_1 - \nabla p_2 + \nabla\cdot\Bigg(\beta_1\sum_{i+j+k=2} Q_i(Q_j:\mathbf{D}_k)\\ & + \beta_4\mathbf{D}_2 + \beta_5(\mathbf{D}_2\cdot Q_0 + \mathbf{D}_1\cdot Q_1 + \mathbf{D}_0\cdot Q_2)\\ & + \beta_6(Q_0\cdot\mathbf{D}_2 + Q_1\cdot\mathbf{D}_1 + Q_2\cdot\mathbf{D}_0)\\ & + \beta_7\sum_{i+j+k=2}\big(\mathbf{D}_i\cdot Q_j\cdot Q_k + Q_i\cdot Q_j\cdot\mathbf{D}_k\big) + \frac{\mu_2}{2}\overline{\mathcal{N}}_2\\ & + \mu_1\big([Q_2,\mathcal{N}_0] + [Q_1,\overline{\mathcal{N}}_1] + [Q_0,\overline{\mathcal{N}}_2]\big)\\ & + \sigma^d(Q_2,Q_0) + \sigma^d(Q_1,Q_1) + \sigma^d(Q_0,Q_2)\Bigg),\end{aligned} \tag{2.17}$$

$$\nabla\cdot\mathbf{v}_2 = 0, \tag{2.18}$$

where

$$\begin{aligned}&\ddot{Q}_2 = (\partial_t + \mathbf{v}_0\cdot\nabla)\dot{Q}_2, \quad \dot{Q}_2 = (\partial_t + \mathbf{v}_0\cdot\nabla)Q_2, \quad \mathcal{N}_2 = \dot{Q}_2 - [\mathbf{\Omega}_0, Q_2],\\ &\overline{\mathcal{N}}_2 = \mathcal{N}_2 + \mathbf{v}_2\cdot\nabla Q_0 + \mathbf{v}_1\cdot\nabla Q_1 - [\mathbf{\Omega}_2, Q_0] - [\mathbf{\Omega}_1, Q_1].\end{aligned}$$

In what follows, we will show how to solve $(Q_i, \mathbf{v}_i)(0 \le i \le 2)$ and Q_3. First of all, combining (2.9) with Proposition 2.1, we deduce that Q_0 is a critical point and could be taken as

$$Q_0(t,\mathbf{x}) = s\left(\mathbf{n}(t,\mathbf{x})\mathbf{n}(t,\mathbf{n}) - \frac{1}{3}\mathbf{I}\right) \tag{2.19}$$

for some $\mathbf{n}(t,\mathbf{x}) \in \mathbb{S}^2$ and $s = s_1$.

PROPOSITION 2.3. *Suppose that* $(Q_0, \mathbf{v}_0)$ *is a smooth solution of the system* (2.10)–(2.12)*; then* $(\mathbf{n}, \mathbf{v}_0)$ *must be a solution of the full inertial Ericksen–Leslie system* (1.2)–(1.4)*, where the coefficients are determined by* (1.23)–(1.25).

Proof. Recalling the first property $\mathscr{H}_{\mathbf{n}}(Q_1) \in (\mathrm{Ker}\mathscr{H}_{\mathbf{n}})^\perp$ in Proposition 2.2, we can deduce from (2.10) that

$$\left(J\ddot{Q}_0 + \mu_1\mathcal{N}_0 + \mathscr{L}(Q_0) + \frac{\mu_2}{2}\mathbf{D}_0\right) : (\mathbf{n}\mathbf{n}^\perp + \mathbf{n}^\perp\mathbf{n}) = 0. \tag{2.20}$$

Substituting (2.19) into (2.20), we get by a subtle calculation that

$$\begin{aligned}
\ddot{Q}_0 : (\mathbf{n}\mathbf{n}^\perp + \mathbf{n}^\perp\mathbf{n}) &= s(\ddot{\mathbf{n}}\mathbf{n} + 2\dot{\mathbf{n}}\dot{\mathbf{n}} + \mathbf{n}\ddot{\mathbf{n}}) : (\mathbf{n}\mathbf{n}^\perp + \mathbf{n}^\perp\mathbf{n})\\
&= 2s\ddot{\mathbf{n}} \cdot \mathbf{n}^\perp,\\
\mathcal{N}_0 : (\mathbf{n}\mathbf{n}^\perp + \mathbf{n}^\perp\mathbf{n}) &= [s(\dot{\mathbf{n}}\mathbf{n} + \mathbf{n}\dot{\mathbf{n}}) + s(\mathbf{n}\mathbf{n}\cdot\mathbf{\Omega}_0 - \mathbf{\Omega}_0\cdot\mathbf{n}\mathbf{n})] : (\mathbf{n}\mathbf{n}^\perp + \mathbf{n}^\perp\mathbf{n})\\
&= 2s\mathbf{N}\cdot\mathbf{n}^\perp,\\
\mathscr{L}(Q_0) : (\mathbf{n}\mathbf{n}^\perp + \mathbf{n}^\perp\mathbf{n}) &= -\frac{1}{s}\mathbf{h}\cdot\mathbf{n}^\perp,\\
\mathbf{D}_0 : (\mathbf{n}\mathbf{n}^\perp + \mathbf{n}^\perp\mathbf{n}) &= 2(\mathbf{D}_0\cdot\mathbf{n})\cdot\mathbf{n}^\perp,
\end{aligned}$$

from which it follows that

$$\mathbf{n}^\perp \cdot \left(2s^2J\ddot{\mathbf{n}} + 2s^2\mu_1\mathbf{N} - \mathbf{h} + s\mu_2\mathbf{D}_0\cdot\mathbf{n}\right) = 0,$$

which implies

$$\mathbf{n} \times \left(I\ddot{\mathbf{n}} - \mathbf{h} + \gamma_1\mathbf{N} + \gamma_2\mathbf{D}_0\cdot\mathbf{n}\right) = 0, \tag{2.21}$$

where

$$I = 2s^2J, \quad \gamma_1 = 2s^2\mu_1, \quad \gamma_2 = s\mu_2.$$

Applying the definition of $\mathrm{Ker}\mathscr{H}_{\mathbf{n}}$ and (2.19) yields

$$\begin{aligned}
\mathcal{N}_0 &= \frac{\partial Q_0}{\partial t} + \mathbf{v}_0\cdot\nabla Q_0 + Q_0\cdot\mathbf{\Omega}_0 - \mathbf{\Omega}_0\cdot Q_0\\
&= s(\mathbf{n}\mathbf{N} + \mathbf{N}\mathbf{n}) \in \mathrm{Ker}\mathscr{H}_{\mathbf{n}}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\sigma_0 &\stackrel{\text{def}}{=} \beta_1 Q_0(Q_0 : \mathbf{D}_0) + \beta_4\mathbf{D}_0 + \beta_5\mathbf{D}_0\cdot Q_0 + \beta_6 Q_0\cdot\mathbf{D}_0\\
&\quad + \beta_7(\mathbf{D}_0\cdot Q_0^2 + Q_0^2\cdot\mathbf{D}_0) + \frac{\mu_2}{2}\mathcal{N}_0 + \mu_1(Q_0\cdot\mathcal{N}_0 - \mathcal{N}_0\cdot Q_0)\\
&= \beta_1 s^2(\mathbf{n}\mathbf{n} : \mathbf{D}_0)\mathbf{n}\mathbf{n} - \frac{1}{3}\beta_1 s^2(\mathbf{n}\mathbf{n} : \mathbf{D}_0)\mathbf{I} + \beta_4\mathbf{D}_0 + \beta_5 s\mathbf{D}_0\cdot\mathbf{n}\mathbf{n}\\
&\quad + \beta_6 s\mathbf{n}\mathbf{n}\cdot\mathbf{D}_0 - \frac{1}{3}(\beta_5 + \beta_6)s\mathbf{D}_0 + \frac{1}{3}\beta_7 s^2(\mathbf{D}_0\cdot\mathbf{n}\mathbf{n} + \mathbf{n}\mathbf{n}\cdot\mathbf{D}_0)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{9}\beta_7 s^2\mathbf{D}_0+\frac{1}{2}\mu_2 s(\mathbf{nN}+\mathbf{Nn})+\mu_1 s^2(\mathbf{nN}-\mathbf{Nn})\\
=&\ \beta_1 s^2(\mathbf{nn}:\mathbf{D}_0)\mathbf{nn}+\left(\frac{1}{2}\mu_2 s-\mu_1 s^2\right)\mathbf{Nn}+\left(\frac{1}{2}\mu_2 s+\mu_1 s^2\right)\mathbf{nN}\\
&+\left(\beta_4-\frac{1}{3}(\beta_5+\beta_6)s+\frac{2}{9}\beta_7 s^2\right)\mathbf{D}_0+\left(\beta_5 s+\frac{1}{3}\beta_7 s^2\right)\mathbf{nn}\cdot\mathbf{D}_0\\
&+\left(\beta_6 s+\frac{1}{3}\beta_7 s^2\right)\mathbf{D}_0\cdot\mathbf{nn}+\text{pressure terms}\\
=&\ \sigma^L+\text{pressure terms}.
\end{aligned}
$$

In addition, from Lemma 3.5 in [34] we know that $\sigma^E=\sigma^d(Q_0,Q_0)$. Here σ^E and σ^L (see (1.5) and (1.6)) are just the elastic stress and the viscous stress in the full inertial Ericksen–Leslie system, respectively. This completes the proof of Proposition 2.3. ∎

2.3. Existence of the Hilbert expansion. In this subsection, we are going to elucidate the existence of the Hilbert expansion. In other words, we will show how to solve $(Q_i,\mathbf{v}_i)(1\le i\le 2)$ and Q_3 from the system (2.13)–(2.18). To be more specific, we have the following Proposition 2.4.

PROPOSITION 2.4. *Let* $(\mathbf{n},\mathbf{v}_0)$ *be a smooth solution of the full inertial Ericksen–Leslie system* (1.2)–(1.4) *on* $[0,T]$ *and satisfy*

$$(\mathbf{v}_0,\partial_t\mathbf{n},\nabla\mathbf{n})\in L^\infty([0,T];H^k)\quad \textit{for}\quad k\ge 20.$$

Then there exists the solution $(Q_i,\mathbf{v}_i)(i=0,1,2)$ *and* $Q_3\in(\mathrm{Ker}\mathscr{H}_{\mathbf{n}})^\perp$ *of the system* (2.13)–(2.18) *satisfying*

$$(\mathbf{v}_i,\partial_t Q_i,\nabla Q_i)\in L^\infty([0,T];H^{k-4i})(i=0,1,2),\quad Q_3\in L^\infty([0,T];H^{k-11}).$$

Before proving Proposition 2.4, we need the following Lemma 2.5 from [33] and Lemma 2.6.

LEMMA 2.5. *The dissipation relation*

$$\hat\beta_1|\mathbf{nn}:\mathbf{D}|^2+\hat\beta_2|\mathbf{D}|^2+\hat\beta_3|\mathbf{n}\cdot\mathbf{D}|^2\geqq 0 \tag{2.22}$$

holds for any symmetric traceless matric $\mathbf{D}$ *and unit vector* $\mathbf{n}$ *if and only if*

$$\hat\beta_2\geqq 0,\quad 2\hat\beta_2+\hat\beta_3\geqq 0,\quad \frac{3}{2}\hat\beta_2+\hat\beta_3+\hat\beta_1\geqq 0. \tag{2.23}$$

LEMMA 2.6. *Assume that* $Q_1=Q_1^\top+Q_1^\perp$ *with* $Q_1^\top\in\mathrm{Ker}\mathscr{H}_{\mathbf{n}}$ *and* $Q_1^\perp\in(\mathrm{Ker}\mathscr{H}_{\mathbf{n}})^\perp$. *Then it follows that*

$$\mathscr{P}^{out}(\dot Q_1)=L\left(Q_1^\top\right)+R,\quad \mathscr{P}^{in}(\dot Q_1)=\dot Q_1^\top+L\left(Q_1^\top\right)+R, \tag{2.24}$$

$$\mathscr{P}^{out}(\ddot Q_1)=L(\dot Q_1^\top)+L\left(Q_1^\top\right)+R,\quad \mathscr{P}^{in}(\ddot Q_1)=\ddot Q_1^\top+L(\dot Q_1^\top)+L\left(Q_1^\top\right)+R, \tag{2.25}$$

where $\dot Q_1^\top\overset{def}{=}(\partial_t+\mathbf{v}_0\cdot\nabla)Q_1^\top$ *and* $\ddot Q_1^\top\overset{def}{=}(\partial_t+\mathbf{v}_0\cdot\nabla)\dot Q_1^\top$. *In addition,* $L(\cdot)$ *represents the linear function with the coefficients belonging to* $L^\infty([0,T];H^{k-1})$ *and* $R\in L^\infty([0,T];H^{k-3})$ *some function depending only on* $\mathbf{n},\mathbf{v}_0,Q_1^\perp$.

Proof. For the proof of (2.24) see [34] for the details. It remains to prove (2.25). Let $Q_1^\top = \mathbf{n}\mathbf{n}^\perp + \mathbf{n}^\perp\mathbf{n}$ with $\mathbf{n}^\perp \cdot \mathbf{n} = 0$; then it follows that

$$\begin{aligned}\dot{Q}_1^\top &= \mathbf{n}\dot{\mathbf{n}}^\perp + \dot{\mathbf{n}}\mathbf{n}^\perp + \dot{\mathbf{n}}^\perp\mathbf{n} + \mathbf{n}^\perp\dot{\mathbf{n}},\\ \ddot{Q}_1^\top &= 2(\dot{\mathbf{n}}\dot{\mathbf{n}}^\perp + \dot{\mathbf{n}}^\perp\dot{\mathbf{n}}) + \mathbf{n}\ddot{\mathbf{n}}^\perp + \ddot{\mathbf{n}}^\perp\mathbf{n} + \ddot{\mathbf{n}}\mathbf{n}^\perp + \mathbf{n}^\perp\ddot{\mathbf{n}},\end{aligned}$$

where $\dot{\mathbf{n}} = (\partial_t + \mathbf{v}_0 \cdot \nabla)\mathbf{n}$ and $\dot{\mathbf{n}}^\perp = (\partial_t + \mathbf{v}_0 \cdot \nabla)\mathbf{n}^\perp$. Note that

$$\begin{aligned}&\mathbf{n}^\perp \cdot \mathbf{n} = \dot{\mathbf{n}} \cdot \mathbf{n} = 0, \quad \overline{(\mathbf{n}^\perp \cdot \mathbf{n})}^{\cdot} = \dot{\mathbf{n}}^\perp \cdot \mathbf{n} + \mathbf{n}^\perp \cdot \dot{\mathbf{n}} = 0,\\ &\overline{(\mathbf{n}^\perp \cdot \mathbf{n})}^{\cdot\cdot} = \ddot{\mathbf{n}}^\perp \cdot \mathbf{n} + 2\dot{\mathbf{n}}^\perp \cdot \dot{\mathbf{n}} + \mathbf{n}^\perp \cdot \ddot{\mathbf{n}} = 0.\end{aligned}$$

By a simple computation, we have

$$\begin{aligned}(\mathbf{I} - \mathbf{n}\mathbf{n}) \cdot \ddot{Q}_1^\top \cdot \mathbf{n} &= (\delta_{ij} - n_i n_j)\left(2(\dot{n}_j\dot{n}_k^\perp + \dot{n}_j^\perp\dot{n}_k) + n_j\ddot{n}_k^\perp + \ddot{n}_j^\perp n_k + \ddot{n}_j n_k^\perp + n_j^\perp\ddot{n}_k\right) n_k\\ &= 2\dot{n}_i\dot{n}_k^\perp n_k + \ddot{n}_i^\perp + n_i^\perp\ddot{n}_k n_k - n_i\ddot{n}_k^\perp n_k\\ &= \ddot{n}_i^\perp + \left(n_i n_k^\perp + n_i^\perp n_k\right)\ddot{n}_k - 2\dot{n}_i n_k^\perp\dot{n}_k + 2n_i\dot{n}_k^\perp\dot{n}_k.\end{aligned}$$

Consequently, using the fact $\mathbf{n}^\perp = Q_1^\top \cdot \mathbf{n}$ and the definition of the projection operator $\mathscr{P}^{in}$, we obtain

$$\begin{aligned}\mathscr{P}^{in}(\ddot{Q}_1^\top) &= \mathbf{n}\left((\mathbf{I} - \mathbf{n}\mathbf{n}) \cdot \ddot{Q}_1^\top \cdot \mathbf{n}\right) + \left((\mathbf{I} - \mathbf{n}\mathbf{n}) \cdot \ddot{Q}_1^\top \cdot \mathbf{n}\right)\mathbf{n}\\ &= \mathbf{n}\ddot{\mathbf{n}}^\perp + \ddot{\mathbf{n}}^\perp\mathbf{n} + 2\left(\dot{Q}_1^\top : \mathbf{n}\dot{\mathbf{n}}\right)\mathbf{n}\mathbf{n} + L\left(Q_1^\top\right),\end{aligned}$$

which yields that

$$\begin{aligned}\mathscr{P}^{out}(\ddot{Q}_1) &= \mathscr{P}^{out}(\ddot{Q}_1^\top) + R\\ &= 2\left(\dot{\mathbf{n}}\mathbf{n} \cdot \dot{Q}_1^\top + \dot{Q}_1^\top \cdot \mathbf{n}\dot{\mathbf{n}}\right) + 2\left(\dot{Q}_1^\top : \mathbf{n}\dot{\mathbf{n}}\right)\mathbf{n}\mathbf{n} + L\left(Q_1^\top\right) + R\\ &= L(\dot{Q}_1^\top) + L\left(Q_1^\top\right) + R.\end{aligned}$$

Therefore, we can deduce that

$$\mathscr{P}^{in}(\ddot{Q}_1) = \ddot{Q}_1 - \mathscr{P}^{out}(\ddot{Q}_1) = \ddot{Q}_1^\top + L(\dot{Q}_1^\top) + L\left(Q_1^\top\right) + R. \qquad \square$$

Proof of Proposition 2.4. Suppose that $(\mathbf{n}, \mathbf{v}_0)$ is a smooth solution of the full inertial Ericksen–Leslie model (1.2)–(1.4) on $[0, T]$ such that

$$(\mathbf{v}_0, \partial_t\mathbf{n}, \nabla\mathbf{n}) \in L^\infty([0, T]; H^k) \text{ for } k \geq 20.$$

Thanks to $Q_0 = s(\mathbf{n}(t, \mathbf{x})\mathbf{n}(t, \mathbf{x}) - \frac{1}{3}\mathbf{I})$, we know $Q_0 \in L^\infty([0, T], H^{k+1})$. Note that we could solve $Q_1^\perp$ from (2.10) and easily get $Q_1^\perp \in L^\infty([0, T]; H^{k-1})$ by Proposition 2.2. Thus, the existence of $(Q_1, \mathbf{v}_1)$ can be reduced to solving $(Q_1^\top, \mathbf{v}_1)$.

The key observation is that $(Q_1^\top, \mathbf{v}_1)$ satisfies a linear dissipative system, although the system seems nonlinear at first glance due to the term $\mathbf{H}_1$ in (2.13) which contains $\mathbf{B}_1$. For this end, we derive the linear system of $(\mathbf{v}_1, Q_1^\top)$. We denote

$$\widehat{\mathbf{B}}_1(Q, \overline{Q}) = -b\left(Q \cdot \overline{Q} - \frac{1}{3}\left(Q : \overline{Q}\mathbf{I}\right)\right) + c\left(2(Q : Q_0)\overline{Q} + \left(Q : \overline{Q}\right)Q_0\right).$$

Thus we have

$$\mathbf{B}_1 = \widehat{\mathbf{B}}_1(Q_1,Q_1) = \widehat{\mathbf{B}}_1\left(Q_1^\top,Q_1^\top\right) + \widehat{\mathbf{B}}_1\left(Q_1^\top,Q_1^\perp\right) + \widehat{\mathbf{B}}_1\left(Q_1^\perp,Q_1^\top\right) + \widehat{\mathbf{B}}_1\left(Q_1^\perp,Q_1^\perp\right)$$
$$= \widehat{\mathbf{B}}_1\left(Q_1^\top,Q_1^\top\right) + L\left(Q_1^\top,\mathbf{v}_1\right).$$

By a simple calculation we get

$$\widehat{\mathbf{B}}_1\left(Q_1^\top,Q_1^\top\right) \in (\mathrm{Ker}\mathscr{H}_{\mathbf{n}})^\perp. \tag{2.26}$$

We denote

$$\mathscr{A} = \mathscr{P}^{in}(\mathscr{L}\left(Q_1^\top\right)), \quad \mathscr{C}_1 = \mathscr{P}^{in}([\mathbf{\Omega}_1,Q_0]), \quad \mathscr{D}_1 = \mathscr{P}^{in}(\mathbf{D}_1),$$
$$\mathscr{U} = \mathscr{P}^{in}(\dot{\mathbf{v}}_1\cdot\nabla Q_0), \quad \mathscr{C}_2 = \mathscr{P}^{out}([\mathbf{\Omega}_1,Q_0]), \quad \mathscr{D}_2 = \mathscr{P}^{out}(\mathbf{D}_1).$$

Taking the projection $\mathscr{P}^{in}$ on both sides of (2.13), and noticing that $\mathscr{H}_{\mathbf{n}}(Q_2) \in (\mathrm{Ker}\mathscr{H}_{\mathbf{n}})^\perp$ and $\mathscr{L}(Q_1) = \mathscr{L}(Q_1^\top) + R$, from Lemma 2.6 and (2.26) we obtain that

$$J\ddot{Q}_1^\top + \mu_1\dot{Q}_1^\top = -\mathscr{A} - \frac{\mu_2}{2}\mathscr{D}_1 + \mu_1\mathscr{C}_1 - J\mathscr{U} + L(\dot{Q}_1^\top) + L\left(Q_1^\top,\mathbf{v}_1\right) + R.$$

Note that, due to (2.26), the nonlinear term $\widehat{\mathbf{B}}_1(Q_1^\top,Q_1^\top)$ vanishes in the above equation.

Thus, we have the following closed linear system of $(Q_1^\top,\mathbf{v}_1)$:

$$J\ddot{Q}_1^\top + \mu_1\dot{Q}_1^\top = -\mathscr{A} - \frac{\mu_2}{2}\mathscr{D}_1 + \mu_1\mathscr{C}_1 - J\mathscr{U} + L(\dot{Q}_1^\top) + L\left(Q_1^\top,\mathbf{v}_1\right) + R, \tag{2.27}$$

$$\begin{aligned}\frac{\partial\mathbf{v}_1}{\partial t} + \mathbf{v}_0\cdot\nabla\mathbf{v}_1 = -\nabla p_1 + \nabla\cdot\Big(&\beta_1 Q_0(Q_0:\mathbf{D}_1) + \beta_4\mathbf{D}_1 + \beta_5\mathbf{D}_1\cdot Q_0\\ &+ \beta_6 Q_0\cdot\mathbf{D}_1 + \beta_7\left(\mathbf{D}_1\cdot Q_0^2 + Q_0^2\cdot\mathbf{D}_1\right)\\ &+ \frac{\mu_2}{2}\Big(\dot{Q}_1^\top - [\mathbf{\Omega}_1,Q_0]\Big) + \mu_1\Big[Q_0,\Big(\dot{Q}_1^\top - [\mathbf{\Omega}_1,Q_0]\Big)\Big]\\ &+ \sigma^d(Q_1^\top,Q_0) + \sigma^d(Q_0,Q_1^\top) + L\left(Q_1^\top,\mathbf{v}_1\right) + R\Big),\end{aligned} \tag{2.28}$$

$$\nabla\mathbf{v}_1 = 0. \tag{2.29}$$

In order to prove the unique solvability of the linear system (2.27)–(2.29), we need to present an a priori estimate for the energy

$$\mathscr{E}(t) \overset{\mathrm{def}}{=} \|\mathbf{v}_1\|_{L^2}^2 + \left\langle Q_1^\top,\mathscr{L}\left(Q_1^\top\right)\right\rangle + \|\dot{Q}_1^\top\|_{L^2}^2 + \|Q_1^\top\|_{L^2}^2,$$

that is, to prove the energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(t) \le C\left(\mathscr{E}(t) + \|R(t)\|_{L^2}\right), \tag{2.30}$$

where the solution $(Q_1^\top,\mathbf{v}_1)$ satisfies $(\mathbf{v}_1,\partial_t Q_1^\top,\nabla Q_1^\top) \in L^\infty([0,T];H^{k-4})$.

First of all, from (2.27) and (1.19) we have

$$\begin{aligned}&J\left\langle\ddot{Q}_1^\top,Q_1^\top\right\rangle + \mu_1\left\langle\dot{Q}_1^\top,Q_1^\top\right\rangle\\ &= \left\langle -\mathscr{L}\left(Q_1^\top\right) - \frac{\mu_2}{2}\mathbf{D}_1 + \mu_1[\mathbf{\Omega}_1,Q_0],Q_1^\top\right\rangle - J\left\langle\dot{\mathbf{v}}_1\cdot\nabla Q_0,Q_1^\top\right\rangle\\ &\quad + \left\langle L(\dot{Q}_1^\top) + L\left(Q_1^\top,\mathbf{v}_1\right) + R,Q_1^\top\right\rangle\\ &\le -\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle\mathbf{v}_1\cdot\nabla Q_0,Q_1^\top\Big\rangle + \delta\|\nabla\mathbf{v}_1\|_{L^2}^2 + C_\delta\Big(\|\mathbf{v}_1\|_{L^2}^2\\ &\quad + \|\dot{Q}_1^\top\|_{L^2}^2 + \|Q_1^\top\|_{H^1}^2 + \|R\|_{L^2}^2\Big),\end{aligned} \tag{2.31}$$

where we have been obliged to estimate the term $-J\langle \dot{\mathbf{v}}_1 \cdot \nabla Q_0, Q_1^\top \rangle$. In fact, from integration by parts we know that

$$\begin{aligned} &- J\left\langle \dot{\mathbf{v}}_1 \cdot \nabla Q_0, Q_1^\top \right\rangle \\ &\quad = -\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{v}_1 \cdot \nabla Q_0, Q_1^\top \rangle + \langle \mathbf{v}_1 \cdot (\partial_t + \mathbf{v}_0 \cdot \nabla)\nabla Q_0, Q_1^\top \rangle + \langle \mathbf{v}_1 \cdot \nabla Q_0, \dot{Q}_1^\top \rangle \\ &\quad \le -\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \mathbf{v}_1 \cdot \nabla Q_0, Q_1^\top \right\rangle + C\left(\|\mathbf{v}_1\|_{L^2}^2 + \|Q_1^\top\|_{L^2}^2 + \|\dot{Q}_1^\top\|_{L^2}^2 \right). \end{aligned}$$

It can be observed that, for any $Q \in \mathbb{S}_0^3$, there holds

$$\begin{aligned} \langle \ddot{Q}, Q \rangle &= \int_{\mathbb{R}^3} \partial_t \dot{Q}_{ij} Q_{ij} + v_k \partial_k \dot{Q}_{ij} Q_{ij} \mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^3} \Big(\partial_t (\dot{Q}_{ij} Q_{ij}) + v_k \partial_k (\dot{Q}_{ij} Q_{ij}) - \dot{Q}_{ij}\dot{Q}_{ij} \Big) \mathrm{d}\mathbf{x} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \dot{Q} : Q \mathrm{d}\mathbf{x} - \int_{\mathbb{R}^3} |\dot{Q}|^2 \mathrm{d}\mathbf{x}. \end{aligned} \tag{2.32}$$

From (2.31) and (2.32) we thus obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \left(J\dot{Q}_1^\top : Q_1^\top + J(\mathbf{v}_1 \cdot \nabla Q_0) : Q_1^\top + \frac{\mu_1}{2}|Q_1^\top|^2 \right) \mathrm{d}\mathbf{x} \\ &\quad \le \delta \|\nabla \mathbf{v}_1\|_{L^2}^2 + C_\delta \left(\|\mathbf{v}_1\|_{L^2}^2 + \|\dot{Q}_1^\top\|_{L^2}^2 + \|Q_1^\top\|_{H^1}^2 + \|R\|_{L^2}^2 \right). \end{aligned} \tag{2.33}$$

Taking advantage of the linear system (2.27)–(2.29) and integration by parts over $\mathbb{R}^3$, we know

$$\begin{aligned} &\langle \partial_t \mathbf{v}_1, \mathbf{v}_1 \rangle + J\langle \ddot{Q}_1^\top, \dot{Q}_1^\top \rangle \\ &\quad = \underbrace{-\Big\langle \beta_1 Q_0(Q_0 : \mathbf{D}_1) + \beta_4 \mathbf{D}_1 + \beta_5 \mathbf{D}_1 \cdot Q_0 + \beta_6 Q_0 \cdot \mathbf{D}_1, \nabla \mathbf{v}_1 \Big\rangle}_{I_1} \\ &\qquad \underbrace{- \Big\langle \beta_7 (\mathbf{D}_1 \cdot Q_0^2 + Q_0^2 \cdot \mathbf{D}_1), \nabla \mathbf{v}_1 \Big\rangle}_{I_2} \\ &\qquad \underbrace{- \Big\langle \frac{\mu_2}{2}(\dot{Q}_1^\top - [\mathbf{\Omega}_1, Q_0]) + \mu_1 \big[Q_0, (\dot{Q}_1^\top - [\mathbf{\Omega}_1, Q_0])\big], \nabla \mathbf{v}_1 \Big\rangle}_{I_3} \\ &\qquad \underbrace{- \mu_1 \langle \dot{Q}_1^\top - \mathscr{C}_1, \dot{Q}_1^\top \rangle - \Big\langle \mathscr{A} + \frac{\mu_2}{2}\mathscr{D}_1, \dot{Q}_1^\top \Big\rangle}_{I_4} \underbrace{- J\langle \mathscr{U}, \dot{Q}_1^\top \rangle}_{I_5} \\ &\qquad + \underbrace{\Big\langle L(\dot{Q}_1^\top) + L\left(Q_1^\top, \mathbf{v}_1\right) + R, \dot{Q}_1^\top \Big\rangle + \Big\langle \sigma^d(Q_1^\top, Q_0) + \sigma^d(Q_0, Q_1^\top), \nabla \mathbf{v}_1 \Big\rangle}_{I_6} \\ &\qquad + \underbrace{\Big\langle L\left(Q_1^\top, \mathbf{v}_1\right) + R, \nabla \mathbf{v}_1 \Big\rangle}_{I_7}. \end{aligned} \tag{2.34}$$

We next estimate the right-hand side of (2.34) term by term. Using $Q_0 = s(\mathbf{nn} - \frac{1}{3}\mathbf{I})$ and the relation $\beta_6 - \beta_5 = \mu_2$ in (1.16), noting that $\langle [\mathbf{D}_1, Q_0], \mathbf{D}_1 \rangle = 0$, we obtain that

$$
\begin{aligned}
I_1+I_2=&-\left\langle \beta_1 Q_0(Q_0:\mathbf{D}_1)+\beta_4\mathbf{D}_1+\frac{\beta_5+\beta_6}{2}(Q_0\cdot\mathbf{D}_1+\mathbf{D}_1\cdot Q_0),\mathbf{D}_1\right\rangle\\
&-\left\langle \beta_7(\mathbf{D}_1\cdot Q_0^2+Q_0^2\cdot\mathbf{D}_1),\mathbf{D}_1\right\rangle\\
&+\left\langle\left(\frac{\beta_5+\beta_6}{2}-\beta_5\right)\mathbf{D}_1\cdot Q_0+\left(\frac{\beta_5+\beta_6}{2}-\beta_6\right)Q_0\cdot\mathbf{D}_1,\mathbf{D}_1+\mathbf{\Omega}_1\right\rangle\\
=&-\beta_1 s^2\|\mathbf{nn}:\mathbf{D}_1\|_{L^2}^2-\left(\beta_4-\frac{s(\beta_5+\beta_6)}{3}+\frac{2}{9}\beta_7 s^2\right)\|\mathbf{D}_1\|_{L^2}^2\\
&-\left(s(\beta_5+\beta_6)+\frac{2}{3}\beta_7 s^2\right)\|\mathbf{n}\cdot\mathbf{D}_1\|_{L^2}^2+\frac{\mu_2}{2}\langle[\mathbf{\Omega}_1,Q_0],\mathbf{D}_1\rangle.
\end{aligned}
$$

Making use of $\mathscr{P}^{in}(\dot{Q}_1^\top)=\dot{Q}_1^\top+L(Q_1^\top)$ and the self-adjoint property of the projection operator yields that

$$
\begin{aligned}
-\langle\mathscr{A},\dot{Q}_1^\top\rangle&=-\left\langle\mathscr{L}\left(Q_1^\top\right),\dot{Q}_1^\top+L\left(Q_1^\top\right)\right\rangle\\
&=-\left\langle\mathscr{L}\left(Q_1^\top\right),\partial_t Q_1^\top\right\rangle-\left\langle\mathscr{L}\left(Q_1^\top\right),\mathbf{v}_0\cdot\nabla Q_1^\top\right\rangle-\left\langle\mathscr{L}\left(Q_1^\top\right),L\left(Q_1^\top\right)\right\rangle\\
&\le-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle Q_1^\top,\mathscr{L}\left(Q_1^\top\right)\right\rangle+C\|Q_1^\top\|_{H^1}^2. \qquad (2.35)
\end{aligned}
$$

Here we have employed the following fact that, for any $Q\in\mathbb{S}_0^3$,

(2.36)

$$
\begin{aligned}
&-\langle\mathscr{L}(Q),\mathbf{v}_0\cdot\nabla Q\rangle\\
&=\int_{\mathbb{R}^3}v_{0j}Q_{kl,j}\left(L_1\Delta Q_{kl}+\frac{1}{2}(L_2+L_3)\left(Q_{km,ml}+Q_{lm,mk}-\frac{2}{3}\delta_{kl}Q_{ij,ij}\right)\right)\mathrm{d}\mathbf{x}\\
&=\int_{\mathbb{R}^3}\Big(-L_1 v_{0j}Q_{kl,mj}Q_{kl,m}-\frac{1}{2}(L_2+L_3)\left(v_{0j}Q_{kl,lj}Q_{km,m}+v_{0j}Q_{kl,kj}Q_{lm,m}\right)\\
&\qquad -L_1 v_{0j,m}Q_{kl,j}Q_{kl,m}-\frac{1}{2}(L_2+L_3)\left(v_{0j,l}Q_{kl,j}Q_{km,m}+v_{0j,k}Q_{kl,j}Q_{lm,m}\right)\Big)\mathrm{d}\mathbf{x}\\
&=\int_{\mathbb{R}^3}\left(-L_1 v_{0j,m}Q_{kl,j}Q_{kl,m}-\frac{1}{2}(L_2+L_3)\left(v_{0j,l}Q_{kl,j}Q_{km,m}+v_{0j,k}Q_{kl,j}Q_{lm,m}\right)\right)\mathrm{d}\mathbf{x}\\
&\le C\|Q\|_{H^1}^2.
\end{aligned}
$$

In addition, we have

$$
\begin{aligned}
-\frac{\mu_2}{2}\left\langle\mathscr{D}_1,\dot{Q}_1^\top\right\rangle&=-\frac{\mu_2}{2}\left\langle\mathbf{D}_1,\mathscr{P}^{in}\left(\dot{Q}_1^\top\right)\right\rangle\\
&=-\frac{\mu_2}{2}\left\langle\mathbf{D}_1,\dot{Q}_1^\top+L\left(Q_1^\top\right)\right\rangle\\
&\le-\frac{\mu_2}{2}\left\langle\mathbf{D}_1,\dot{Q}_1^\top\right\rangle+\delta\|\nabla\mathbf{v}_1\|_{L^2}^2+C_\delta\|Q_1^\top\|_{L^2}^2. \qquad (2.37)
\end{aligned}
$$

For terms I_3 and I_4, we notice that

$$
\begin{aligned}
\mu_1\left\langle\mathscr{C}_1,\dot{Q}_1^\top\right\rangle&=\mu_1\left\langle[\mathbf{\Omega}_1,Q_0],\mathscr{P}^{in}\left(\dot{Q}_1^\top\right)\right\rangle=\mu_1\left\langle[\mathbf{\Omega}_1,Q_0],\dot{Q}_1^\top+L\left(Q_1^\top\right)\right\rangle\\
&\le\mu_1\left\langle[\mathbf{\Omega}_1,Q_0],\dot{Q}_1^\top\right\rangle+\delta\|\nabla\mathbf{v}_1\|_{L^2}^2+C_\delta\|Q_1^\top\|_{L^2}^2;
\end{aligned}
$$

then from (2.35) and (2.37), we get

$$\begin{aligned}
I_3+I_4 \le& -\frac{\mu_2}{2}\left\langle\left(\dot{Q}_1^\top-[\mathbf{\Omega}_1,Q_0]\right),\mathbf{D}_1\right\rangle-\mu_1\left\langle[Q_0,\mathbf{\Omega}_1],\left(\dot{Q}_1^\top-[\mathbf{\Omega}_1,Q_0]\right)\right\rangle\\
&-\mu_1\left\langle\dot{Q}_1^\top-[\mathbf{\Omega}_1,Q_0],\dot{Q}_1^\top\right\rangle-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle Q_1^\top,\mathscr{L}\left(Q_1^\top\right)\right\rangle-\frac{\mu_2}{2}\left\langle\mathbf{D}_1,\dot{Q}_1^\top\right\rangle\\
&+\delta\|\nabla\mathbf{v}_1\|_{L^2}^2+C_\delta\|Q_1^\top\|_{H^1}^2\\
=& -\mu_2\left\langle\dot{Q}_1^\top-[\mathbf{\Omega}_1,Q_0],\mathbf{D}_1\right\rangle-\frac{\mu_2}{2}\left\langle[\mathbf{\Omega}_1,Q_0],\mathbf{D}_1\right\rangle-\mu_1\|\dot{Q}_1^\top-[\mathbf{\Omega}_1,Q_0]\|_{L^2}^2\\
&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle Q_1^\top,\mathscr{L}\left(Q_1^\top\right)\right\rangle+\delta\|\nabla\mathbf{v}_1\|_{L^2}^2+C_\delta\|Q_1^\top\|_{H^1}^2\\
\le& -\mu_1\left\|\dot{Q}_1^\top-[\mathbf{\Omega}_1,Q_0]+\frac{\mu_2}{2\mu_1}\mathbf{D}_1\right\|_{L^2}^2+\frac{\mu_2^2}{4\mu_1}\|\mathbf{D}_1\|_{L^2}^2-\frac{\mu_2}{2}\left\langle[\mathbf{\Omega}_1,Q_0],\mathbf{D}_1\right\rangle\\
&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle Q_1^\top,\mathscr{L}\left(Q_1^\top\right)\right\rangle+\delta\|\nabla\mathbf{v}_1\|_{L^2}^2+C_\delta\|Q_1^\top\|_{H^1}^2.
\end{aligned}$$

For term I_5, using (2.27) and basic properties of the projection operator $\mathscr{P}^{in}$, and integration by parts, we deduce that

$$\begin{aligned}
I_5 =& -J\left\langle\dot{\mathbf{v}}_1\cdot\nabla Q_0,\mathscr{P}^{in}\left(\dot{Q}_1^\top\right)\right\rangle\\
=& -J\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\mathbf{v}_1\cdot\nabla Q_0,\mathscr{P}^{in}\left(\dot{Q}_1^\top\right)\right\rangle+J\left\langle\mathbf{v}_1\cdot\dot{\overline{\nabla Q_0}},\mathscr{P}^{in}\left(\dot{Q}_1^\top\right)\right\rangle\\
&+J\left\langle\mathbf{v}_1\cdot\nabla Q_0,(\partial_t+\mathbf{v}_0\cdot\nabla)\,\mathscr{P}^{in}\left(\dot{Q}_1^\top\right)\right\rangle\\
\le& -J\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\mathbf{v}_1\cdot\nabla Q_0,\mathscr{P}^{in}\left(\dot{Q}_1^\top\right)\right\rangle+J\left\langle\mathscr{P}^{in}\left(\mathbf{v}_1\cdot\nabla Q_0\right),\ddot{Q}_1^\top\right\rangle+C\left(\|\mathbf{v}_1\|_{L^2}^2+\|\dot{Q}_1^\top\|_{L^2}^2\right)\\
=& -J\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\mathbf{v}_1\cdot\nabla Q_0,\mathscr{P}^{in}\left(\dot{Q}_1^\top\right)\right\rangle-\mu_1\left\langle\mathscr{P}^{in}\left(\mathbf{v}_1\cdot\nabla Q_0\right),\dot{Q}_1^\top\right\rangle-\left\langle\mathbf{v}_1\cdot\nabla Q_0,\mathscr{L}\left(Q_1^\top\right)\right\rangle\\
&-\left\langle\mathbf{v}_1\cdot\nabla Q_0,\frac{\mu_2}{2}\mathbf{D}_1+\mu_1[\mathbf{\Omega}_1,Q_0]\right\rangle-J\left\langle\mathbf{v}_1\cdot\nabla Q_0,\dot{\mathbf{v}}_1\cdot\nabla Q_0\right\rangle\\
&+\left\langle\mathscr{P}^{in}(\mathbf{v}_1\cdot\nabla Q_0),L\left(\dot{Q}_1^\top\right)+L\left(Q_1^\top,\mathbf{v}_1\right)+R\right\rangle+C\left(\|\mathbf{v}_1\|_{L^2}^2+\|\dot{Q}_1^\top\|_{L^2}^2\right)\\
\le& -J\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\mathbf{v}_1\cdot\nabla Q_0,\mathscr{P}^{in}\left(\dot{Q}_1^\top\right)\right\rangle-\frac{J}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}_1\cdot\nabla Q_0\|_{L^2}^2\\
&+\delta\|\nabla\mathbf{v}_1\|_{L^2}^2+C_\delta\left(\|\mathbf{v}_1\|_{L^2}^2+\|\dot{Q}_1^\top\|_{L^2}^2+\|Q_1^\top\|_{H^1}^2+\|R\|_{L^2}^2\right),
\end{aligned}$$

where $\dot{\overline{\nabla Q_0}}=(\partial_t+\mathbf{v}_0\cdot\nabla)\nabla Q_0$ and we have utilized the following estimate

$$\begin{aligned}
-J\langle\mathbf{v}_1\cdot\nabla Q_0,\dot{\mathbf{v}}_1\cdot\nabla Q_0\rangle =& -\frac{J}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}_1\cdot\nabla Q_0\|_{L^2}^2+J\left\langle\mathbf{v}_1\cdot\nabla Q_0,\mathbf{v}_1\cdot\dot{\overline{\nabla Q_0}}\right\rangle\\
\le& -\frac{J}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}_1\cdot\nabla Q_0\|_{L^2}^2+C\|\mathbf{v}_1\|_{L^2}^2.
\end{aligned}$$

For terms I_6 and I_7, we have

$$I_6+I_7\le\delta\|\nabla\mathbf{v}_1\|_{L^2}^2+C_\delta\left(\|\mathbf{v}_1\|_{L^2}^2+\|\dot{Q}_1^\top\|_{L^2}^2+\|Q_1^\top\|_{H^1}^2+\|R\|_{L^2}^2\right).$$

Putting all the above estimates together and using Lemma 2.5, we obtain that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left(|\mathbf{v}_1|^2+J\Big(|\dot{Q}_1^\top|^2+|\mathbf{v}_1\cdot\nabla Q_0|^2\Big)+Q_1^\top:\mathscr{L}\left(Q_1^\top\right)\right)\mathrm{d}\mathbf{x}\\
&\quad+\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\mathbf{v}_1\cdot\nabla Q_0,\mathscr{P}^{in}\left(\dot{Q}_1^\top\right)\right\rangle\\
&\leq-\tilde{\beta}_1\|\mathbf{nn}:\mathbf{D}_1\|_{L^2}^2-\tilde{\beta}_2\|\mathbf{D}_1\|_{L^2}^2-\tilde{\beta}_3\|\mathbf{n}\cdot\mathbf{D}_1\|_{L^2}^2-5\delta\|\nabla\mathbf{v}_1\|_{L^2}^2\\
&\quad+4\delta\|\nabla\mathbf{v}_1\|_{L^2}^2+C_\delta\left(\|\mathbf{v}_1\|_{L^2}^2+\|\dot{Q}_1^\top\|_{L^2}^2+\|Q_1^\top\|_{H^1}^2+\|R\|_{L^2}^2\right)\\
&\leq-\delta\|\nabla\mathbf{v}_1\|_{L^2}^2+C_\delta\left(\|\mathbf{v}_1\|_{L^2}^2+\|\dot{Q}_1^\top\|_{L^2}^2+\|Q_1^\top\|_{H^1}^2+\|R\|_{L^2}^2\right),
\end{aligned}
\tag{2.38}
$$

where the coefficients $\tilde{\beta}_i(i=1,2,3)$ are given by

$$
\begin{cases}
\tilde{\beta}_1=\beta_1 s^2,\quad \tilde{\beta}_2=\beta_4-5\delta-\dfrac{s(\beta_5+\beta_6)}{3}+\dfrac{2}{9}\beta_7 s^2-\dfrac{\mu_2^2}{4\mu_1},\\
\tilde{\beta}_3=s(\beta_5+\beta_6)+\dfrac{2}{3}\beta_7 s^2,
\end{cases}
\tag{2.39}
$$

and $\delta>0$ is small enough, such that $\tilde{\beta}_1,\tilde{\beta}_2,\tilde{\beta}_3$ satisfy the relation (2.23) (notice that (2.23) holds with strictly positive sign when $\delta=0$). Notice that

$$
\begin{aligned}
&|\mathbf{v}_1|^2+J\Big(|\dot{Q}_1^\top|^2+|\mathbf{v}_1\cdot\nabla Q_0|^2\Big)+2J(\mathbf{v}_1\cdot\nabla Q_0):\mathscr{P}^{in}(\dot{Q}_1^\top)\\
&\quad=|\mathbf{v}_1|^2+J\Big(|\mathscr{P}^{in}(\dot{Q}_1^\top)+\mathbf{v}_1\cdot\nabla Q_0|^2+|\mathscr{P}^{out}(\dot{Q}_1^\top)|^2\Big)\\
&\quad\geq\frac{1}{2}|\mathbf{v}_1|^2+C(|\nabla Q_0|)|\dot{Q}_1^\top|^2.
\end{aligned}
$$

Therefore, combining (2.33) and (2.38), and choosing suitable $M>0$, such that

$$
\begin{aligned}
M\left(\frac{1}{2}|\mathbf{v}_1|^2+C(|\nabla Q_0|)|\dot{Q}_1^\top|^2\right)+J\dot{Q}_1^\top:Q_1^\top+J(\mathbf{v}_1\cdot\nabla Q_0):Q_1^\top+\frac{\mu_1}{2}|Q_1^\top|^2\\
\geq C(\|\nabla Q_0\|_{L^\infty})\left(|\mathbf{v}_1|^2+|\dot{Q}_1^\top|^2+|Q_1^\top|^2\right),
\end{aligned}
$$

we obtain the following energy estimate:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(t)\leq C(\|\nabla Q_0\|_{L^\infty})(\mathscr{E}(t)+\|R(t)\|_{L^2}).
$$

The estimate of the higher order derivative for $(\mathbf{v}_1,Q_1)$ can also be established by introducing a similar energy functional. Therefore, the solution $(\mathbf{v}_1,Q_1)$ is uniquely determined. In a similar argument, we can solve $(\mathbf{v}_2,Q_2)$ and Q_3 by (2.16)–(2.17). Here we omit the details. ∎

3. The estimate for the remainder.

The main task of this section is to derive the remainder system and the uniform estimates for the remainder. Proposition 2.4 tells us that $(\mathbf{v}_i,\partial_t Q_i,\nabla Q_i)\in L^\infty([0,T];H^{k-4i})$ for $i=0,1,2$ and $Q_3\in L^\infty([0,T];H^{k-11})$. Hence, in what follows, $\mathbf{v}_i$ and Q_i will be treated as known functions. We denote by C a constant depending on $\sum_{i=0}^2\sup_{t\in[0,T]}\|\mathbf{v}_i(t)\|_{H^{k-4i}}$ and $\sum_{i=0}^3\sup_{t\in[0,T]}\|Q_i(t)\|_{H^{k+1-4i}}$, and independent of ε.

3.1. The system for the remainder.

Recalling the Hilbert expansions (2.5)–(2.6), we have

$$
Q_R=\varepsilon^{-3}(Q^\varepsilon-\widetilde{Q}),\quad \mathbf{v}_R=\varepsilon^{-3}(\mathbf{v}^\varepsilon-\tilde{\mathbf{v}}),
\tag{3.1}
$$

where Q_R and $\mathbf{v}_R$ depend on ε. In order to derive the system of the remainder (3.1), we denote

$$\begin{aligned}\widetilde{\mathbf{D}} =&\mathbf{D}_0+\varepsilon\mathbf{D}_1+\varepsilon^2\mathbf{D}_2, \quad \widetilde{\mathbf{\Omega}}=\mathbf{\Omega}_0+\varepsilon\mathbf{\Omega}+\varepsilon^2\mathbf{\Omega}_2,\\ \dot{Q}_R =&(\partial_t+\tilde{\mathbf{v}}\cdot\nabla)Q_R, \quad \ddot{Q}_R=(\partial_t+\tilde{\mathbf{v}}\cdot\nabla)\dot{Q}_R.\end{aligned}$$

From (2.7)–(2.8) and the definitions of $\mathbf{H}_i (i=0,1,2)$, the molecular field $\mathbf{H}(Q^\varepsilon)$ can be expanded into

$$\mathbf{H}(Q^\varepsilon)=-\varepsilon^{-1}\mathscr{T}(Q^\varepsilon)-\mathscr{L}(Q^\varepsilon)=-\mathbf{H}_0-\varepsilon\mathbf{H}_1-\varepsilon^2\mathbf{H}_2-\varepsilon^2\mathbf{H}_R-\varepsilon^3\mathscr{T}_R^\varepsilon,$$

where $\mathbf{H}_R=\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R)\overset{\text{def}}{=}\mathscr{H}_{\mathbf{n}}(Q_R)+\varepsilon\mathscr{L}(Q_R)$.

Therefore, from (1.20)–(1.22) and (2.9)–(2.18), the system for the remainder can be derived as follows:

$$J\ddot{Q}_R+\mu_1\dot{Q}_R=-\varepsilon^{-1}\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R)-\frac{\mu_2}{2}\mathbf{D}_R+\mu_1[\mathbf{\Omega}_R,Q_0]+\mathbf{F}_R+\widetilde{\mathbf{F}}_R, \tag{3.2}$$

$$\begin{aligned}\frac{\partial\mathbf{v}_R}{\partial t}+\tilde{\mathbf{v}}\cdot\nabla\mathbf{v}_R=&-\nabla p_R+\nabla\cdot\Big(\beta_1Q_0(Q_0:\mathbf{D}_R)+\beta_4\mathbf{D}_R+\beta_5Q_0\cdot\mathbf{D}_R\\ &+\beta_6\mathbf{D}_R\cdot Q_0+\beta_7(\mathbf{D}_R\cdot Q_0^2+Q_0^2\cdot\mathbf{D}_R)\\ &+\frac{\mu_2}{2}(\dot{Q}_R-[\mathbf{\Omega}_R,Q_0])+\mu_1\big[Q_0,(\dot{Q}_R-[\mathbf{\Omega}_R,Q_0])\big]\Big)\\ &+\nabla\cdot\mathbf{G}_R+\mathbf{G}'_R,\end{aligned} \tag{3.3}$$

$$\nabla\cdot\mathbf{v}_R=0. \tag{3.4}$$

The term $\mathbf{F}_R$ is given by

$$\mathbf{F}_R=\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3+\mathbf{F}_4+\mathbf{F}_5,$$

where $\mathbf{F}_1$ is independent of $(\mathbf{v}_R,Q_R)$,

$$\begin{aligned}\mathbf{F}_1=J\Bigg(&-\partial_t^2Q_3-2\mathbf{v}_0\cdot\nabla\partial_tQ_3-2\mathbf{v}_1\cdot\nabla\partial_t(Q_2+\varepsilon Q_3)-2\mathbf{v}_2\cdot\nabla\partial_t\widehat{Q}^\varepsilon-\partial_t\mathbf{v}_0\cdot\nabla Q_3\\ &-\partial_t\mathbf{v}_1\cdot\nabla(Q_2+\varepsilon Q_3)-\partial_t\mathbf{v}_2\cdot\nabla\widehat{Q}^\varepsilon-\sum_{i+j+k\geq3}\varepsilon^{i+j+k-3}\mathbf{v}_i\cdot\nabla(\mathbf{v}_j\cdot\nabla Q_k)\Bigg)\\ &+\mu_1\Big(-\partial_tQ_3-\mathbf{v}_0\cdot\nabla Q_3-\mathbf{v}_1\cdot\nabla(Q_2+\varepsilon Q_3)-\mathbf{v}_2\cdot\nabla\widehat{Q}^\varepsilon\Big)\\ &+\mu_1\Bigg(\sum_{i+j\geq3}\varepsilon^{i+j-3}(\mathbf{\Omega}_i\cdot Q_j-Q_j\cdot\mathbf{\Omega}_i)\Bigg)-\mathbf{B}^\varepsilon-\mathscr{L}(Q_3)\\ \equiv&-J\mathbf{F}_{11}-\mu_1\mathbf{F}_{12}-\mathbf{B}^\varepsilon-\mathscr{L}(Q_3),\end{aligned}$$

and $\mathbf{F}_2,\mathbf{F}_3$ linearly depend on $(\mathbf{v}_R,Q_R)$,

$$\begin{aligned}\mathbf{F}_2=&\mu_1(\widetilde{\mathbf{\Omega}}\cdot Q_R-Q_R\cdot\widetilde{\mathbf{\Omega}})-\Big(-b\mathscr{B}(\widehat{Q}^\varepsilon,Q_R)+c\mathscr{C}(Q_R,\widehat{Q}^\varepsilon,Q_0)+\frac{c}{2}\varepsilon\mathscr{C}(Q_R,\widehat{Q}^\varepsilon,\widehat{Q}^\varepsilon)\Big),\\ \mathbf{F}_3=&-J\mathbf{v}_R\cdot\nabla(\partial_t\widetilde{Q}+\tilde{\mathbf{v}}\cdot\nabla\widetilde{Q})+\mu_1\Big(-\mathbf{v}_R\cdot\nabla\widetilde{Q}+\varepsilon\mathbf{\Omega}_R\cdot\widehat{Q}^\varepsilon-\varepsilon\widehat{Q}^\varepsilon\cdot\mathbf{\Omega}_R\Big)\\ \equiv&-J\mathbf{F}_{31}-\mu_1\mathbf{F}_{32},\end{aligned}$$

and $\mathbf{F}_4,\mathbf{F}_5$ nonlinearly depend on $(\mathbf{v}_R,Q_R)$,

$$\begin{aligned}\mathbf{F}_4 =& -\varepsilon^3 J\mathbf{v}_R\cdot\nabla(\mathbf{v}_R\cdot\nabla\widetilde{Q}) + \varepsilon^3\mu_1\Big(-\mathbf{v}_R\cdot\nabla Q_R + \mathbf{\Omega}_R\cdot Q_R - Q_R\cdot\mathbf{\Omega}_R\Big)\\ \equiv& -\varepsilon^3 J\mathbf{F}_{41} - \varepsilon^3\mu_1\mathbf{F}_{42},\\ \mathbf{F}_5 =& -\left(-\frac{b}{2}\varepsilon^2\mathscr{B}(Q_R,Q_R) + c\varepsilon^2\mathscr{C}(Q_R,Q_R,\widetilde{Q}) + c\varepsilon^5\mathscr{C}(Q_R,Q_R,Q_R)\right).\end{aligned}$$

The term $\widetilde{\mathbf{F}}_R$, including the derivative term with respect to time t, is given by

$$\begin{aligned}\widetilde{\mathbf{F}}_R =& -J(\partial_t + \tilde{\mathbf{v}}\cdot\nabla)(\mathbf{v}_R\cdot\nabla\widetilde{Q})\\ &+ \varepsilon^3 J\Big(-(\partial_t+\tilde{\mathbf{v}}\cdot\nabla)(\mathbf{v}_R\cdot\nabla Q_R) - \mathbf{v}_R\cdot\nabla\dot{Q}_R - \varepsilon^3\mathbf{v}_R\cdot\nabla(\mathbf{v}_R\cdot\nabla Q_R)\Big).\end{aligned}$$

On the other hand, the term $\mathbf{G}'_R$ takes the following form:

$$\mathbf{G}'_R = -\mathbf{v}_1\cdot\nabla\mathbf{v}_2 - \mathbf{v}_2\cdot\nabla\mathbf{v}_1 - \varepsilon\mathbf{v}_2\cdot\nabla\mathbf{v}_2 - \mathbf{v}_R\cdot\nabla\tilde{\mathbf{v}} - \varepsilon^3\mathbf{v}_R\cdot\nabla\mathbf{v}_R.$$

Similarly, the term $\mathbf{G}_R$ can be written as

$$\mathbf{G}_R = \mathbf{G}_1 + \mathbf{G}_2 + \mathbf{G}_3,$$

where $\mathbf{G}_1$ is given by

$$\begin{aligned}\mathbf{G}_1 =& \sum_{i+j+k\geq 3}\varepsilon^{i+j+k-3}\Big(\beta_1 Q_i(Q_j:\mathbf{D}_k) + \beta_7\Big(\mathbf{D}_i\cdot Q_j\cdot Q_k + Q_i\cdot Q_j\cdot\mathbf{D}_k\Big)\Big)\\ &+ \sum_{i+j\geq 3}\varepsilon^{i+j-3}\Big(\beta_5\mathbf{D}_i\cdot Q_j + \beta_6 Q_i\cdot\mathbf{D}_j + \sigma^d(Q_i,Q_j)\Big) + \frac{\mu_2}{2}\mathbf{F}_{12}\\ &+ \mu_1\left(\sum_{i+j\geq 3}\varepsilon^{i+j-3}[Q_i,\partial_t Q_j] + \sum_{i+j+k\geq 3}\varepsilon^{i+j+k-3}\Big[Q_i,\big(\mathbf{v}_j\cdot\nabla Q_k - [\mathbf{\Omega}_j, Q_k]\big)\Big]\right),\end{aligned}$$

and $\mathbf{G}_2, \mathbf{G}_3$ are given by

$$\begin{aligned}\mathbf{G}_2 =& \beta_1\Big(\widetilde{Q}(Q_R:\widetilde{\mathbf{D}}) + Q_R(\widetilde{Q}:\widetilde{\mathbf{D}}) + \varepsilon Q_0(\widehat{Q}^\varepsilon:\mathbf{D}_R) + \varepsilon\widehat{Q}^\varepsilon(\widetilde{Q}:\mathbf{D}_R)\Big)\\ &+ \beta_5(\widetilde{\mathbf{D}}\cdot Q_R + \varepsilon\mathbf{D}_R\cdot\widehat{Q}^\varepsilon) + \beta_6(\varepsilon\widehat{Q}^\varepsilon\cdot\mathbf{D}_R + Q_R\cdot\widetilde{\mathbf{D}})\\ &+ \beta_7\Big(\widetilde{\mathbf{D}}\cdot Q_R\cdot\widetilde{Q} + \widetilde{\mathbf{D}}\cdot\widetilde{Q}\cdot Q_R + \varepsilon\mathbf{D}_R\cdot\widehat{Q}^\varepsilon\cdot\widetilde{Q} + \varepsilon\mathbf{D}_R\cdot Q_0\cdot\widehat{Q}^\varepsilon\Big)\\ &+ \beta_7\Big(\widetilde{Q}\cdot Q_R\cdot\widetilde{\mathbf{D}} + Q_R\cdot\widetilde{Q}\cdot\widetilde{\mathbf{D}} + \varepsilon\widehat{Q}^\varepsilon\cdot Q_0\cdot\mathbf{D}_R + \varepsilon\widetilde{Q}\cdot\widehat{Q}^\varepsilon\cdot\mathbf{D}_R\Big)\\ &+ \frac{\mu_2}{2}(\mathbf{F}_{32} - [\widetilde{\mathbf{\Omega}}, Q_R]) + \mu_1\Big[Q_R,\big(\partial_t\widetilde{Q} + \tilde{\mathbf{v}}\cdot\nabla\widetilde{Q} - [\widetilde{\mathbf{\Omega}},\widetilde{Q}]\big)\Big]\\ &+ \mu_1\Big[\widetilde{Q},\big(\mathbf{v}_R\cdot\nabla\widetilde{Q} - [\widetilde{\mathbf{\Omega}}, Q_R]\big)\Big] + \mu_1\big[\varepsilon\widehat{Q}^\varepsilon,(\dot{Q}_R - [\mathbf{\Omega}_R, Q_0])\big]\\ &- \mu_1\big[\widetilde{Q},[\mathbf{\Omega}_R,\varepsilon\widehat{Q}^\varepsilon]\big] + \sigma^d(\widetilde{Q},Q_R) + \sigma^d(Q_R,\widetilde{Q}),\\ \mathbf{G}_3 =& \varepsilon^3\Big(\beta_1\Big(\widetilde{Q}(Q_R:\mathbf{D}_R) + Q_R(\widetilde{Q}:\mathbf{D}_R) + Q_R(Q_R:\widetilde{\mathbf{D}}) + \varepsilon^3 Q_R(Q_R:\mathbf{D}_R)\Big)\\ &+ \beta_7\Big(\widetilde{\mathbf{D}}\cdot Q_R\cdot Q_R + \mathbf{D}_R\cdot\widetilde{Q}\cdot Q_R + \mathbf{D}_R\cdot Q_R\cdot\widetilde{Q} + \varepsilon^3\mathbf{D}_R\cdot Q_R\cdot Q_R\Big)\\ &+ \beta_7\Big(\widetilde{Q}\cdot Q_R\cdot\mathbf{D}_R + Q_R\cdot\widetilde{Q}\cdot\mathbf{D}_R + Q_R\cdot Q_R\cdot\widetilde{\mathbf{D}} + \varepsilon^3 Q_R\cdot Q_R\cdot\mathbf{D}_R\Big)\\ &+ \beta_5\mathbf{D}_R\cdot Q_R + \beta_6 Q_R\cdot\mathbf{D}_R + \frac{\mu_2}{2}\mathbf{F}_{42} + \mu_1\Big[\widetilde{Q},\big(\mathbf{v}_R\cdot\nabla Q_R - [\mathbf{\Omega}_R, Q_R]\big)\Big]\end{aligned}$$

$$+\mu_1\Big[Q_R,\big(\dot{Q}_R+\mathbf{v}_R\cdot\nabla\widetilde{Q}-[\widetilde{\mathbf{\Omega}},Q_R]-[\mathbf{\Omega}_R,\widetilde{Q}]\big)\Big]$$
$$+\mu_1\varepsilon^3\Big[Q_R,\big(\mathbf{v}_R\cdot\nabla Q_R-[\mathbf{\Omega}_R,Q_R]\big)\Big]+\sigma^d(Q_R,Q_R)\Big).$$

3.2. Uniform estimates for the remainder. In this subsection, we derive the uniform estimates for the remainder. We assume that $(\mathbf{v}_R,Q_R)$ is a smooth solution of the remainder system (3.2)–(3.4) and introduce the following energy functional:

$$\begin{aligned}\mathfrak{E}(t)\stackrel{\text{def}}{=}\int_{\mathbb{R}^3}&\Big(|\mathbf{v}_R|^2+|\dot{Q}_R|^2+|Q_R|^2+\varepsilon^{-1}\mathscr{H}^{\varepsilon}_{\mathbf{n}}(Q_R):Q_R\Big)\\&+\varepsilon^2\Big(|\nabla\mathbf{v}_R|^2+|\partial_i\dot{Q}_R|^2+\varepsilon^{-1}\mathscr{H}^{\varepsilon}(\partial_iQ_R):\partial_iQ_R\Big)\\&+\varepsilon^4\Big(|\Delta\mathbf{v}_R|^2+|\Delta\dot{Q}_R|^2+\varepsilon^{-1}\mathscr{H}^{\varepsilon}(\Delta Q_R):\Delta Q_R\Big)\mathrm{d}\mathbf{x},\end{aligned}\tag{3.5}$$

$$\mathfrak{F}(t)\stackrel{\text{def}}{=}\int_{\mathbb{R}^3}\delta\Big(|\nabla\mathbf{v}_R|^2+\varepsilon^2|\Delta\mathbf{v}_R|^2+\varepsilon^4|\nabla\Delta\mathbf{v}_R|^2\Big)\mathrm{d}\mathbf{x}.\tag{3.6}$$

By using the definitions of $\mathfrak{E}$ and $\mathfrak{F}$, we can immediately obtain the following.

Lemma 3.1. *The following estimates hold:*

$$\begin{aligned}\|(\varepsilon\nabla^2Q_R,\varepsilon^2\nabla^3Q_R)\|_{L^2}+\|(\mathbf{v}_R,\varepsilon\nabla\mathbf{v}_R,\varepsilon^2\nabla^2\mathbf{v}_R)\|_{L^2}&\le C\mathfrak{E}^{\frac12},\\\|Q_R\|_{H^1}+\|(\dot{Q}_R,\varepsilon\partial_i\dot{Q}_R,\varepsilon^2\Delta\dot{Q}_R)\|_{L^2}&\le C\mathfrak{E}^{\frac12},\\\|(\nabla\mathbf{v}_R,\varepsilon\nabla^2\mathbf{v}_R,\varepsilon^2\nabla^3\mathbf{v}_R)\|_{L^2}&\le C\mathfrak{F}^{\frac12}.\end{aligned}$$

In order to establish the estimates of the remainder terms $(\mathbf{F}_R,\mathbf{G}_R)$ and $\mathbf{G}'_R$, it is desirable to utilize the following inequality:

$$\|fg\|_{H^k}\le C\|f\|_{H^2}\|g\|_{H^k},\quad k=0,1,2.\tag{3.7}$$

Lemma 3.2. *For the remainder term $\mathbf{F}_R$, the following estimate holds:*

$$\|(\mathbf{F}_R,\varepsilon\nabla\mathbf{F}_R,\varepsilon^2\Delta\mathbf{F}_R)\|_{L^2}\le C\left(1+\mathfrak{E}^{\frac12}+\varepsilon\mathfrak{E}+\varepsilon^3\mathfrak{E}^{\frac32}+\varepsilon\mathfrak{F}^{\frac12}+\varepsilon\mathfrak{E}^{\frac12}\mathfrak{F}^{\frac12}\right).$$

Proof. Applying Lemma 3.1, we see at once that

$$\begin{aligned}&\|(\mathbf{F}_1,\varepsilon\nabla\mathbf{F}_1,\varepsilon^2\Delta\mathbf{F}_1)\|_{L^2}\le C,\\&\|(\mathbf{F}_2,\varepsilon\nabla\mathbf{F}_2,\varepsilon^2\Delta\mathbf{F}_2)\|_{L^2}\le C\mathfrak{E}^{\frac12},\\&\|(\mathbf{F}_3,\varepsilon\nabla\mathbf{F}_3,\varepsilon^2\Delta\mathbf{F}_3)\|_{L^2}\le C\big(\mathfrak{E}^{\frac12}+\varepsilon\mathfrak{F}^{\frac12}\big).\end{aligned}$$

Using the inequality (3.7), we have

$$\begin{aligned}\|\mathbf{F}_4\|_{H^k}\le{}&C\varepsilon\|\mathbf{v}_R\|_{H^k}\big(\|\varepsilon^2\nabla\mathbf{v}_R\|_{H^2}+\|\varepsilon^2\mathbf{v}_R\|_{H^2}\big)\\&+C\varepsilon\|\mathbf{v}_R\|_{H^k}\|\varepsilon^2\nabla Q_R\|_{H^2}+C\varepsilon^2\|\varepsilon Q_R\|_{H^2}\|\nabla\mathbf{v}_R\|_{H^k},\end{aligned}$$

which implies

$$\|(\mathbf{F}_4,\varepsilon\nabla\mathbf{F}_4,\varepsilon^2\Delta\mathbf{F}_4)\|_{L^2}\le C\varepsilon(\mathfrak{E}+\mathfrak{E}^{\frac12}\mathfrak{F}^{\frac12}).$$

Similarly, from Lemma 3.1 and (3.7) again, we can infer that

$$\|(\mathbf{F}_5,\varepsilon\nabla\mathbf{F}_5,\varepsilon^2\Delta\mathbf{F}_5)\|_{L^2}\le C\varepsilon(\mathfrak{E}+\varepsilon^2\mathfrak{E}^{\frac32}).$$

The proof is finished. ∎

LEMMA 3.3. *For the remainder term $\mathbf{G}_R$, the following estimates hold:*

$$\|(\mathbf{G}_R, \varepsilon\nabla\mathbf{G}_R, \varepsilon^2\Delta\mathbf{G}_R)\|_{L^2} \le C\left(1 + \mathfrak{E}^{\frac{1}{2}} + \varepsilon\mathfrak{E} + \varepsilon^3\mathfrak{E}^{\frac{3}{2}} + \varepsilon\mathfrak{F}^{\frac{1}{2}} + \varepsilon^2\mathfrak{E}^{\frac{1}{2}}\mathfrak{F}^{\frac{1}{2}} + \varepsilon^4\mathfrak{E}\mathfrak{F}^{\frac{1}{2}}\right),$$
$$\|(\mathbf{G}'_R, \varepsilon\nabla\mathbf{G}'_R, \varepsilon^2\Delta\mathbf{G}'_R)\|_{L^2} \le C\left(1 + \mathfrak{E}^{\frac{1}{2}} + \mathfrak{F}^{\frac{1}{2}} + \varepsilon\mathfrak{E}^{\frac{1}{2}}\mathfrak{F}^{\frac{1}{2}}\right).$$

Proof. It is straightforward to show from Lemma 3.1 that

$$\|\left(\mathbf{G}_1, \varepsilon\nabla\mathbf{G}_1, \varepsilon^2\Delta\mathbf{G}_1\right)\|_{L^2} \le C,$$
$$\|\left(\mathbf{G}_2, \varepsilon\nabla\mathbf{G}_2, \varepsilon^2\Delta\mathbf{G}_2\right)\|_{L^2} \le C\big(\mathfrak{E}^{\frac{1}{2}} + \varepsilon\mathfrak{F}^{\frac{1}{2}}\big).$$

By the inequality (3.7), we obtain

$$\begin{aligned}\|\mathbf{G}_3\|_{H^k} \le C\varepsilon^2\|\varepsilon Q_R\|_{H^2}\Big(&\|\nabla\mathbf{v}_R\|_{H^k} + \|Q_R\|_{H^k} + \varepsilon^3\|Q_R : \mathbf{D}_R\|_{H^k} + \|\dot{Q}_R\|_{H^k}\\ &+ \|\mathbf{v}_R\|_{H^k} + \varepsilon^3\left(\|\mathbf{v}_R\cdot\nabla Q_R\|_{H^k} + \|\mathbf{\Omega}_R\cdot Q_R\|_{H^k}\right)\Big)\\ &+ C\varepsilon\|\varepsilon^2\nabla Q_R\|_{H^2}(\|\mathbf{v}_R\|_{H^k} + \|\nabla Q_R\|_{H^k}),\end{aligned}$$

which gives

$$\|(\mathbf{G}_3, \varepsilon\nabla\mathbf{G}_3, \varepsilon^2\Delta\mathbf{G}_3)\|_{L^2} \le C\big(\varepsilon\mathfrak{E} + \varepsilon^3\mathfrak{E}^{\frac{3}{2}} + \varepsilon^2\mathfrak{E}^{\frac{1}{2}}\mathfrak{F}^{\frac{1}{2}} + \varepsilon^4\mathfrak{E}\mathfrak{F}^{\frac{1}{2}}\big).$$

Then the conclusion follows. ∎

We point out that for $m = 0, 1, 2$, highly singular terms $\langle\frac{1}{\varepsilon}\mathscr{H}^{\varepsilon}_{\mathbf{n}}(\partial_i^m Q_R), \partial_i^m Q_R\rangle$ in (3.5) come from the L^2-inner products $\langle\frac{1}{\varepsilon}\partial_i^m\mathscr{H}^{\varepsilon}_{\mathbf{n}}(Q_R), \partial_i^m\dot{Q}_R\rangle$. Fortunately, Lemma 3.4 will play a crucial role in dealing with these singular terms, which makes the whole machinery work.

LEMMA 3.4. *Assume that $(\mathbf{v}_R, Q_R)$ is a smooth solution of the remainder system (3.2)–(3.4). Then for any $\delta > 0$, there exists a constant C depending on $\mathbf{n}, \nabla_{t,x}\mathbf{n}, \tilde{\mathbf{v}}$, and $\widetilde{Q}$ such that*

$$\begin{aligned}\frac{1}{\varepsilon}\left\langle\dot{\overline{\mathbf{nn}}}\cdot Q_R, Q_R\right\rangle \le& -J\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\mathscr{H}^{-1}_{\mathbf{n}}(\dot{\overline{\mathbf{nn}}}\cdot Q_R^{\top}), \dot{Q}_R + \mathbf{v}_R\cdot\nabla Q^{\varepsilon}\right\rangle\\ &+ C(1 + \mathfrak{E} + \varepsilon^2\mathfrak{E}^2) + \left(\delta + C\varepsilon^2\mathfrak{E}\right)\mathfrak{F},\end{aligned}\tag{3.8}$$

$$\begin{aligned}\frac{1}{\varepsilon}\left\langle\left(Q_R : \dot{\overline{\mathbf{nn}}}\right)\mathbf{nn}, Q_R\right\rangle \le& -J\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\mathscr{H}^{-1}_{\mathbf{n}}\left(Q_R^{\top} : \dot{\overline{\mathbf{nn}}}\left(\mathbf{nn} - \frac{1}{3}\mathbf{I}\right)\right), \dot{Q}_R + \mathbf{v}_R\cdot\nabla Q^{\varepsilon}\right\rangle\\ &+ C\left(1 + \mathfrak{E} + \varepsilon^2\mathfrak{E}^2\right) + \left(\delta + C\varepsilon^2\mathfrak{E}\right)\mathfrak{F},\end{aligned}\tag{3.9}$$

where $\dot{\overline{\mathbf{nn}}} \overset{def}{=} (\partial_t + \tilde{\mathbf{v}}\cdot\nabla)(\mathbf{nn})$, $Q^{\varepsilon} = \widetilde{Q} + \varepsilon^3 Q_R$, and $\mathscr{H}^{-1}_{\mathbf{n}}$ is defined by (2.4). Moreover, for $m = 1, 2$, the following estimates hold:

$$\varepsilon^{2m-1}\left\langle\dot{\overline{\mathbf{nn}}}\cdot\partial_i^m Q_R, \partial_i^m Q_R\right\rangle \le C\mathfrak{E},\tag{3.10}$$

$$\varepsilon^{2m-1}\left\langle\left(\partial_i^m Q_R : \dot{\overline{\mathbf{nn}}}\right)\mathbf{nn}, \partial_i^m Q_R\right\rangle \le C\mathfrak{E},\tag{3.11}$$

where ∂_i^m represents the mth order partial derivative operator with respect to the component x_i.

Proof. We assume $Q_R = Q_R^\top + Q_R^\perp$ with $Q_R^\top \in \mathrm{Ker}\mathscr{H}_{\mathbf{n}}$ and $Q_R^\perp \in (\mathrm{Ker}\mathscr{H}_{\mathbf{n}})^\perp$. Then we obtain

$$\langle \dot{\overline{\mathbf{nn}}} \cdot Q_R, Q_R\rangle = \left\langle \dot{\overline{\mathbf{nn}}} \cdot Q_R^\top, Q_R^\top\right\rangle + 2\left\langle \dot{\overline{\mathbf{nn}}} \cdot Q_R^\top, Q_R^\perp\right\rangle + \left\langle \dot{\overline{\mathbf{nn}}} \cdot Q_R^\perp, Q_R^\perp\right\rangle.$$

Note that there holds $\dot{\overline{\mathbf{nn}}} \cdot Q_R^\top \in (\mathrm{Ker}\mathscr{H}_{\mathbf{n}})^\perp$ since $\dot{\overline{\mathbf{nn}}} = \dot{\mathbf{n}}\mathbf{n} + \mathbf{n}\dot{\mathbf{n}} \in \mathrm{Ker}\mathscr{H}_{\mathbf{n}}$. Then we have $\frac{1}{\varepsilon}\langle \dot{\overline{\mathbf{nn}}} \cdot Q_R^\top, Q_R^\top\rangle = 0$. Using Proposition 2.2, it follows that

$$\begin{aligned}\frac{1}{\varepsilon}\left\langle \dot{\overline{\mathbf{nn}}} \cdot Q_R^\perp, Q_R^\perp\right\rangle &\le C\frac{1}{\varepsilon}\|Q_R^\perp\|_{L^2}^2 \le C\frac{1}{\varepsilon}\langle \mathscr{H}_{\mathbf{n}}(Q_R), Q_R\rangle \\ &\le C\left(\frac{1}{\varepsilon}\langle \mathscr{H}_{\mathbf{n}}^\varepsilon(Q_R), Q_R\rangle - \langle \mathscr{L}(Q_R), Q_R\rangle\right) \le C\mathfrak{E}.\end{aligned}$$

It can be seen from Proposition 2.2 that

$$\begin{aligned}\frac{1}{\varepsilon}\left\langle \dot{\overline{\mathbf{nn}}} \cdot Q_R^\top, Q_R^\perp\right\rangle &= \left\langle \mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}} \cdot Q_R^\top), \frac{1}{\varepsilon}\mathscr{H}_{\mathbf{n}}(Q_R)\right\rangle \\ &= \left\langle \mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}} \cdot Q_R^\top), \frac{1}{\varepsilon}\mathscr{H}_{\mathbf{n}}^\varepsilon(Q_R)\right\rangle - \left\langle \mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}} \cdot Q_R^\top), \mathscr{L}(Q_R)\right\rangle \\ &\le \left\langle \mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}} \cdot Q_R^\top), \frac{1}{\varepsilon}\mathscr{H}_{\mathbf{n}}^\varepsilon(Q_R)\right\rangle + C(\|\nabla Q_R\|_{L^2}^2 + \|Q_R\|_{L^2}^2).\end{aligned}$$

From (3.2), we have

$$\begin{aligned}&\left\langle \mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}} \cdot Q_R^\top), \frac{1}{\varepsilon}\mathscr{H}_{\mathbf{n}}^\varepsilon(Q_R)\right\rangle \\ &\quad = \underbrace{-J\left\langle \mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}} \cdot Q_R^\top), \ddot{Q}_R\right\rangle}_{\mathcal{M}_1} \underbrace{-\mu_1\left\langle \mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}} \cdot Q_R^\top), \dot{Q}_R\right\rangle}_{\mathcal{M}_2} \\ &\qquad \underbrace{-\frac{\mu_2}{2}\left\langle \mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}} \cdot Q_R^\top), \mathbf{D}_R\right\rangle}_{\mathcal{M}_3} + \underbrace{\mu_1\left\langle \mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}} \cdot Q_R^\top), [\mathbf{\Omega}_R, Q_0]\right\rangle}_{\mathcal{M}_4} \\ &\qquad + \underbrace{\left\langle \mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}} \cdot Q_R^\top), \mathbf{F}_R\right\rangle}_{\mathcal{M}_5} + \underbrace{\left\langle \mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}} \cdot Q_R^\top), \widetilde{\mathbf{F}}_R\right\rangle}_{\mathcal{M}_6}.\end{aligned}$$

Using integration by parts, we get

$$\begin{aligned}\mathcal{M}_1 &= -J\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}} \cdot Q_R^\top), \dot{Q}_R\right\rangle + J\left\langle (\partial_t + \tilde{\mathbf{v}}\cdot\nabla)\mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}} \cdot Q_R^\top), \dot{Q}_R\right\rangle \\ &\le -J\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}} \cdot Q_R^\top), \dot{Q}_R\right\rangle + C(\|Q_R\|_{L^2}^2 + \|\dot{Q}_R\|_{L^2}^2).\end{aligned}$$

From Lemma 3.1, we can easily estimate that

$$\begin{aligned}\mathcal{M}_2 &\le C\|Q_R\|_{L^2}\|\dot{Q}_R\|_{L^2} \le C\mathfrak{E}, \\ \mathcal{M}_3 + \mathcal{M}_4 &\le C\|Q_R\|_{L^2}\|\nabla\mathbf{v}_R\| \le C\mathfrak{E}^{\frac{1}{2}}\mathfrak{F}^{\frac{1}{2}}, \\ \mathcal{M}_5 &\le C\|Q_R\|_{L^2}\|\mathbf{F}_R\|_{L^2} \le C\mathfrak{E}^{\frac{1}{2}}\|\mathbf{F}_R\|_{L^2}.\end{aligned}$$

For the term $\mathcal{M}_6$, we have

$$\begin{aligned}\mathcal{M}_6 = &-J\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}} \cdot Q_R^\top), \mathbf{v}_R\cdot\nabla\widetilde{Q} + \varepsilon^3\mathbf{v}_R\cdot\nabla Q_R\right\rangle \\ &+ \underbrace{J\left\langle (\partial_t + \tilde{\mathbf{v}}\cdot\nabla)\mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}} \cdot Q_R^\top), \mathbf{v}_R\cdot\nabla\widetilde{Q}\right\rangle}_{\widetilde{\mathcal{M}}_1}\end{aligned}$$

$$+\underbrace{\varepsilon^3 J\Big\langle(\partial_t+\tilde{\mathbf{v}}\cdot\nabla)\mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}}\cdot Q_R^\top),\mathbf{v}_R\cdot\nabla Q_R\Big\rangle}_{\widetilde{\mathcal{M}}_2}$$
$$+\underbrace{\varepsilon^3 J\Big\langle(\mathbf{v}_R\cdot\nabla)\mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}}\cdot Q_R^\top),\dot{Q}_R\Big\rangle}_{\widetilde{\mathcal{M}}_3}$$
$$+\underbrace{\varepsilon^6 J\Big\langle(\mathbf{v}_R\cdot\nabla)\mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}}\cdot Q_R^\top),\mathbf{v}_R\cdot\nabla Q_R\Big\rangle}_{\widetilde{\mathcal{M}}_4}.$$

Using Lemma 3.1, we can infer that

$$\begin{aligned}
&\widetilde{\mathcal{M}}_1\le C(\|Q_R\|_{L^2}+\|\dot{Q}\|_{L^2})\|\mathbf{v}_R\|_{L^2}\le C\mathfrak{E},\\
&\widetilde{\mathcal{M}}_2\le C\varepsilon^3(\|Q_R\|_{L^2}+\|\dot{Q}\|_{L^2})\|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{L^2}\le C\varepsilon\mathfrak{E}^{\frac32},\\
&\widetilde{\mathcal{M}}_3\le C\varepsilon^3\|\mathbf{v}_R\|_{H^2}\|Q_R\|_{H^1}\|\dot{Q}_R\|_{L^2}\le C\varepsilon\mathfrak{E}^{\frac32},\\
&\widetilde{\mathcal{M}}_4\le C\varepsilon^6\|\mathbf{v}_R\|_{H^2}^2\|Q_R\|_{H^1}^2\le C\varepsilon^2\mathfrak{E}^2.
\end{aligned}$$

Thus, we obtain the following estimate:

$$\begin{aligned}
\frac{1}{\varepsilon}\langle\dot{\overline{\mathbf{nn}}}\cdot Q_R,Q_R\rangle\le{}&-J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle\mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}}\cdot Q_R^\top),\dot{Q}_R\Big\rangle-J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle\mathscr{H}_{\mathbf{n}}^{-1}(\dot{\overline{\mathbf{nn}}}\cdot Q_R^\top),\mathbf{v}_R\cdot\nabla Q^\varepsilon\Big\rangle\\
&+C(1+\mathfrak{E}+\varepsilon^2\mathfrak{E}^2)+(\delta+C\varepsilon^2\mathfrak{E})\mathfrak{F}.
\end{aligned}$$

Similarly, we get

$$\langle Q_R:\dot{\overline{\mathbf{nn}}},Q_R:\mathbf{nn}\rangle=\langle Q_R^\top:\dot{\overline{\mathbf{nn}}},Q_R^\perp:\mathbf{nn}\rangle+\langle Q_R^\perp:\dot{\overline{\mathbf{nn}}},Q_R^\perp:\mathbf{nn}\rangle.$$

Therefore, the analogous argument leads to the second estimate (3.9).

For the case of $m=1$ in (3.10) and (3.11), we first assume that $\partial_i Q_R=(\partial_i Q_R)^\top+(\partial_i Q_R)^\perp$ with $(\partial_i Q_R)^\top\in\mathrm{Ker}\mathscr{H}_{\mathbf{n}}$ and $(\partial_i Q_R)^\perp\in(\mathrm{Ker}\mathscr{H}_{\mathbf{n}})^\perp$. Then we have

$$\big\langle\dot{\overline{\mathbf{nn}}}\cdot\partial_i Q_R,\partial_i Q_R\big\rangle=2\big\langle\dot{\overline{\mathbf{nn}}}\cdot(\partial_i Q_R)^\top,(\partial_i Q_R)^\perp\big\rangle+\big\langle\dot{\overline{\mathbf{nn}}}\cdot(\partial_i Q_R)^\perp,(\partial_i Q_R)^\perp\big\rangle. \tag{3.12}$$

By Proposition 2.2, the third term in (3.12) can be estimated as

$$\begin{aligned}
\varepsilon\big\langle\dot{\overline{\mathbf{nn}}}\cdot(\partial_i Q_R)^\perp,(\partial_i Q_R)^\perp\big\rangle&\le C\varepsilon\|(\partial_i Q_R)^\perp\|_{L^2}^2\le C\varepsilon\langle\mathscr{H}_{\mathbf{n}}(\partial_i Q_R),\partial_i Q_R\rangle\\
&\le C\Big(\varepsilon\langle\mathscr{H}_{\mathbf{n}}^\varepsilon(\partial_i Q_R),\partial_i Q_R\rangle-\varepsilon^2\langle\mathscr{L}(\partial_i Q_R),\partial_i Q_R\rangle\Big)\\
&\le C\mathfrak{E}.
\end{aligned}$$

For the second term in (3.12), using Proposition 2.2, we obtain

$$\begin{aligned}
\varepsilon\left\langle\dot{\overline{\mathbf{nn}}}\cdot(\partial_i Q_R)^\top,(\partial_i Q_R)^\perp\right\rangle&=\varepsilon\Big\langle\mathscr{H}_{\mathbf{n}}^{-1}\big(\dot{\overline{\mathbf{nn}}}\cdot(\partial_i Q_R)^\perp\big),\mathscr{H}_{\mathbf{n}}(\partial_i Q_R)\Big\rangle\\
&\le C\varepsilon\left(\|(\partial_i Q_R)^\perp\|_{L^2}^2+\|\partial_i Q_R\|_{L^2}^2\right)\le C\mathfrak{E}.
\end{aligned}$$

Likewise, we can prove that

$$\varepsilon\big\langle\partial_i Q_R:\dot{\overline{\mathbf{nn}}},\partial_i Q_R:\mathbf{nn}\big\rangle\le C\mathfrak{E}.$$

For the case of $m=2$, we suppose that $\Delta Q_R=(\Delta Q_R)^\top+(\Delta Q_R)^\perp$ with $(\Delta Q_R)^\top\in\mathrm{Ker}\mathscr{H}_{\mathbf{n}}$ and $(\Delta Q_R)^\perp\in(\mathrm{Ker}\mathscr{H}_{\mathbf{n}})^\perp$. Adopting an analogous argument yields (3.10) and (3.11) for $m=2$. ∎

We next deal with the estimates for the remainder term $\widetilde{\mathbf{F}}_R$. For convenience, the remainder term $\widetilde{\mathbf{F}}_R$, involving the derivatives with respect to time t, is denoted by

$$\begin{aligned}\widetilde{\mathbf{F}}_R =& -J(\partial_t + \tilde{\mathbf{v}}\cdot\nabla)(\mathbf{v}_R\cdot\nabla\widetilde{Q})\\ &+\varepsilon^3 J\Big(-(\partial_t+\tilde{\mathbf{v}}\cdot\nabla)(\mathbf{v}_R\cdot\nabla Q_R) - \mathbf{v}_R\cdot\nabla\dot{Q}_R - \varepsilon^3\mathbf{v}_R\cdot\nabla(\mathbf{v}_R\cdot\nabla Q_R)\Big)\\ &\stackrel{\text{def}}{=} \widetilde{\mathbf{F}}_1 + \widetilde{\mathbf{F}}_2.\end{aligned}$$

LEMMA 3.5. *For the remainder term $\widetilde{\mathbf{F}}_R$, it follows that*

$$\langle\widetilde{\mathbf{F}}_R, Q_R\rangle \le -J\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, Q_R\rangle + C\left(\mathfrak{E}+\varepsilon^2\mathfrak{E}^2\right). \tag{3.13}$$

Proof. Using integration by parts, it is easy to calculate that

$$\begin{aligned}\langle\widetilde{\mathbf{F}}_1, Q_R\rangle =& -J\langle(\partial_t+\tilde{\mathbf{v}}\cdot\nabla)(\mathbf{v}_R\cdot\nabla\widetilde{Q}), Q_R\rangle\\ =& -J\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, Q_R\rangle + J\langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, \dot{Q}_R\rangle\\ \le& -J\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, Q_R\rangle + C\mathfrak{E}.\end{aligned}$$

By virtue of the incompressibility $\nabla\cdot\mathbf{v}_R = 0$, the following fact holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\mathbf{v}_R\cdot\nabla Q_R, Q_R\big\rangle = 0;$$

this combines with Lemma 3.1 to get

$$\begin{aligned}\langle\widetilde{\mathbf{F}}_2, Q_R\rangle =& -\varepsilon^3 J\langle(\partial_t+\tilde{\mathbf{v}}\cdot\nabla)(\mathbf{v}_R\cdot\nabla Q_R), Q_R\rangle + \varepsilon^3 J\langle\mathbf{v}_R\cdot\nabla Q_R, \dot{Q}_R\rangle\\ &+\varepsilon^6 J\langle\mathbf{v}_R\cdot\nabla Q_R, \mathbf{v}_R\cdot\nabla Q_R\rangle\\ =& -\varepsilon^3 J\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\mathbf{v}_R\cdot\nabla Q_R, Q_R\big\rangle + 2\varepsilon^3 J\big\langle\mathbf{v}_R\cdot\nabla Q_R, \dot{Q}_R\big\rangle\\ &+\varepsilon^6 J\langle\mathbf{v}_R\cdot\nabla Q_R, \mathbf{v}_R\cdot\nabla Q_R\rangle\\ \le& C\varepsilon^3\|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{L^2}\|\dot{Q}_R\|_{L^2} + C\varepsilon^6\|\mathbf{v}_R\|_{L^2}^2\|\nabla Q_R\|_{H^2}^2\\ \le& C(\varepsilon\mathfrak{E}^{\frac{3}{2}} + \varepsilon^2\mathfrak{E}^2).\end{aligned}$$

Consequently, we conclude the proof of the lemma. ∎

LEMMA 3.6. *For the remainder term $\widetilde{\mathbf{F}}_R$ and $m = 0, 1, 2$, there holds*

$$\begin{aligned}&\varepsilon^{2m}\langle\partial_i^m\widetilde{\mathbf{F}}_R, \partial_i^m\dot{Q}_R\rangle\\ &\quad\le -\varepsilon^{2m}\frac{J}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\partial_i^m(\mathbf{v}_R\cdot\nabla Q^\varepsilon)\big\|_{L^2}^2 - \varepsilon^{2m}J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle\partial_i^m(\mathbf{v}_R\cdot\nabla Q^\varepsilon), \partial_i^m\dot{Q}_R\Big\rangle\\ &\qquad + C\left(1+\mathfrak{E}+\varepsilon^2\mathfrak{E}^2+\varepsilon^8\mathfrak{E}^5\right) + \left(\delta + C\varepsilon^2\mathfrak{E}\right)\mathfrak{F},\end{aligned} \tag{3.14}$$

where ∂_i^m represents the mth order partial derivative operator with respect to the component x_i, and $Q^\varepsilon = \widetilde{Q} + \varepsilon^3 Q_R$.

Proof. We only provide here the arguments of (3.14) for the case $m = 0$. We relegate the proof of the cases $m = 1, 2$ in (3.14) to the appendix so as not to distract from the main body of this paper.

First, we control the term $\langle\widetilde{\mathbf{F}}_1, \dot{Q}_R\rangle$. Note that there holds $\langle\mathbf{v}_R\cdot\nabla Q_0, \mathscr{H}_{\mathbf{n}}(Q_R)\rangle = 0$ since $\mathbf{v}_R\cdot\nabla Q_0 \in \mathrm{Ker}\mathscr{H}_{\mathbf{n}}$ and $\mathscr{H}_{\mathbf{n}}(Q_R) \in (\mathrm{Ker}\mathscr{H}_{\mathbf{n}})^\perp$. Then we have

$$\begin{aligned}\Big\langle \mathbf{v}_R\cdot\nabla\widetilde{Q}, \frac{1}{\varepsilon}\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R)\Big\rangle &= \Big\langle \mathbf{v}_R\cdot\nabla\widehat{Q}^\varepsilon, \mathscr{H}_{\mathbf{n}}(Q_R)+\varepsilon\mathscr{L}(Q_R)\Big\rangle\\ &\le C\|\mathbf{v}_R\|_{L^2}(\|Q_R\|_{L^2}+\varepsilon\|Q_R\|_{H^2}) \le C\mathfrak{E},\end{aligned} \tag{3.15}$$

where $\widetilde{Q} = Q_0+\varepsilon\widehat{Q}^\varepsilon = Q_0+\varepsilon(Q_1+\varepsilon Q_2+\varepsilon^2 Q_3)$.

From (3.2) and the bound (3.15), utilizing integration by parts and Lemma 3.1 yield

$$\begin{aligned}\langle\widetilde{\mathbf{F}}_1, \dot{Q}_R\rangle =& -J\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, \dot{Q}_R\rangle + J\langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, \ddot{Q}_R\rangle\\ =& -J\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, \dot{Q}_R\rangle - \mu_1\langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, \dot{Q}_R\rangle - \Big\langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, \frac{1}{\varepsilon}\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R)\Big\rangle\\ &+\Big\langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, -\frac{\mu_2}{2}\mathbf{D}_R+\mu_1[\mathbf{\Omega}_R, Q_0]\Big\rangle + \langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, \mathbf{F}_R+\widetilde{\mathbf{F}}_R\rangle\\ \le& -J\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, \dot{Q}_R\rangle + C(\mathfrak{E}+\mathfrak{E}^{\frac12}\mathfrak{F}^{\frac12})\\ &+C\mathfrak{E}^{\frac12}\|\mathbf{F}_R\|_{L^2} + \langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, \widetilde{\mathbf{F}}_R\rangle.\end{aligned} \tag{3.16}$$

It is easy to check that

$$\langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, \widetilde{\mathbf{F}}_1\rangle = -\frac{J}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\mathbf{v}_R\cdot\nabla\widetilde{Q}\big\|^2_{L^2}. \tag{3.17}$$

By using integration by parts, we deduce from Lemma 3.1 that

$$\begin{aligned}\langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, \widetilde{\mathbf{F}}_2\rangle =& -\varepsilon^3 J\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, \mathbf{v}_R\cdot\nabla Q_R\big\rangle + \varepsilon^3 J\big\langle\mathbf{v}_R\cdot\nabla(\mathbf{v}_R\cdot\nabla\widetilde{Q}), \dot{Q}_R\big\rangle\\ &+\underbrace{\varepsilon^3 J\Big\langle(\partial_t+\tilde{\mathbf{v}}\cdot\nabla)(\mathbf{v}_R\cdot\nabla\widetilde{Q}), \mathbf{v}_R\cdot\nabla Q_R\Big\rangle}_{\mathbf{S}_1}\\ &+\varepsilon^6 J\Big\langle\mathbf{v}_R\cdot\nabla(\mathbf{v}_R\cdot\nabla\widetilde{Q}), \mathbf{v}_R\cdot\nabla Q_R\Big\rangle\\ \le& -\varepsilon^3 J\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, \mathbf{v}_R\cdot\nabla Q_R\big\rangle + \mathbf{S}_1\\ &+C\varepsilon^3\|\mathbf{v}_R\|_{H^2}\|\mathbf{v}_R\|_{H^1}\|\dot{Q}_R\|_{L^2} + C\varepsilon^6\|\mathbf{v}_R\|^2_{H^2}\|\mathbf{v}_R\|_{H^1}\|\nabla Q_R\|_{L^2}\\ \le& -J\varepsilon^3\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, \mathbf{v}_R\cdot\nabla Q_R\big\rangle + \mathbf{S}_1\\ &+C\left(\varepsilon\mathfrak{E}^{\frac32}+\varepsilon^2\mathfrak{E}^2+\varepsilon\mathfrak{E}\mathfrak{F}^{\frac12}+\varepsilon^2\mathfrak{E}^{\frac32}\mathfrak{F}^{\frac12}\right).\end{aligned} \tag{3.18}$$

Then from (3.16)–(3.18) and Lemma 3.2 we obtain

$$\begin{aligned}\langle\widetilde{\mathbf{F}}_1, \dot{Q}_R\rangle \le& -J\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, \dot{Q}_R\rangle - \frac{J}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\mathbf{v}_R\cdot\nabla\widetilde{Q}\big\|^2_{L^2} - \varepsilon^3 J\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\mathbf{v}_R\cdot\nabla\widetilde{Q}, \mathbf{v}_R\cdot\nabla Q_R\big\rangle\\ &+\mathbf{S}_1 + C\left(1+\mathfrak{E}+\varepsilon^2\mathfrak{E}^2+\mathfrak{E}^{\frac12}\mathfrak{F}^{\frac12}\right)+C\varepsilon^2\mathfrak{E}\mathfrak{F}.\end{aligned} \tag{3.19}$$

Now we derive the estimate of $\langle\widetilde{\mathbf{F}}_2, \dot{Q}_R\rangle$. Similarly we obtain from (3.2) that

$$
\begin{aligned}
\langle \widetilde{\mathbf{F}}_2, \dot{Q}_R\rangle =& -\varepsilon^3 J\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{v}_R\cdot\nabla Q_R, \dot{Q}_R\rangle + \varepsilon^3 J\langle \mathbf{v}_R\cdot\nabla Q_R, \ddot{Q}_R\rangle \\
& -\varepsilon^3 J\langle \mathbf{v}_R\cdot\nabla\dot{Q}_R, \dot{Q}_R\rangle + \varepsilon^6 J\big\langle \mathbf{v}_R\cdot\nabla Q_R, \mathbf{v}_R\cdot\nabla\dot{Q}_R\big\rangle \\
=& -\varepsilon^3 J\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{v}_R\cdot\nabla Q_R, \dot{Q}_R\rangle \underbrace{-\varepsilon^3\mu_1\langle \mathbf{v}_R\cdot\nabla Q_R, \dot{Q}_R\rangle}_{\mathcal{A}_1} \\
& \underbrace{-\varepsilon^3\Big\langle \mathbf{v}_R\cdot\nabla Q_R, \frac{1}{\varepsilon}\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R)\Big\rangle}_{\mathcal{A}_2} + \underbrace{\varepsilon^3\Big\langle \mathbf{v}_R\cdot\nabla Q_R, -\frac{\mu_2}{2}\mathbf{D}_R + \mu_1[\mathbf{\Omega}_R, Q_0]\Big\rangle}_{\mathcal{A}_3} \\
& + \varepsilon^3\langle \mathbf{v}_R\cdot\nabla Q_R, \mathbf{F}_R + \widetilde{\mathbf{F}}_R\rangle + \underbrace{\varepsilon^6 J\big\langle \mathbf{v}_R\cdot\nabla Q_R, \mathbf{v}_R\cdot\nabla\dot{Q}_R\big\rangle}_{\mathbf{W}_1}.
\end{aligned}
$$

By Lemma 3.1, we have

$$
\begin{aligned}
&\mathcal{A}_1 \le C\varepsilon^3\|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{L^2}\|\dot{Q}_R\|_{L^2} \le C\varepsilon\mathfrak{E}^{\frac{3}{2}}, \\
&\mathcal{A}_2 = -\varepsilon^2\Big\langle \mathbf{v}_R\cdot\nabla Q_R, \mathscr{H}_{\mathbf{n}}(Q_R) + \varepsilon\mathscr{L}(Q_R)\Big\rangle \\
&\qquad \le C\varepsilon^2\|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{L^2}(\|Q_R\|_{L^2} + \|\varepsilon Q_R\|_{H^2}) \le C(\varepsilon^2\mathfrak{E}^{\frac{3}{2}} + \varepsilon\mathfrak{E}\mathfrak{F}^{\frac{1}{2}}), \\
&\mathcal{A}_3 \le C\varepsilon^3\|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{L^2}\|\nabla\mathbf{v}_R\|_{L^2} \le C\varepsilon\mathfrak{E}\mathfrak{F}^{\frac{1}{2}}.
\end{aligned}
$$

By Lemma 3.2, we get

$$
\begin{aligned}
\varepsilon^3\langle \mathbf{v}_R\cdot\nabla Q_R, \mathbf{F}_R\rangle &\le \varepsilon^3\|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{L^2}\|\mathbf{F}_R\|_{L^2} \le \varepsilon\mathfrak{E}\|\mathbf{F}_R\|_{L^2} \\
&\le C\left(1 + \mathfrak{E} + \varepsilon^2\mathfrak{E}^2 + \varepsilon^8\mathfrak{E}^5\right) + C\varepsilon^2\mathfrak{E}\mathfrak{F}.
\end{aligned}
$$

Using integration by parts, it follows immediately by Lemma 3.1 that

$$
\begin{aligned}
&\varepsilon^3\langle \mathbf{v}_R\cdot\nabla Q_R, \widetilde{\mathbf{F}}_R\rangle \\
&\quad = -\varepsilon^6\frac{J}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}_R\cdot\nabla Q_R\|^2_{L^2} - \mathbf{S}_1 - \mathbf{W}_1 - \varepsilon^9 J\Big\langle \mathbf{v}_R\cdot\nabla Q_R, \mathbf{v}_R\cdot\nabla(\mathbf{v}_R\cdot\nabla Q_R)\Big\rangle \\
&\quad = -\varepsilon^6\frac{J}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}_R\cdot\nabla Q_R\|^2_{L^2} - \mathbf{S}_1 - \mathbf{W}_1.
\end{aligned}
$$

Thus we find

$$
\begin{aligned}
\langle \widetilde{\mathbf{F}}_2, \dot{Q}_R\rangle \le& -\varepsilon^3 J\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{v}_R\cdot\nabla Q_R, \dot{Q}_R\rangle - \varepsilon^6\frac{J}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}_R\cdot\nabla Q_R\|^2_{L^2} \\
& - \mathbf{S}_1 + C\left(1 + \mathfrak{E} + \varepsilon^2\mathfrak{E}^2 + \varepsilon^8\mathfrak{E}^5\right) + C\varepsilon^2\mathfrak{E}\mathfrak{F}.
\end{aligned} \tag{3.20}
$$

Recalling $Q^\varepsilon = \widetilde{Q} + \varepsilon^3 Q_R$, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}_R\cdot\nabla\widetilde{Q}\|^2_{L^2} + 2\varepsilon^3\frac{\mathrm{d}}{\mathrm{d}t}\big\langle \mathbf{v}_R\cdot\nabla\widetilde{Q}, \mathbf{v}_R\cdot\nabla Q_R\big\rangle + \varepsilon^6\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}_R\cdot\nabla Q_R\|^2_{L^2} \\
&\qquad = \frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}_R\cdot\nabla(\widetilde{Q} + \varepsilon^3 Q_R)\|^2_{L^2} = \frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}_R\cdot\nabla Q^\varepsilon\|^2_{L^2}.
\end{aligned} \tag{3.21}
$$

Therefore, summarizing (3.19) and (3.20), and using (3.21), we obtain

$$
\begin{aligned}
\langle \widetilde{\mathbf{F}}_R, \dot{Q}_R\rangle \le& -\frac{J}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}_R\cdot\nabla Q^\varepsilon\|^2_{L^2} - J\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{v}_R\cdot\nabla Q^\varepsilon, \dot{Q}_R\rangle \\
& + C\left(1 + \mathfrak{E} + \varepsilon^2\mathfrak{E}^2 + \varepsilon^8\mathfrak{E}^5\right) + (\delta + C\varepsilon^2\mathfrak{E})\mathfrak{F}.
\end{aligned}
$$

∎

3.3. The uniform energy estimate. In this subsection, we derive the uniform energy estimate for the remainder system.

PROPOSITION 3.7. *Let* $(\mathbf{v}_R, Q_R)$ *be a smooth solution of the remainder system* (3.2)–(3.4) *on* $[0,T]$*; then for any* $t \in [0,T]$*, it holds that*

$$\frac{d}{dt}\widetilde{\mathfrak{E}}(t) + \mathfrak{F}(t) \le C\left(1 + \mathfrak{E} + \varepsilon^2\mathfrak{E}^2 + \varepsilon^8\mathfrak{E}^5\right) + C\big(\varepsilon + \varepsilon^2\mathfrak{E}^{\frac{1}{2}} + \varepsilon^2\mathfrak{E}\big)\mathfrak{F},$$

where the energy functional $\widetilde{\mathfrak{E}}(t)$ *is defined by*

$$\widetilde{\mathfrak{E}}(t) \overset{def}{=} \widetilde{\mathfrak{E}}_0(t) + \widetilde{\mathfrak{E}}_1(t) + \widetilde{\mathfrak{E}}_2(t), \tag{3.22}$$

and $\widetilde{\mathfrak{E}}_i(t)(i = 0,1,2)$ *are given as follows:*

$$\begin{cases} \widetilde{\mathfrak{E}}_0(t) = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^3}\Big(|\mathbf{v}_R|^2 + J|\dot{Q}_R + \mathbf{v}_R\cdot\nabla Q^\varepsilon + Q_R|^2 + (\mu_1 - J)|Q_R|^2 \\ \qquad\qquad + \dfrac{1}{\varepsilon}\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R) : Q_R\Big)\mathrm{d}\mathbf{x} + J\Big\langle \mathscr{H}^{-1}_{\mathbf{n}}\mathcal{G}(Q_R^\top), \dot{Q}_R + \mathbf{v}_R\cdot\nabla Q^\varepsilon\Big\rangle, \\ \widetilde{\mathfrak{E}}_1(t) = \dfrac{\varepsilon^2}{2}\displaystyle\int_{\mathbb{R}^3}\Big(|\partial_i\mathbf{v}_R|^2 + J|\partial_i\dot{Q}_R + \partial_i(\mathbf{v}_R\cdot\nabla Q^\varepsilon)|^2 + \dfrac{1}{\varepsilon}\mathscr{H}^\varepsilon_{\mathbf{n}}(\partial_i Q_R) : \partial_i Q_R\Big)\mathrm{d}\mathbf{x}, \\ \widetilde{\mathfrak{E}}_2(t) = \dfrac{\varepsilon^4}{2}\displaystyle\int_{\mathbb{R}^3}\Big(|\Delta\mathbf{v}_R|^2 + J|\Delta\dot{Q}_R + \Delta(\mathbf{v}_R\cdot\nabla Q^\varepsilon)|^2 + \dfrac{1}{\varepsilon}\mathscr{H}^\varepsilon_{\mathbf{n}}(\Delta Q_R) : \Delta Q_R\Big)\mathrm{d}\mathbf{x}. \end{cases}$$

Here $\mathcal{G}(Q) \overset{def}{=} 2bs\dot{\overline{\mathbf{nn}}}\cdot Q - 4cs^2 Q : \dot{\overline{\mathbf{nn}}}(\mathbf{nn} - \frac{1}{3}\mathbf{I})$ *and* $Q^\varepsilon = \widetilde{Q} + \varepsilon^3 Q_R$*.*

Proof. Step 1. L^2*-estimate.* On the one hand, multiplying (3.2) by Q_R, taking the trace and integrating over the space $\mathbb{R}^3$, and using the fact that $\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R) : Q_R \ge 0$ yield

(3.23)

$$\begin{aligned} &J\langle \ddot{Q}_R, Q_R\rangle + \mu_1\langle \dot{Q}_R, Q_R\rangle \\ &\quad = -\Big\langle \frac{1}{\varepsilon}\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R), Q_R\Big\rangle - \frac{\mu_2}{2}\langle \mathbf{D}_R, Q_R\rangle + \mu_1\langle[\mathbf{\Omega}_R, Q_0], Q_R\rangle + \langle \mathbf{F}_R + \widetilde{\mathbf{F}}_R, Q_R\rangle \\ &\quad \le C\|\nabla\mathbf{v}_R\|_{L^2}\|Q_R\|_{L^2} + \mathfrak{E}^{\frac{1}{2}}\|\mathbf{F}_R\|_{L^2} + \langle\widetilde{\mathbf{F}}_R, Q_R\rangle. \end{aligned}$$

Considering the previous equality (2.32), then (3.23) can be reduced to

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\Big(2J\dot{Q}_R : Q_R + \mu_1|Q_R|^2\Big)\mathrm{d}\mathbf{x} \\ &\qquad \le C\big(\mathfrak{E} + \mathfrak{E}^{\frac{1}{2}}\mathfrak{F}^{\frac{1}{2}}\big) + \mathfrak{E}^{\frac{1}{2}}\|\mathbf{F}_R\|_{L^2} + \langle\widetilde{\mathbf{F}}_R, Q_R\rangle. \end{aligned} \tag{3.24}$$

On the other hand, multiplying (3.2) by $\dot{Q}_R$ and (3.3) by $\mathbf{v}_R$, integrating by parts over the space $\mathbb{R}^3$, we hence obtain

$$\begin{aligned} &\langle \dot{\mathbf{v}}_R, \mathbf{v}_R\rangle + J\langle \ddot{Q}_R, \dot{Q}_R\rangle \\ &\quad = \underbrace{-\Big\langle \beta_1 Q_0(Q_0 : \mathbf{D}_R) + \beta_4\mathbf{D}_R + \beta_5\mathbf{D}_R\cdot Q_0 + \beta_6 Q_0\cdot\mathbf{D}_R, \nabla\mathbf{v}_R\Big\rangle}_{\mathcal{I}_1} \\ &\qquad \underbrace{-\Big\langle \beta_7(\mathbf{D}_R\cdot Q_0^2 + Q_0^2\cdot\mathbf{D}_R), \nabla\mathbf{v}_R\Big\rangle}_{\mathcal{I}_2} \underbrace{-\frac{\mu_2}{2}\big\langle \dot{Q}_R - [\mathbf{\Omega}_R, Q_0], \nabla\mathbf{v}_R\big\rangle}_{\mathcal{I}_3} \end{aligned}$$

$$
\underbrace{-\mu_1\Big\langle\big[Q_0,(\dot{Q}_R-[\mathbf{\Omega}_R,Q_0])\big],\nabla\mathbf{v}_R\Big\rangle}_{\mathcal{I}_4}\underbrace{-\frac{\mu_2}{2}\langle\mathbf{D}_R,\dot{Q}_R\rangle}_{\mathcal{I}_5}
$$

$$
\underbrace{-\mu_1\big\langle\dot{Q}_R-[\mathbf{\Omega}_R,Q_0],\dot{Q}_R\big\rangle}_{\mathcal{I}_6}\underbrace{-\Big\langle\frac{1}{\varepsilon}\mathscr{H}^{\varepsilon}_{\mathbf{n}}(Q_R),\dot{Q}_R\Big\rangle}_{\mathcal{I}_7}
$$

$$
+\langle\nabla\cdot\mathbf{G}_R+\mathbf{G}'_R,\mathbf{v}_R\rangle+\langle\mathbf{F}_R+\widetilde{\mathbf{F}}_R,\dot{Q}_R\rangle. \tag{3.25}
$$

Now we estimate (3.25) term by term as follows. We will frequently use a simple fact that $\langle A,B\rangle=0$ if the tensor A is symmetric but B skew symmetric. Remembering the relation $\beta_6-\beta_5=\mu_2$, and noting $Q_0=s(\mathbf{nn}-\frac{1}{3}\mathbf{I})$, it follows that

$$
\begin{aligned}
\mathcal{I}_1+\mathcal{I}_2=&-\Big\langle\beta_1Q_0(Q_0:\mathbf{D}_R)+\beta_4\mathbf{D}_R+\frac{\beta_5+\beta_6}{2}(Q_0\cdot\mathbf{D}_R+\mathbf{D}_R\cdot Q_0),\mathbf{D}_R\Big\rangle\\
&-\Big\langle\beta_7(\mathbf{D}_R\cdot Q_0^2+Q_0^2\cdot\mathbf{D}_R),\mathbf{D}_R\Big\rangle\\
&+\Big\langle\Big(\frac{\beta_5+\beta_6}{2}-\beta_5\Big)\mathbf{D}_R\cdot Q_0+\Big(\frac{\beta_5+\beta_6}{2}-\beta_6\Big)Q_0\cdot\mathbf{D}_R,\mathbf{D}_R+\mathbf{\Omega}_R\Big\rangle\\
=&-\beta_1s^2\|\mathbf{nn}:\mathbf{D}_R\|^2_{L^2}-\Big(\beta_4-\frac{s(\beta_5+\beta_6)}{3}+\frac{2}{9}\beta_7s^2\Big)\|\mathbf{D}_R\|^2_{L^2}\\
&-\Big(s(\beta_5+\beta_6)+\frac{2}{3}\beta_7s^2\Big)\|\mathbf{n}\cdot\mathbf{D}_R\|^2_{L^2}+\underbrace{\frac{\mu_2}{2}\Big\langle[\mathbf{D}_R,Q_0],\mathbf{\Omega}_R\Big\rangle}_{\mathcal{I}'_1}.
\end{aligned} \tag{3.26}
$$

Due to the symmetry of the commutator $[\mathbf{\Omega}_R,Q_0]$, it follows that

$$
\begin{aligned}
\mathcal{I}'_1+\mathcal{I}_3+\mathcal{I}_5&=\frac{\mu_2}{2}\langle[\mathbf{D}_R,Q_0],\mathbf{\Omega}_R\rangle-\frac{\mu_2}{2}\big\langle\dot{Q}_R-[\mathbf{\Omega}_R,Q_0],\mathbf{D}_R\big\rangle-\frac{\mu_2}{2}\langle\mathbf{D}_R,\dot{Q}_R\rangle\\
&=-\mu_2\big\langle\dot{Q}_R-[\mathbf{\Omega}_R,Q_0],\mathbf{D}_R\big\rangle.
\end{aligned}
$$

Simultaneously, we have

$$
\begin{aligned}
\mathcal{I}_4+\mathcal{I}_6=&-\mu_1\big\langle\big[Q_0,(\dot{Q}_R-[\mathbf{\Omega}_R,Q_0])\big],\mathbf{\Omega}_R\big\rangle-\mu_1\big\langle\dot{Q}_R-[\mathbf{\Omega}_R,Q_0],\dot{Q}_R\big\rangle\\
=&-\mu_1\big\langle[Q_0,\mathbf{\Omega}_R],\dot{Q}_R-[\mathbf{\Omega}_R,Q_0]\big\rangle-\mu_1\big\langle\dot{Q}_R-[\mathbf{\Omega}_R,Q_0],\dot{Q}_R\big\rangle\\
=&-\mu_1\|\dot{Q}_R-[\mathbf{\Omega}_R,Q_0]\|^2_{L^2}.
\end{aligned}
$$

It may be observed that

$$
\mathcal{I}'_1+\mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5+\mathcal{I}_6=-\mu_1\Big\|\dot{Q}_R-[\mathbf{\Omega}_R,Q_0]+\frac{\mu_2}{2\mu_1}\mathbf{D}_R\Big\|^2_{L^2}+\frac{\mu_2^2}{4\mu_1}\|\mathbf{D}_R\|^2_{L^2},
$$

which combines with (3.26) and the dissipation relation (2.22) yields

$$
\begin{aligned}
&\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5+\mathcal{I}_6\\
&=-\mu_1\Big\|\dot{Q}_R-[\mathbf{\Omega}_R,Q_0]+\frac{\mu_2}{2\mu_1}\mathbf{D}_R\Big\|^2_{L^2}+\frac{\mu_2^2}{4\mu_1}\|\mathbf{D}_R\|^2_{L^2}\\
&\quad-\tilde{\beta}_1\|\mathbf{nn}:\mathbf{D}_R\|^2_{L^2}-\tilde{\beta}_2\|\mathbf{D}_R\|^2_{L^2}-\tilde{\beta}_3\|\mathbf{n}\cdot\mathbf{D}_R\|^2_{L^2}-4\delta\|\mathbf{D}_R\|^2_{L^2}\\
&\leq-4\delta\|\mathbf{D}_R\|^2_{L^2},
\end{aligned} \tag{3.27}
$$

where $\delta>0$ is small enough, such that $\tilde{\beta}_i(i=1,2,3)$ given by (2.39) satisfy (2.23).

For the term $\mathcal{I}_7$, using $Q_R:\mathbf{I}=\mathrm{Tr}Q_R=0$, we can write

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\frac{1}{\varepsilon}\mathscr{H}^{\varepsilon}_{\mathbf{n}}(Q_R),Q_R\right\rangle&=\frac{2}{\varepsilon}\langle\mathscr{H}^{\varepsilon}_{\mathbf{n}}(Q_R),\dot{Q}_R\rangle+\frac{1}{\varepsilon}\Big\langle bs\big(\dot{\overline{\mathbf{nn}}}\cdot Q_R+Q_R\cdot\dot{\overline{\mathbf{nn}}}\big)\\&\quad-2cs^2\big(Q_R:\dot{\overline{\mathbf{nn}}}(\mathbf{nn})+(Q_R:\mathbf{nn})\dot{\overline{\mathbf{nn}}}\big),Q_R\Big\rangle\\&=-2\mathcal{I}_7+\frac{2}{\varepsilon}\Big\langle bs\dot{\overline{\mathbf{nn}}}\cdot Q_R-2cs^2Q_R:\dot{\overline{\mathbf{nn}}}(\mathbf{nn}),Q_R\Big\rangle,\end{aligned}$$

which implies from Lemma 3.4 that

$$\begin{aligned}\mathcal{I}_7\le&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\frac{1}{\varepsilon}\mathscr{H}^{\varepsilon}_{\mathbf{n}}(Q_R),Q_R\right\rangle-J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle\mathscr{H}^{-1}_{\mathbf{n}}\mathcal{G}(Q_R^{\top}),\dot{Q}_R+\mathbf{v}_R\cdot\nabla Q^{\varepsilon}\Big\rangle\\&+C(1+\mathfrak{E}+\varepsilon^2\mathfrak{E}^2)+(\delta+C\varepsilon^2\mathfrak{E})\mathfrak{F}.\end{aligned}\tag{3.28}$$

Hence, summarizing (3.25) and the estimates (3.27)–(3.28), we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\Big(|\mathbf{v}_R|^2+J|\dot{Q}_R|^2+\frac{1}{\varepsilon}\mathscr{H}^{\varepsilon}_{\mathbf{n}}(Q_R):Q_R\Big)\mathrm{d}\mathbf{x}\\&\quad+J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle\mathscr{H}^{-1}_{\mathbf{n}}\mathcal{G}(Q_R^{\top}),\dot{Q}_R+\mathbf{v}_R\cdot\nabla Q^{\varepsilon}\Big\rangle+4\delta\|\nabla\mathbf{v}_R\|^2_{L^2}\\&\le C\Big(\|\mathbf{G}'_R\|_{L^2}\mathfrak{E}^{\frac12}+\|\mathbf{F}_R\|_{L^2}\mathfrak{E}^{\frac12}+\|\mathbf{G}_R\|_{L^2}\mathfrak{F}^{\frac12}\Big)+\langle\widetilde{\mathbf{F}}_R,\dot{Q}_R\rangle\\&\quad+C\left(1+\mathfrak{E}+\varepsilon^2\mathfrak{E}^2\right)+\left(\delta+C\varepsilon^2\mathfrak{E}\right)\mathfrak{F}.\end{aligned}\tag{3.29}$$

Then, adding (3.24) to (3.29), and using Lemmas 3.2–3.3 and Lemmas 3.5–3.6, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathfrak{E}}_0(t)+4\delta\|\nabla\mathbf{v}_R\|^2_{L^2}\\&\quad\le C\left(1+\mathfrak{E}+\varepsilon^2\mathfrak{E}^2+\varepsilon^8\mathfrak{E}^5\right)+\delta\mathfrak{F}+C\big(\varepsilon+\varepsilon^2\mathfrak{E}^{\frac12}+\varepsilon^2\mathfrak{E}\big)\mathfrak{F}.\end{aligned}\tag{3.30}$$

Step 2. H^1-*estimate*. We apply the derivative ∂_i on (3.2) and take the L^2-inner product with $\partial_i\dot{Q}_R$. Again by acting ∂_i on (3.3) and taking the L^2-inner product with $\partial_i\mathbf{v}_R$, we then have

$$\begin{aligned}&\varepsilon^2\Big\langle\partial_t(\partial_i\mathbf{v}_R),\partial_i\mathbf{v}_R\Big\rangle+\varepsilon^2J\Big\langle\partial_t(\partial_i\dot{Q}_R),\partial_i\dot{Q}_R\Big\rangle\\&=\underbrace{-\varepsilon^2\Big\langle\partial_i\Big(\beta_1Q_0(Q_0:\mathbf{D}_R)+\beta_4\mathbf{D}_R+\beta_5\mathbf{D}_R\cdot Q_0+\beta_6Q_0\cdot\mathbf{D}_R\Big),\nabla\partial_i\mathbf{v}_R\Big\rangle}_{\mathcal{J}_1}\\&\quad\underbrace{-\varepsilon^2\beta_7\Big\langle\partial_i(\mathbf{D}_R\cdot Q_0^2+Q_0^2\cdot\mathbf{D}_R),\nabla\partial_i\mathbf{v}_R\Big\rangle}_{\mathcal{J}_2}\\&\quad\underbrace{-\varepsilon^2\frac{\mu_2}{2}\Big\langle\partial_i(\dot{Q}_R-[\mathbf{\Omega}_R,Q_0]),\nabla\partial_i\mathbf{v}_R\Big\rangle}_{\mathcal{J}_3}\underbrace{-\varepsilon^2\mu_1\Big\langle\partial_i\big[Q_0,(\dot{Q}_R-[\mathbf{\Omega}_R,Q_0])\big],\nabla\partial_i\mathbf{v}_R\Big\rangle}_{\mathcal{J}_4}\\&\quad\underbrace{-\varepsilon^2\langle\partial_i\tilde{\mathbf{v}}\cdot\nabla\mathbf{v}_R,\partial_i\mathbf{v}_R\rangle-\varepsilon^2\langle\partial_i\mathbf{G}_R,\nabla\partial_i\mathbf{v}_R\rangle+\varepsilon^2\langle\partial_i\mathbf{G}'_R,\partial_i\mathbf{v}_R\rangle}_{\mathcal{J}_5}\end{aligned}$$

$$\underbrace{-\varepsilon^2\frac{\mu_2}{2}\langle\partial_i\mathbf{D}_R,\partial_i\dot{Q}_R\rangle}_{\mathscr{J}_6}\underbrace{-\varepsilon^2\mu_1\langle\partial_i\dot{Q}_R-\partial_i[\mathbf{\Omega}_R,Q_0],\partial_i\dot{Q}_R\rangle}_{\mathscr{J}_7}\underbrace{-\varepsilon^2\Big\langle\frac{1}{\varepsilon}\partial_i\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R),\partial_i\dot{Q}_R\Big\rangle}_{\mathscr{J}_8}$$
$$\underbrace{-\varepsilon^2\langle\partial_i\tilde{\mathbf{v}}\cdot\nabla\dot{Q}_R,\partial_i\dot{Q}_R\rangle}_{\mathscr{J}_9}+\varepsilon^2\langle\partial_i\mathbf{F}_R+\partial_i\widetilde{\mathbf{F}}_R,\partial_i\dot{Q}_R\rangle.$$

Via employing the analogous method in (3.26), we derive that

$$\begin{aligned}\mathscr{J}_1+\mathscr{J}_2\le&-\varepsilon^2\Big\langle\beta_1Q_0(Q_0:\partial_i\mathbf{D}_R)+\beta_4\partial_i\mathbf{D}_R+\beta_5\partial_i\mathbf{D}_R\cdot Q_0+\beta_6Q_0\cdot\partial_i\mathbf{D}_R,\nabla\partial_i\mathbf{v}_R\Big\rangle\\&-\varepsilon^2\Big\langle\beta_7(\partial_i\mathbf{D}_R\cdot Q_0^2+Q_0^2\cdot\partial_i\mathbf{D}_R),\nabla\partial_i\mathbf{v}_R\Big\rangle+C\|\varepsilon\nabla\mathbf{v}_R\|_{L^2}\|\varepsilon\nabla\partial_i\mathbf{v}_R\|_{L^2}\\\le&-\varepsilon^2\Big\langle\beta_1Q_0(Q_0:\partial_i\mathbf{D}_R)+\beta_4\partial_i\mathbf{D}_R+\frac{\beta_5+\beta_6}{2}(Q_0\cdot\partial_i\mathbf{D}_R+\partial_i\mathbf{D}_R\cdot Q_0),\partial_i\mathbf{D}_R\Big\rangle\\&-\varepsilon^2\Big\langle\beta_7(\partial_i\mathbf{D}_R\cdot Q_0^2+Q_0^2\cdot\partial_i\mathbf{D}_R),\partial_i\mathbf{D}_R\Big\rangle+\underbrace{\varepsilon^2\frac{\mu_2}{2}\langle[\partial_i\mathbf{D}_R,Q_0],\nabla\partial_i\mathbf{v}_R\rangle}_{\mathscr{J}_1'}+C\mathfrak{E}^{\frac12}\mathfrak{F}^{\frac12}.\end{aligned}$$

Direct calculation enables us to get

$$\begin{aligned}\mathscr{J}_1'+\mathscr{J}_3+\mathscr{J}_6\le&\,\varepsilon^2\frac{\mu_2}{2}\langle[\partial_i\mathbf{D}_R,Q_0],\partial_i\mathbf{\Omega}_R\rangle-\varepsilon^2\mu_2\langle\partial_i\mathbf{D}_R,\partial_i\dot{Q}_R\rangle\\&+\varepsilon^2\frac{\mu_2}{2}\langle[\partial_i\mathbf{\Omega}_R,Q_0],\partial_i\mathbf{D}_R\rangle+C\|\varepsilon\nabla\mathbf{v}_R\|_{L^2}\|\varepsilon\nabla\partial_i\mathbf{v}_R\|_{L^2}\\\le&-\varepsilon^2\mu_2\langle\partial_i\dot{Q}_R-[\partial_i\mathbf{\Omega}_R,Q_0],\partial_i\mathbf{D}_R\rangle+C\mathfrak{E}^{\frac12}\mathfrak{F}^{\frac12}.\end{aligned}$$

For the estimates of $\mathscr{J}_4$ and $\mathscr{J}_7$, it is easy to deduce that

$$\begin{aligned}\mathscr{J}_4+\mathscr{J}_7\le&-\varepsilon^2\mu_1\Big\langle\big[Q_0,(\partial_i\dot{Q}_R-[\partial_i\mathbf{\Omega}_R,Q_0])\big],\nabla\partial_i\mathbf{v}_R\Big\rangle\\&-\varepsilon^2\mu_1\langle\partial_i\dot{Q}_R-[\partial_i\mathbf{\Omega}_R,Q_0],\partial_i\dot{Q}_R\rangle\\&+C\Big(\big\|\varepsilon(\dot{Q}_R-[\mathbf{\Omega}_R,Q_0])\big\|_{L^2}+\big\|\varepsilon(\partial_i\dot{Q}_R-[\mathbf{\Omega}_R,\partial_iQ_0])\big\|_{L^2}\Big)\|\varepsilon\nabla\partial_i\mathbf{v}_R\|_{L^2}\\&+C\|\varepsilon\nabla\mathbf{v}_R\|_{L^2}\|\varepsilon\partial_i\dot{Q}_R\|_{L^2}\\\le&-\varepsilon^2\mu_1\big\|\partial_i\dot{Q}_R-[\partial_i\mathbf{\Omega}_R,Q_0]\big\|_{L^2}^2+C\big(\mathfrak{E}^{\frac12}\mathfrak{F}^{\frac12}+\mathfrak{E}\big).\end{aligned}$$

Noticing the equality

$$\begin{aligned}&-\varepsilon^2\mu_1\big\|\partial_i\dot{Q}_R-[\partial_i\mathbf{\Omega}_R,Q_0]\big\|_{L^2}^2-\varepsilon^2\mu_2\langle\partial_i\dot{Q}_R-[\partial_i\mathbf{\Omega}_R,Q_0],\partial_i\mathbf{D}_R\rangle\\&=-\varepsilon^2\mu_1\Big\|\partial_i\dot{Q}_R-[\partial_i\mathbf{\Omega}_R,Q_0]+\frac{\mu_2}{2\mu_1}\partial_i\mathbf{D}_R\Big\|_{L^2}^2+\frac{\mu_2^2}{4\mu_1}\|\partial_i\mathbf{D}_R\|_{L^2}^2,\end{aligned}$$

and taking advantage of the dissipation relation (2.22), we can infer that

$$\begin{aligned}&\mathscr{J}_1+\mathscr{J}_2+\mathscr{J}_3+\mathscr{J}_4+\mathscr{J}_6+\mathscr{J}_7\\&\le-\varepsilon^2\beta_1s^2\|\mathbf{nn}:\partial_i\mathbf{D}_R\|_{L^2}^2-\varepsilon^2\left(\beta_4-\frac{s(\beta_5+\beta_6)}{3}\right)\|\partial_i\mathbf{D}_R\|_{L^2}^2\\&\quad-\varepsilon^2s(\beta_5+\beta_6)\|\mathbf{n}\cdot\partial_i\mathbf{D}_R\|_{L^2}^2-\varepsilon^2\mu_2\langle\partial_i\dot{Q}_R-[\partial_i\mathbf{\Omega}_R,Q_0],\partial_i\mathbf{D}_R\rangle\\&\quad-\varepsilon^2\mu_1\big\|\partial_i\dot{Q}_R-[\partial_i\mathbf{\Omega}_R,Q_0]\big\|_{L^2}^2+C(\mathfrak{E}^{\frac12}\mathfrak{F}^{\frac12}+\mathfrak{E})\\&\le-\varepsilon^2\tilde{\beta}_1\|\mathbf{nn}:\partial_i\mathbf{D}_R\|_{L^2}^2-\varepsilon^2\tilde{\beta}_2\|\partial_i\mathbf{D}_R\|_{L^2}^2-\varepsilon^2\tilde{\beta}_3\|\mathbf{n}\cdot\partial_i\mathbf{D}_R\|_{L^2}^2\\&\quad-4\varepsilon^2\delta\|\partial_i\mathbf{D}_R\|_{L^2}^2+C(\mathfrak{E}^{\frac12}\mathfrak{F}^{\frac12}+\mathfrak{E})\\&\le-4\varepsilon^2\delta\|\partial_i\mathbf{D}_R\|_{L^2}^2+C\mathfrak{E}+\delta\mathfrak{F},\end{aligned}$$

where $\delta > 0$ is small enough such that the coefficients $\tilde{\beta}_i (i = 1, 2, 3)$ given by (2.39) satisfy the relation (2.23). In addition, the terms $\mathscr{J}_5$ and $\mathscr{J}_9$ can be controlled as

$$\begin{aligned}\mathscr{J}_5 + \mathscr{J}_9 \le\, & C\Big(\|\varepsilon\nabla\mathbf{v}_R\|_{L^2}^2 + \|\varepsilon\partial_i\mathbf{G}_R\|_{L^2}\|\varepsilon\nabla\partial_i\mathbf{v}_R\|_{L^2} \\ & + \|\varepsilon\partial_i\mathbf{G}'_R\|_{L^2}\|\varepsilon\partial_i\mathbf{v}_R\|_{L^2} + \|\varepsilon\nabla\dot{Q}_R\|_{L^2}\|\varepsilon\partial_i\dot{Q}_R\|_{L^2}\Big) \\ \le\, & C\mathfrak{E} + C(\|\varepsilon\partial_i\mathbf{G}_R\|_{L^2}\mathfrak{F}^{\frac{1}{2}} + \|\varepsilon\partial_i\mathbf{G}'_R\|_{L^2}\mathfrak{E}^{\frac{1}{2}}).\end{aligned}$$

We next deal with the term $\mathscr{J}_8$. First, we can observe that

$$\begin{aligned}\mathscr{J}_8 \le & -\varepsilon\Big\langle\mathscr{H}^\varepsilon_{\mathbf{n}}(\partial_i Q_R), \partial_i\dot{Q}_R\Big\rangle + \varepsilon\|Q_R\|_{L^2}\|\partial_i\dot{Q}_R\|_{L^2} \\ \le & -\varepsilon\Big\langle\mathscr{H}^\varepsilon_{\mathbf{n}}(\partial_i Q_R), \dot{\overline{\partial_i Q_R}}\Big\rangle - \varepsilon\Big\langle\mathscr{H}^\varepsilon_{\mathbf{n}}(\partial_i Q_R), \partial_i\tilde{\mathbf{v}}\cdot\nabla Q_R\Big\rangle + C\mathfrak{E}^{\frac{1}{2}}\mathfrak{F}^{\frac{1}{2}} \\ \le & \underbrace{-\varepsilon\Big\langle\mathscr{H}^\varepsilon_{\mathbf{n}}(\partial_i Q_R), \dot{\overline{\partial_i Q_R}}\Big\rangle}_{\mathscr{J}'_8} + C(\mathfrak{E} + \mathfrak{E}^{\frac{1}{2}}\mathfrak{F}^{\frac{1}{2}}).\end{aligned}$$

Using Lemma 3.4, we get

$$\begin{aligned}\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathscr{H}^\varepsilon_{\mathbf{n}}(\partial_i Q_R), \partial_i Q_R\rangle = & \, 2\varepsilon\langle\mathscr{H}^\varepsilon_{\mathbf{n}}(\partial_i Q_R), \dot{\overline{\partial_i Q_R}}\rangle + \frac{1}{\varepsilon}\Big\langle bs\big(\dot{\overline{\mathbf{nn}}}\cdot\partial_i Q_R + \partial_i Q_R\cdot\dot{\overline{\mathbf{nn}}}\big) \\ & - 2cs^2\big(\partial_i Q_R : \dot{\overline{\mathbf{nn}}}(\mathbf{nn}) + (\partial_i Q_R : \mathbf{nn})\dot{\overline{\mathbf{nn}}}\big), \partial_i Q_R\Big\rangle \\ \le & -2\mathscr{J}'_8 + C\mathfrak{E},\end{aligned}$$

which implies

$$\mathscr{J}_8 \le -\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathscr{H}^\varepsilon_{\mathbf{n}}(\partial_i Q_R), \partial_i Q_R\rangle + \delta\mathfrak{F} + C\mathfrak{E}. \tag{3.31}$$

Summarizing the above estimates, we get

$$\begin{aligned}& \varepsilon^2\Big\langle\partial_t(\partial_i\mathbf{v}_R), \partial_i\mathbf{v}_R\Big\rangle + \varepsilon^2 J\Big\langle\partial_t(\partial_i\dot{Q}_R), \partial_i\dot{Q}_R\Big\rangle \\ & \quad + \frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathscr{H}^\varepsilon_{\mathbf{n}}(\partial_i Q_R), \partial_i Q_R\rangle + 4\varepsilon^2\delta\|\partial_i\nabla\mathbf{v}_R\|_{L^2}^2 \\ & \le C\Big(\|\varepsilon\partial_i\mathbf{G}_R\|_{L^2}\mathfrak{F}^{\frac{1}{2}} + \|\varepsilon\partial_i\mathbf{G}'_R\|_{L^2}\mathfrak{E}^{\frac{1}{2}} + \|\varepsilon\partial_i\mathbf{F}_R\|_{L^2}\mathfrak{E}^{\frac{1}{2}}\Big) \\ & \quad + \varepsilon^2\langle\partial_i\widetilde{\mathbf{F}}_R, \partial_i\dot{Q}_R\rangle + C\mathfrak{E} + \delta\mathfrak{F}.\end{aligned}$$

Then using Lemmas 3.2–3.3 and Lemma 3.6, we obtain

$$\begin{aligned}& \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathfrak{E}}_1(t) + 4\varepsilon^2\delta\|\partial_i\nabla\mathbf{v}_R\|_{L^2}^2 \\ & \quad \le C\left(1 + \mathfrak{E} + \varepsilon^2\mathfrak{E}^2 + \varepsilon^8\mathfrak{E}^5\right) + \delta\mathfrak{F} + C(\varepsilon + \varepsilon^2\mathfrak{E}^{\frac{1}{2}} + \varepsilon^2\mathfrak{E})\mathfrak{F}.\end{aligned} \tag{3.32}$$

Step 3. *H^2-estimate*. Similar to Step 2, one can deduce that

$$\begin{aligned}& \varepsilon^4\Big\langle\partial_t(\Delta\mathbf{v}_R), \Delta\mathbf{v}_R\Big\rangle + \varepsilon^4 J\Big\langle\partial_t(\Delta\dot{Q}_R), \Delta\dot{Q}_R\Big\rangle \\ & \quad + \frac{\varepsilon^3}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathscr{H}^\varepsilon_{\mathbf{n}}(\Delta Q_R), \Delta Q_R\rangle + 4\varepsilon^4\delta\|\nabla\Delta\mathbf{v}_R\|_{L^2}^2 \\ & \le C\Big(\|\varepsilon^2\Delta\mathbf{G}_R\|_{L^2}\mathfrak{F}^{\frac{1}{2}} + \|\varepsilon^2\Delta\mathbf{G}'_R\|_{L^2}\mathfrak{E}^{\frac{1}{2}} + \|\varepsilon^2\Delta\mathbf{F}_R\|_{L^2}\mathfrak{E}^{\frac{1}{2}}\Big) \\ & \quad + \varepsilon^4\langle\Delta\widetilde{\mathbf{F}}_R, \Delta\dot{Q}_R\rangle + C\mathfrak{E} + \delta\mathfrak{F}.\end{aligned} \tag{3.33}$$

The proof of (3.33) is relegated to the appendix. Likewise, using Lemmas 3.2–3.3 and Lemma 3.6 yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathfrak{E}}_2(t)+4\varepsilon^4\delta\|\nabla\Delta\mathbf{v}_R\|_{L^2}^2\\&\quad\le C\left(1+\mathfrak{E}+\varepsilon^2\mathfrak{E}^2+\varepsilon^8\mathfrak{E}^5\right)+\delta\mathfrak{F}+C\big(\varepsilon+\varepsilon^2\mathfrak{E}^{\frac12}+\varepsilon^2\mathfrak{E}\big)\mathfrak{F}.\end{aligned}\tag{3.34}$$

Combining (3.30), (3.32), and (3.34), we finish the proof of Proposition 3.7. ☐

The following lemma shows that $\widetilde{\mathfrak{E}}(t)$ defined by (3.22) and $\mathfrak{E}(t)$ defined by (1.26) can be controlled by each other.

LEMMA 3.8. *If $\mu_1 \gg J$, then there exist constants $c_0>0$ and $C_0>0$ such that*

$$c_0(1-\varepsilon\mathfrak{E}(t))\mathfrak{E}(t)\le\widetilde{\mathfrak{E}}(t)\le C_0(1+\varepsilon\mathfrak{E}(t))\mathfrak{E}(t).\tag{3.35}$$

Proof. It suffices to prove the first inequality in (3.35). Let $S_R=\dot{Q}_R+\mathbf{v}_R\cdot\nabla Q^\varepsilon+Q_R$, $Q^\varepsilon=\widetilde{Q}+\varepsilon^3Q_R$, and $Q_R=Q_R^\top+Q_R^\perp$ with $Q_R^\top\in\mathrm{Ker}\mathscr{H}_\mathbf{n}$ and $Q_R^\perp\in(\mathrm{Ker}\mathscr{H}_\mathbf{n})^\perp$. Then using Proposition 2.2 we have

$$\begin{aligned}\widetilde{\mathfrak{E}}_0(t)&=\frac12\int_{\mathbb{R}^3}\Big(|\mathbf{v}_R|^2+J|S_R|^2+(\mu_1-J)|Q_R|^2+\frac1\varepsilon\mathscr{H}^\varepsilon_\mathbf{n}(Q_R):Q_R\Big)\mathrm{d}\mathbf{x}\\&\quad+J\big\langle\mathscr{H}_\mathbf{n}^{-1}\mathcal{G}(Q_R^\top),S_R\big\rangle-J\big\langle\mathscr{H}_\mathbf{n}^{-1}\mathcal{G}(Q_R^\top),Q_R^\perp\big\rangle\\&\ge C\int_{\mathbb{R}^3}\Big(|\mathbf{v}_R|^2+|S_R|^2+\frac{\mu_1}{2}|Q_R|^2+\frac1\varepsilon\mathscr{H}^\varepsilon_\mathbf{n}(Q_R):Q_R\Big)\mathrm{d}\mathbf{x}\\&\ge C(1-\varepsilon^3\|\nabla Q_R\|_{L^\infty})\int_{\mathbb{R}^3}\Big(|\mathbf{v}_R|^2+|\dot{Q}_R|^2+|Q_R|^2+\frac1\varepsilon\mathscr{H}^\varepsilon_\mathbf{n}(Q_R):Q_R\Big)\mathrm{d}\mathbf{x}.\end{aligned}$$

Note that for $m=1,2$, by using the Hölder inequality, we estimate

$$\begin{aligned}&\varepsilon^{2m}\big\|\partial_i^m(\mathbf{v}_R\cdot\nabla Q^\varepsilon)\big\|_{L^2}^2\\&\quad\le C\varepsilon^{2m}\|\mathbf{v}_R\|_{H^m}^2+C\varepsilon^{2m+3}\left(\|\mathbf{v}_R\|^2_{H^m}\|\nabla Q_R\|^2_{H^2}+\|\mathbf{v}_R\|^2_{H^2}\|\nabla Q_R\|^2_{H^m}\right)\\&\quad\le C(1+\varepsilon\mathfrak{E})\mathfrak{E},\end{aligned}$$

which yields that

$$\begin{aligned}\widetilde{\mathfrak{E}}_1(t)+\widetilde{\mathfrak{E}}_2(t)\ge C\int_{\mathbb{R}^3}\Big(&|\partial_i\mathbf{v}_R|^2+|\partial_i\dot{Q}_R|^2+\frac1\varepsilon\mathscr{H}^\varepsilon_\mathbf{n}(\partial_iQ_R):\partial_iQ_R\\&+|\Delta\mathbf{v}_R|^2+|\Delta\dot{Q}_R|^2+\frac1\varepsilon\mathscr{H}^\varepsilon_\mathbf{n}(\Delta Q_R):\Delta Q_R\Big)\mathrm{d}\mathbf{x}-\delta_0(1+\varepsilon\mathfrak{E})\mathfrak{E}.\end{aligned}$$

Therefore, there exists a constant $c_0>0$ such that $c_0(1-\varepsilon\mathfrak{E}(t))\mathfrak{E}(t)\le\widetilde{\mathfrak{E}}(t)$. ☐

3.4. The proof Theorem 1.1. Given the initial data $(\mathbf{v}_0^\varepsilon,\partial_tQ_0^\varepsilon,\nabla Q_0^\varepsilon)\in H^2$, it can be proved from the similar energy method in [5] that there exists a maximal time $T_\varepsilon>0$ and a unique solution $(\mathbf{v}^\varepsilon,Q^\varepsilon)$ of the system (1.20)–(1.22) such that

$$(\partial_tQ^\varepsilon,\nabla Q^\varepsilon)\in L^\infty([0,T_\varepsilon);H^2)\cap L^2(0,T_\varepsilon;H^2),\quad\mathbf{v}^\varepsilon\in L^\infty([0,T_\varepsilon);H^2)\cap L^2(0,T_\varepsilon;H^3).$$

From Proposition 3.7 and Lemma 3.8 we have

$$\frac{d}{dt}\widetilde{\mathfrak{E}}(t)+\mathfrak{F}(t)\le C(1+\widetilde{\mathfrak{E}}+\varepsilon^2\widetilde{\mathfrak{E}}^2+\varepsilon^8\widetilde{\mathfrak{E}}^5)+C(\varepsilon+\varepsilon^2\widetilde{\mathfrak{E}}^{\frac12}+\varepsilon^2\widetilde{\mathfrak{E}})\mathfrak{F}$$

for any $t \in [0, T_\varepsilon]$. Under the assumptions of Theorem 1.1, it follows that

$$\widetilde{\mathfrak{E}}(0) \le C_1\Big(\|\mathbf{v}^\varepsilon_{R,0}\|_{H^2} + \|Q^\varepsilon_{R,0}\|_{H^3} + \|\partial_t Q^\varepsilon_{R,0}\|_{H^2} + \varepsilon^{-1}\|\mathscr{P}^{\mathrm{out}}(Q^\varepsilon_{R,0})\|_{L^2}\Big) \le C_1 E_0.$$

Let $\widetilde{E}_1 = (1 + C_1E_0)e^{2CT} > \widetilde{\mathfrak{E}}(0)$, and

$$T_1 = \sup\{t \in [0, T_\varepsilon] : \widetilde{\mathfrak{E}}(t) \le \widetilde{E}_1\}.$$

If we take ε_0 small enough such that

$$4\varepsilon_0\widetilde{E}_1 < c_0, \quad \varepsilon_0^2\widetilde{E}_1 + \varepsilon_0^8\widetilde{E}_1^4 \le 1, \quad C\left(\varepsilon_0 + \varepsilon_0^2\widetilde{E}_1^{\frac{1}{2}} + \varepsilon_0^2\widetilde{E}_1\right) \le 1/2,$$

then for $t \le T_1$, there holds

$$\frac{d}{dt}\widetilde{\mathfrak{E}}(t) \le 2C(1 + \widetilde{\mathfrak{E}}).$$

Therefore, we can infer by means of a continuous argument that $T_1 = T_\varepsilon$, $T \le T_\varepsilon$, and $\widetilde{\mathfrak{E}}(t) \le \widetilde{E}_1$ for $t \in [0, T]$. Moreover, as $c_0(1 - \varepsilon\mathfrak{E}(t))\mathfrak{E}(t) \le \widetilde{\mathfrak{E}}(t) \le \widetilde{E}_1 < c_0/(4\varepsilon_0)$ and $\mathfrak{E}(t)$ is continuous, we know that $\mathfrak{E}(t)$ cannot attain $1/(2\varepsilon)$. Otherwise $\widetilde{E}_1 \ge c_0/(4\varepsilon)$, which yields a contradiction. Therefore, we have $\mathfrak{E}(t) \le 2\widetilde{E}_1/c_0 \triangleq E_1$ for $t \in [0, T]$. This completes the proof of Theorem 1.1.

Appendix A. The energy dissipation relation.

LEMMA A.1. *Assume that* $\beta_1, \beta_4, \mu_1 > 0$*, and* $\beta_4 - \frac{\mu_2^2}{4\mu_1} > 0$*. Then for any smooth solution* $(\mathbf{v}, Q)$ *of the inertial Qian–Sheng system* (1.11)–(1.13)*, it holds that*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^3}\frac{1}{2}(|\mathbf{v}|^2 + J|\dot{Q}|^2)\mathrm{d}\mathbf{x} + \mathcal{F}(Q, \nabla Q)\right)\\
&= -\beta_1\|Q : \mathbf{D}\|^2_{L^2} - \left(\beta_4 - \frac{\mu_2^2}{4\mu_1}\right)\|\mathbf{D}\|^2_{L^2} - (\beta_5 + \beta_6)\langle \mathbf{D}\cdot Q, \mathbf{D}\rangle\\
&\quad - 2\beta_7\|\mathbf{D}\cdot Q\|^2_{L^2} - \mu_1\left\|\dot{Q} - [\mathbf{\Omega}, Q] + \frac{\mu_2}{2\mu_1}\mathbf{D}\right\|^2_{L^2}.
\end{aligned} \tag{A.1}$$

Moreover, assume one of the following assumptions holds: (i) $\beta_5 + \beta_6 = 0$ *if* $\beta_7 = 0$, (ii) $(\beta_5 + \beta_6)^2 < 8\beta_7(\beta_4 - \frac{\mu_2^2}{4\mu_1})$ *if* $\beta_7 \ne 0$*; then the right-hand side in* (A.1) *is nonpositive.*

Proof. First, using either of the assumptions, it is easy to obtain that

$$\left(\beta_4 - \frac{\mu_2^2}{4\mu_1}\right)|\mathbf{D}|^2 + (\beta_5 + \beta_6)(\mathbf{D}\cdot Q : \mathbf{D}) + 2\beta_7|\mathbf{D}\cdot Q|^2_{L^2} > c_0|\mathbf{D}|^2$$

for some $c_0 > 0$. Now we prove (A.1). Taking the L^2-inner product with $\dot{Q}$ in (1.11), and taking the L^2-inner product with $\mathbf{v}$ in (1.12), we get

$$\begin{aligned}
&J\langle \ddot{Q}, \dot{Q}\rangle + \langle \partial_t\mathbf{v}, \mathbf{v}\rangle\\
&= -\mu_1\big\langle \dot{Q} - [\mathbf{\Omega}, Q], \dot{Q}\big\rangle + \langle \mathbf{H}, \dot{Q}\rangle - \frac{\mu_2}{2}\langle \mathbf{D}, \dot{Q}\rangle + \langle \nabla\cdot\sigma^d, \mathbf{v}\rangle\\
&\quad - \Big\langle \beta_1 Q(Q : \mathbf{D}) + \beta_4\mathbf{D} + \beta_5\mathbf{D}\cdot Q + \beta_6 Q\cdot\mathbf{D} + \beta_7(\mathbf{D}\cdot Q^2 + Q^2\cdot\mathbf{D}), \nabla\mathbf{v}\Big\rangle\\
&\quad - \frac{\mu_2}{2}\langle \dot{Q} - [\mathbf{\Omega}, Q], \nabla\mathbf{v}\rangle - \mu_1\Big\langle [Q, (\dot{Q} - [\mathbf{\Omega}, Q])], \nabla\mathbf{v}\Big\rangle\\
&\stackrel{\text{def}}{=} I + II + III + IV + V + VI + VII.
\end{aligned}$$

For terms I and VII, we have

$$\begin{aligned} I+VII =& -\mu_1\big\langle \dot{Q}-[\mathbf{\Omega},Q],\dot{Q}\big\rangle - \mu_1\Big\langle [Q,(\dot{Q}-[\mathbf{\Omega},Q])],\mathbf{D}+\mathbf{\Omega}\Big\rangle \\ =& -\mu_1\big\langle \dot{Q}-[\mathbf{\Omega},Q],\dot{Q}\big\rangle - \mu_1\Big\langle [Q,\mathbf{\Omega}],(\dot{Q}-[\mathbf{\Omega},Q])\Big\rangle \\ =& -\mu_1\|\dot{Q}-[\mathbf{\Omega},Q]\|_{L^2}^2. \end{aligned}$$

Recalling the relation $\beta_6-\beta_5=\mu_2$, we can deduce that

$$\begin{aligned} III+V+VI =& -\Big\langle \beta_1 Q(Q:\mathbf{D})+\beta_4\mathbf{D}+\beta_5\mathbf{D}\cdot Q+\beta_6 Q\cdot\mathbf{D},\mathbf{D}+\mathbf{\Omega}\Big\rangle \\ & -\Big\langle \beta_7(\mathbf{D}\cdot Q^2+Q^2\cdot\mathbf{D}),\mathbf{D}\Big\rangle - \frac{\mu_2}{2}\big\langle 2\dot{Q}-[\mathbf{\Omega},Q],\mathbf{D}\big\rangle \\ =& -\Big\langle \beta_1 Q(Q:\mathbf{D})+\beta_4\mathbf{D}+\frac{\beta_5+\beta_6}{2}(\mathbf{D}\cdot Q+Q\cdot\mathbf{D}),\mathbf{D}\Big\rangle \\ & +\Big\langle\Big(\frac{\beta_5+\beta_6}{2}-\beta_5\Big)\mathbf{D}\cdot Q+\Big(\frac{\beta_5+\beta_6}{2}-\beta_6\Big)Q\cdot\mathbf{D},\mathbf{D}+\mathbf{\Omega}\Big\rangle \\ & -\Big\langle \beta_7(\mathbf{D}\cdot Q^2+Q^2\cdot\mathbf{D}),\mathbf{D}\Big\rangle - \frac{\mu_2}{2}\big\langle 2\dot{Q}-[\mathbf{\Omega},Q],\mathbf{D}\big\rangle \\ =& -\beta_1\|Q:\mathbf{D}\|_{L^2}^2-\beta_4\|\mathbf{D}\|_{L^2}^2-(\beta_5+\beta_6)\langle\mathbf{D}\cdot Q,\mathbf{D}\rangle \\ & -2\beta_7\|\mathbf{D}\cdot Q\|_{L^2}^2-\mu_2\big\langle\dot{Q}-[\mathbf{\Omega},Q],\mathbf{D}\big\rangle. \end{aligned}$$

Further, it follows that

$$\begin{aligned} & I+III+V+VI+VII \\ &\quad = -\beta_1\|Q:\mathbf{D}\|_{L^2}^2-\beta_4\|\mathbf{D}\|_{L^2}^2-(\beta_5+\beta_6)\langle\mathbf{D}\cdot Q,\mathbf{D}\rangle \\ &\qquad -2\beta_7\|\mathbf{D}\cdot Q\|_{L^2}^2-\mu_2\big\langle\dot{Q}-[\mathbf{\Omega},Q],\mathbf{D}\big\rangle-\mu_1\|\dot{Q}-[\mathbf{\Omega},Q]\|_{L^2}^2 \\ &\quad = -\beta_1\|Q:\mathbf{D}\|_{L^2}^2-\Big(\beta_4-\frac{\mu_2^2}{4\mu_1}\Big)\|\mathbf{D}\|_{L^2}^2-(\beta_5+\beta_6)\langle\mathbf{D}\cdot Q,\mathbf{D}\rangle \\ &\qquad -2\beta_7\|\mathbf{D}\cdot Q\|_{L^2}^2-\mu_1\Big\|\dot{Q}-[\mathbf{\Omega},Q]+\frac{\mu_2}{2\mu_1}\mathbf{D}\Big\|_{L^2}^2. \end{aligned}$$

For the second term II, noting that $\mathbf{H}(Q)=-\frac{\delta\mathcal{F}}{\delta Q}$ and $\nabla\cdot\mathbf{v}=0$, we have

$$\begin{aligned} II =& -\Big\langle\frac{\delta\mathcal{F}}{\delta Q},\partial_t Q\Big\rangle+\langle\mathbf{H}(Q),\mathbf{v}\cdot\nabla Q\rangle \\ =& -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(Q,\nabla Q)+\langle\mathbf{H}(Q),\mathbf{v}\cdot\nabla Q\rangle. \end{aligned}$$

Using the definition of the distortion stress σ^d, we can infer that

$$\begin{aligned} IV =& -\int_{\mathbb{R}^3}\partial_j\left(\frac{\partial\mathcal{F}}{\partial Q_{kl,j}}Q_{kl,i}\right)v_i\mathrm{d}\mathbf{x} \\ =& -\int_{\mathbb{R}^3}\left(\partial_j\left(\frac{\partial\mathcal{F}}{\partial Q_{kl,j}}\right)Q_{kl,i}+\frac{\partial\mathcal{F}}{\partial Q_{kl,j}}Q_{kl,ij}\right)v_i\mathrm{d}\mathbf{x} \\ =& -\int_{\mathbb{R}^3}\left(H_{kl}(Q)Q_{kl,i}+\frac{\partial\mathcal{F}}{\partial Q_{kl}}Q_{kl,i}+\frac{\partial\mathcal{F}}{\partial Q_{kl,j}}Q_{kl,ij}\right)v_i\mathrm{d}\mathbf{x} \\ =& -\int_{\mathbb{R}^3}\Big(H_{kl}(Q)Q_{kl,i}+\partial_i\mathcal{F}(Q,\nabla Q)\Big)v_i\mathrm{d}\mathbf{x} \\ =& -\langle\mathbf{H}(Q),\mathbf{v}\cdot\nabla Q\rangle. \end{aligned}$$

In conclusion, under the assumptions of Lemma A.1, we obtain (A.1). ∎

Appendix B. The estimate of $\varepsilon^2\langle\partial_i\widetilde{\mathbf{F}}_R,\partial_i\dot{Q}_R\rangle$. Similar arguments for Lemma 3.6 will be applied to the estimate of higher order derivative terms. First of all, note that $\langle\partial_i\mathbf{v}_R\cdot\nabla Q_0,\mathscr{H}_{\mathbf{n}}(\partial_i Q_R)\rangle=0$; then we have

$$\begin{aligned}
&-\varepsilon^2\Big\langle\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\frac{1}{\varepsilon}\partial_i\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R)\Big\rangle\\
&\quad\le-\varepsilon^2\Big\langle\partial_i\mathbf{v}_R\cdot\nabla(Q_0+\varepsilon\widehat{Q}^\varepsilon),\frac{1}{\varepsilon}\partial_i\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R)\Big\rangle\\
&\qquad+C\varepsilon\|\mathbf{v}_R\|_{L^2}\big(\|\partial_i\mathscr{H}_{\mathbf{n}}(Q_R)\|_{L^2}+\varepsilon\|\partial_i\mathscr{L}(Q_R)\|_{L^2}\big)\\
&\quad\le-\varepsilon^3\Big\langle\partial_i\mathbf{v}_R\cdot\nabla\widehat{Q}^\varepsilon,\frac{1}{\varepsilon}\partial_i\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R)\Big\rangle+C\mathfrak{E}\\
&\quad\le C\varepsilon^2\|\partial_i\mathbf{v}_R\|_{L^2}\big(\|\partial_i\mathscr{H}_{\mathbf{n}}(Q_R)\|_{L^2}+\varepsilon\|\partial_i\mathscr{L}(Q_R)\|_{L^2}\big)+C\mathfrak{E}\\
&\quad\le C\mathfrak{E}.
\end{aligned}\tag{B.1}$$

Recalling (3.2), we derive from the integration by parts over $\mathbf{x}\in\mathbb{R}^3$ that

$$\begin{aligned}
\varepsilon^2\Big\langle\partial_i\widetilde{\mathbf{F}}_1,\partial_i\dot{Q}_R\Big\rangle&=-\varepsilon^2J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\partial_i\dot{Q}_R\Big\rangle-\varepsilon^2J\Big\langle\partial_i\tilde{\mathbf{v}}\cdot\nabla(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\partial_i\dot{Q}_R\Big\rangle\\
&\quad+\varepsilon^2J\Big\langle\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\partial_i\ddot{Q}_R\Big\rangle-\varepsilon^2J\Big\langle\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\partial_i\tilde{\mathbf{v}}\cdot\nabla\dot{Q}_R\Big\rangle\\
&\le-\varepsilon^2J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\partial_i\dot{Q}_R\Big\rangle+\varepsilon^2\Big\langle\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\partial_i\widetilde{\mathbf{F}}_R\Big\rangle\\
&\quad+C\mathfrak{E}^{\frac12}\|\varepsilon\partial_i\mathbf{F}_R\|_{L^2}+C(\mathfrak{E}+\mathfrak{E}^{\frac12}\mathfrak{F}^{\frac12}),
\end{aligned}\tag{B.2}$$

where we have applied Lemma 3.1 and (B.1), and the following estimates:

$$\begin{aligned}
&-\varepsilon^2J\langle\partial_i\tilde{\mathbf{v}}\cdot\nabla(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\partial_i\dot{Q}_R\rangle\le C\varepsilon^2\|\mathbf{v}_R\|_{H^1}\|\partial_i\dot{Q}_R\|_{L^2}\le C\mathfrak{E},\\
&-\varepsilon^2J\langle\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\partial_i\tilde{\mathbf{v}}\cdot\nabla\dot{Q}_R\rangle\le C\varepsilon^2\|\mathbf{v}_R\|_{H^1}\|\nabla\dot{Q}_R\|_{L^2}\le C\mathfrak{E},
\end{aligned}$$

and

$$\begin{aligned}
&\varepsilon^2J\Big\langle\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\partial_i\ddot{Q}_R\Big\rangle\\
&\quad=-\mu_1\varepsilon^2\Big\langle\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\partial_i\dot{Q}_R\Big\rangle-\varepsilon^2\Big\langle\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\frac{1}{\varepsilon}\partial_i\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R)\Big\rangle\\
&\qquad+\varepsilon^2\Big\langle\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}),-\frac{\mu_2}{2}\partial_i\mathbf{D}_R+\mu_1\partial_i[\mathbf{\Omega}_R,Q_0]\Big\rangle\\
&\qquad+\varepsilon^2\langle\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\partial_i\mathbf{F}_R+\partial_i\widetilde{\mathbf{F}}_R\rangle\\
&\quad\le C\varepsilon^2\|\mathbf{v}_R\|_{H^1}(\|\partial_i\dot{Q}_R\|_{L^2}+\|\partial_i\nabla\mathbf{v}_R\|_{L^2})+C\mathfrak{E}\\
&\qquad+C\varepsilon\|\mathbf{v}_R\|_{H^1}\|\varepsilon\partial_i\mathbf{F}_R\|_{L^2}+\varepsilon^2\langle\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\partial_i\widetilde{\mathbf{F}}_R\rangle\\
&\quad\le C(\mathfrak{E}+\mathfrak{E}^{\frac12}\mathfrak{F}^{\frac12})+C\mathfrak{E}^{\frac12}\|\varepsilon\partial_i\mathbf{F}_R\|_{L^2}+\varepsilon^2\langle\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\partial_i\widetilde{\mathbf{F}}_R\rangle.
\end{aligned}\tag{B.3}$$

We proceed to deal with the term $\varepsilon^2\langle\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\partial_i\widetilde{\mathbf{F}}_R\rangle$. Using integration by parts yields

$$\begin{aligned}
&\varepsilon^2\Big\langle\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\partial_i\widetilde{\mathbf{F}}_1\Big\rangle\\
&\quad=-\varepsilon^2J\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q})\|^2_{L^2}-\varepsilon^2J\Big\langle\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\partial_i\tilde{\mathbf{v}}\cdot\nabla(\mathbf{v}_R\cdot\nabla\widetilde{Q})\Big\rangle\\
&\quad\le-\varepsilon^2J\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q})\|^2_{L^2}+C\mathfrak{E}.
\end{aligned}\tag{B.4}$$

It is obvious from integration by parts that

$$
\begin{aligned}
&-\varepsilon^5 J\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}), (\partial_t+\tilde{\mathbf{v}}\cdot\nabla)\partial_i(\mathbf{v}_R\cdot\nabla Q_R)\Big\rangle\\
&\quad= -\varepsilon^5 J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}), \partial_i(\mathbf{v}_R\cdot\nabla Q_R)\Big\rangle\\
&\qquad+\underbrace{\varepsilon^5 J\Big\langle (\partial_t+\tilde{\mathbf{v}}\cdot\nabla)\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}), \partial_i(\mathbf{v}_R\cdot\nabla Q_R)\Big\rangle}_{\mathbf{S}_2}.
\end{aligned}
$$

Then by Lemma 3.1 we have

(B.5)

$$
\begin{aligned}
&\varepsilon^2\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}), \partial_i\widetilde{\mathbf{F}}_2\Big\rangle\\
&\quad= -\varepsilon^5 J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}), \partial_i(\mathbf{v}_R\cdot\nabla Q_R)\Big\rangle+\mathbf{S}_2\\
&\qquad-\varepsilon^5 J\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}), \partial_i\tilde{\mathbf{v}}\cdot\nabla(\mathbf{v}_R\cdot\nabla Q_R)\Big\rangle+\varepsilon^5 J\Big\langle \Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}), \mathbf{v}_R\cdot\nabla\dot{Q}_R\Big\rangle\\
&\qquad-\varepsilon^8 J\Big\langle (\mathbf{v}_R\cdot\nabla)\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}), \mathbf{v}_R\cdot\nabla Q_R\Big\rangle\\
&\quad\le -\varepsilon^5 J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}), \partial_i(\mathbf{v}_R\cdot\nabla Q_R)\Big\rangle+\mathbf{S}_2+C\varepsilon^5\|\mathbf{v}_R\|_{H^1}\|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{H^1}\\
&\qquad+C\varepsilon^5\|\mathbf{v}_R\|_{H^2}^2\|\nabla\dot{Q}_R\|_{L^2}+C\varepsilon^8\|\mathbf{v}_R\|_{H^2}^2\|\mathbf{v}_R\|_{H^3}\|\nabla Q_R\|_{L^2}\\
&\quad\le -\varepsilon^5 J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}), \partial_i(\mathbf{v}_R\cdot\nabla Q_R)\Big\rangle+\mathbf{S}_2\\
&\qquad+C\big(\varepsilon\mathfrak{E}^{\frac32}+\varepsilon^2\mathfrak{E}^2+\varepsilon\mathfrak{E}\mathfrak{F}^{\frac12}+\varepsilon^2\mathfrak{E}^{\frac32}\mathfrak{F}^{\frac12}\big).
\end{aligned}
$$

Thus from (B.4) and (B.5) we conclude that

$$
\begin{aligned}
&\varepsilon^2\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}), \partial_i\widetilde{\mathbf{F}}_R\Big\rangle\\
&\quad\le -\varepsilon^2 J\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q})\|_{L^2}^2-\varepsilon^5 J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q}), \partial_i(\mathbf{v}_R\cdot\nabla Q_R)\Big\rangle\\
&\qquad+\mathbf{S}_2+C\left(\mathfrak{E}+\varepsilon^2\mathfrak{E}^2+\varepsilon^2\mathfrak{E}\mathfrak{F}\right). \qquad\text{(B.6)}
\end{aligned}
$$

We are now in a position to estimate the term $\varepsilon^2\langle\partial_i\widetilde{\mathbf{F}}_2, \partial_i\dot{Q}_R\rangle$. First, via employing integration by parts we find

$$
\begin{aligned}
&\varepsilon^2\Big\langle \partial_i\widetilde{\mathbf{F}}_2, \partial_i\dot{Q}_R\Big\rangle\\
&\quad= -\varepsilon^5 J\Big\langle \partial_i(\partial_t+\tilde{\mathbf{v}}\cdot\nabla)(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\dot{Q}_R\Big\rangle-\varepsilon^5 J\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla\dot{Q}_R), \partial_i\dot{Q}_R\Big\rangle\\
&\qquad-\varepsilon^8 J\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla(\mathbf{v}_R\cdot\nabla Q_R)), \partial_i\dot{Q}_R\Big\rangle\\
&\quad= -\varepsilon^5 J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\dot{Q}_R\Big\rangle+\varepsilon^5 J\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\ddot{Q}_R\Big\rangle\\
&\qquad\underbrace{-\varepsilon^5 J\Big\langle \partial_i\tilde{\mathbf{v}}\cdot\nabla(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\dot{Q}_R\Big\rangle}_{\mathcal{B}_1}\underbrace{-\varepsilon^5 J\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\tilde{\mathbf{v}}\cdot\nabla\dot{Q}_R\Big\rangle}_{\mathcal{B}_2}\\
&\qquad\underbrace{-\varepsilon^5 J\Big\langle \partial_i\mathbf{v}_R\cdot\nabla\dot{Q}_R, \partial_i\dot{Q}_R\Big\rangle}_{\mathcal{B}_3}\underbrace{-\varepsilon^8 J\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla(\mathbf{v}_R\cdot\nabla Q_R)), \partial_i\dot{Q}_R\Big\rangle}_{\mathcal{B}_4}.
\end{aligned}
$$

Using Lemma 3.1, we have

$$\mathcal{B}_1 \le C\varepsilon^5 \|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{H^1}\|\partial_i \dot{Q}_R\|_{L^2} \le C\varepsilon \mathfrak{E}^{\frac{3}{2}},$$
$$\mathcal{B}_2 \le C\varepsilon^5 \|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{H^1}\|\nabla \dot{Q}_R\|_{L^2} \le C\varepsilon \mathfrak{E}^{\frac{3}{2}},$$
$$\mathcal{B}_3 \le C\varepsilon^5 \|\partial_i\mathbf{v}_R\|_{H^2}\|\nabla \dot{Q}_R\|_{L^2}\|\partial_i \dot{Q}_R\|_{L^2} \le C\varepsilon \mathfrak{E}\mathfrak{F}^{\frac{1}{2}}$$

and

$$\begin{aligned}\mathcal{B}_4 =& -\varepsilon^8 J\Big\langle \partial_i \mathbf{v}_R\cdot\nabla(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\dot{Q}_R\Big\rangle - \underbrace{\varepsilon^8 J\Big\langle (\mathbf{v}_R\cdot\nabla)\partial_i(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\dot{Q}_R\Big\rangle}_{\mathbf{W}_2}\\ \le& -\mathbf{W}_2 + C\varepsilon^8\|\partial_i\mathbf{v}_R\|_{H^2}\|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{H^1}\|\partial_i\dot{Q}_R\|_{L^2}\\ \le& -\mathbf{W}_2 + C(\varepsilon^2\mathfrak{E}^2 + \varepsilon^2\mathfrak{E}^{\frac{3}{2}}\mathfrak{F}^{\frac{1}{2}}).\end{aligned}$$

Similar to the estimate of (B.3), from (3.2) we get

$$\begin{aligned}&\varepsilon^5 J\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\ddot{Q}_R\Big\rangle\\ &\quad= -\varepsilon^5\mu_1\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\dot{Q}_R\Big\rangle - \varepsilon^5\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla Q_R), \frac{1}{\varepsilon}\partial_i\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R)\Big\rangle\\ &\qquad+ \varepsilon^5\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla Q_R), -\frac{\mu_2}{2}\partial_i\mathbf{D}_R + \mu_1\partial_i[\mathbf{\Omega}_R, Q_0]\Big\rangle\\ &\qquad+ \varepsilon^5\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\mathbf{F}_R + \partial_i\widetilde{\mathbf{F}}_R\Big\rangle\\ &\quad\le C\varepsilon^5\|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{H^1}\big(\|\partial_i\dot{Q}_R\|_{L^2} + \|\partial_i\nabla\mathbf{v}_R\|_{L^2} + \|\partial_i\mathbf{F}_R\|_{L^2}\big)\\ &\qquad+ C\varepsilon^4\|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{H^1}\big(\|\partial_i\mathscr{H}_{\mathbf{n}}(Q_R)\|_{L^2} + \varepsilon\|\partial_i\mathscr{L}(Q_R)\|_{L^2}\big)\\ &\qquad+ \varepsilon^5\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\widetilde{\mathbf{F}}_R\Big\rangle\\ &\quad\le \varepsilon^5\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\widetilde{\mathbf{F}}_R\Big\rangle + C\big(\varepsilon\mathfrak{E}^{\frac{3}{2}} + \varepsilon\mathfrak{E}\mathfrak{F}^{\frac{1}{2}}\big) + C\varepsilon\mathfrak{E}\|\varepsilon\partial_i\mathbf{F}_R\|_{L^2}.\end{aligned}$$

Thus collecting the above estimates, we can deduce that

$$\begin{aligned}\varepsilon^2\Big\langle \partial_i\widetilde{\mathbf{F}}_2, \partial_i\dot{Q}_R\Big\rangle \le& -\varepsilon^5 J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\dot{Q}_R\Big\rangle + \varepsilon^5\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\widetilde{\mathbf{F}}_R\Big\rangle\\ &- \mathbf{W}_2 + C\left(1 + \mathfrak{E} + \varepsilon^2\mathfrak{E}^2 + \varepsilon^8\mathfrak{E}^5\right) + C\varepsilon^2\mathfrak{E}\mathfrak{F}.\end{aligned} \tag{B.7}$$

Our next task is to calculate the term $\varepsilon^5\langle\partial_i(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\widetilde{\mathbf{F}}_R\rangle$. It is evident to see from integration by parts that

$$\begin{aligned}\varepsilon^5\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\widetilde{\mathbf{F}}_1\Big\rangle =& -\mathbf{S}_2 - J\varepsilon^5\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\tilde{\mathbf{v}}\cdot\nabla(\mathbf{v}_R\cdot\nabla\widetilde{Q})\Big\rangle\\ \le& -\mathbf{S}_2 + C\varepsilon^5\|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{H^1}\|\mathbf{v}_R\|_{H^1}\\ \le& -\mathbf{S}_2 + C\varepsilon\mathfrak{E}^{\frac{3}{2}}.\end{aligned}$$

In addition, by integrating by parts we also have

$$\begin{aligned}&\varepsilon^5\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\widetilde{\mathbf{F}}_2\Big\rangle\\ &\quad= -\varepsilon^8 J\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i(\mathbf{v}_R\cdot\nabla Q_R)\|^2_{L^2} - \varepsilon^8 J\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\tilde{\mathbf{v}}\cdot\nabla(\mathbf{v}_R\cdot\nabla Q_R)\Big\rangle\\ &\qquad+ \mathbf{W}_2 - \varepsilon^8 J\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla Q_R), \partial_i\mathbf{v}_R\cdot\nabla\dot{Q}_R\Big\rangle\\ &\qquad- \varepsilon^{11} J\Big\langle \partial_i(\mathbf{v}_R\cdot\nabla Q_R), \partial_i(\mathbf{v}_R\cdot\nabla(\mathbf{v}_R\cdot\nabla Q_R))\Big\rangle\end{aligned}$$

$$
\begin{aligned}
&\le -\varepsilon^8 J\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i(\mathbf{v}_R\cdot\nabla Q_R)\|_{L^2}^2 + \mathbf{W}_2 + C\varepsilon^8\|\mathbf{v}_R\|_{H^2}^2\|\nabla Q_R\|_{H^1}^2 \\
&\quad + C\varepsilon^8\|\mathbf{v}_R\|_{H^2}^2\|\nabla Q_R\|_{H^2}\|\nabla\dot{Q}_R\|_{L^2} + C\varepsilon^{11}\|\mathbf{v}_R\|_{H^2}^2\|\nabla Q_R\|_{H^1}^2\|\partial_i\mathbf{v}_R\|_{H^2} \\
&\le -\varepsilon^8 J\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i(\mathbf{v}_R\cdot\nabla Q_R)\|_{L^2}^2 + \mathbf{W}_2 + C\big(\varepsilon^2\mathfrak{E}^2 + \varepsilon^4\mathfrak{E}^{\frac52} + \varepsilon^2\mathfrak{E}^{\frac32}\mathfrak{F}^{\frac12} + \varepsilon^3\mathfrak{E}^2\mathfrak{F}^{\frac12}\big).
\end{aligned}
$$

Thus, combining the latest two bounds with (B.2) and (B.6)–(B.7), it follows that

$$
\begin{aligned}
\varepsilon^2\Big\langle\partial_i\mathbf{F}_R,\partial_i\dot{Q}_R\Big\rangle \le{}& -\varepsilon^2\frac{J}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i(\mathbf{v}_R\cdot Q^\varepsilon)\|_{L^2}^2 - \varepsilon^2 J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle\partial_i(\mathbf{v}_R\cdot\nabla Q^\varepsilon),\partial_i\dot{Q}_R\Big\rangle \\
&+ C(1+\mathfrak{E}+\varepsilon^2\mathfrak{E}^2+\varepsilon^8\mathfrak{E}^5) + (\delta + C\varepsilon^2\mathfrak{E})\mathfrak{F}.
\end{aligned}
$$

Appendix C. The estimate of $\boldsymbol{\varepsilon^4\langle\Delta\widetilde{\mathbf{F}}_R,\Delta\dot{Q}_R\rangle}$. First note that

$$
\Delta(\partial_t+\tilde{\mathbf{v}}\cdot\nabla) = (\partial_t+\tilde{\mathbf{v}}\cdot\nabla)\Delta + \Delta\tilde{\mathbf{v}}\cdot\nabla + 2\partial_i\tilde{\mathbf{v}}\cdot\nabla\partial_i;
$$

then from integration by parts we obtain

$$
\begin{aligned}
\varepsilon^4\Big\langle\Delta\widetilde{\mathbf{F}}_1,\Delta\dot{Q}_R\Big\rangle &= -\varepsilon^4 J\Big\langle\Delta(\partial_t+\tilde{\mathbf{v}}\cdot\nabla)(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\Delta\dot{Q}_R\Big\rangle \\
&= -\varepsilon^4 J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\Delta\dot{Q}_R\Big\rangle + \varepsilon^4 J\Big\langle\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\Delta\ddot{Q}_R\Big\rangle \\
&\quad \underbrace{-\varepsilon^4 J\Big\langle(\Delta\tilde{\mathbf{v}}\cdot\nabla+2\partial_i\tilde{\mathbf{v}}\cdot\nabla\partial_i)(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\Delta\dot{Q}_R\Big\rangle}_{\mathcal{C}_1} \\
&\quad \underbrace{-\varepsilon^4 J\Big\langle\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),(\Delta\tilde{\mathbf{v}}\cdot\nabla+2\partial_i\tilde{\mathbf{v}}\cdot\nabla\partial_i)\dot{Q}_R\Big\rangle}_{\mathcal{C}_2}.
\end{aligned}
$$

It can be estimated by Lemma 3.1 that

$$
\begin{aligned}
\mathcal{C}_1 &\le C\varepsilon^4\|\mathbf{v}_R\|_{H^2}\|\Delta\dot{Q}_R\|_{L^2} \le C\mathfrak{E}, \\
\mathcal{C}_2 &\le C\varepsilon^4\|\mathbf{v}_R\|_{H^2}\|\dot{Q}_R\|_{H^2} \le C\mathfrak{E}.
\end{aligned}
$$

Keeping (3.2) in mind, we can deduce that

$$
\begin{aligned}
&\varepsilon^4 J\Big\langle\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\Delta\ddot{Q}_R\Big\rangle \\
&= \underbrace{-\varepsilon^4\mu_1\Big\langle\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\Delta\dot{Q}_R\Big\rangle}_{\mathcal{C}_3} \underbrace{-\varepsilon^4\Big\langle\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\frac{1}{\varepsilon}\Delta\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R)\Big\rangle}_{\mathcal{C}_4} \\
&\quad + \underbrace{\varepsilon^4\Big\langle\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),-\frac{\mu_2}{2}\Delta\mathbf{D}_R+\mu_1\Delta[\mathbf{\Omega}_R,Q_0]\Big\rangle}_{\mathcal{C}_5} + \varepsilon^4\Big\langle\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\Delta\mathbf{F}_R+\Delta\widetilde{\mathbf{F}}_R\Big\rangle.
\end{aligned}
$$

Using Lemma 3.1, we have

$$
\begin{aligned}
\mathcal{C}_3 &\le C\varepsilon^4\|\mathbf{v}_R\|_{H^2}\|\Delta\dot{Q}_R\|_{L^2} \le C\mathfrak{E}, \\
\mathcal{C}_5 &\le C\varepsilon^4\|\mathbf{v}_R\|_{H^2}\|\nabla\Delta\mathbf{v}_R\|_{L^2} \le C\mathfrak{E}^{\frac12}\mathfrak{F}^{\frac12}, \\
\mathcal{C}_4 &= \varepsilon^3\Big\langle\partial_i\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\partial_i\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R)\Big\rangle \\
&\quad \le C\varepsilon^3\|\mathbf{v}_R\|_{H^3}\big(\|\partial_i\mathscr{H}_{\mathbf{n}}(Q_R)\|_{L^2} + \varepsilon\|\partial_i\mathscr{L}(Q_R)\|_{L^2}\big) \le C(\mathfrak{E}+\mathfrak{E}^{\frac12}\mathfrak{F}^{\frac12}).
\end{aligned}
$$

Then we get

$$\varepsilon^4\Big\langle\Delta\widetilde{\mathbf{F}}_1,\Delta\dot{Q}_R\Big\rangle\le-\varepsilon^4J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\Delta\dot{Q}_R\Big\rangle+\varepsilon^4\langle\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\Delta\widetilde{\mathbf{F}}_R\rangle$$
$$+C\mathfrak{E}^{\frac12}\|\varepsilon^2\Delta\mathbf{F}_R\|_{L^2}+C\big(\mathfrak{E}+\mathfrak{E}^{\frac12}\mathfrak{F}^{\frac12}\big). \tag{C.1}$$

Next, we estimate the quantity $\varepsilon^4\langle\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\Delta\widetilde{\mathbf{F}}_R\rangle$. Direct calculations yield that

$$\begin{aligned}&\varepsilon^4\Big\langle\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\Delta\widetilde{\mathbf{F}}_1\Big\rangle\\&=-\varepsilon^4J\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q})\|^2_{L^2}-\varepsilon^4J\Big\langle\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\Delta\tilde{\mathbf{v}}\cdot\nabla(\mathbf{v}_R\cdot\nabla\widetilde{Q})\Big\rangle\\&\quad-\varepsilon^4J\Big\langle\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),2\partial_i\tilde{\mathbf{v}}\cdot\nabla\partial_i(\mathbf{v}_R\cdot\nabla\widetilde{Q})\Big\rangle\\&\le-\varepsilon^4J\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q})\|^2_{L^2}+C\mathfrak{E}.\end{aligned}$$

Using integration by parts, we derive the following bound:

$$\begin{aligned}&\varepsilon^4\Big\langle\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\Delta\widetilde{\mathbf{F}}_2\Big\rangle\\&=-\varepsilon^7J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\Delta(\mathbf{v}_R\cdot\nabla Q_R)\Big\rangle+\underbrace{\varepsilon^7J\Big\langle\partial_i\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\partial_i(\mathbf{v}_R\cdot\nabla\dot{Q}_R)\Big\rangle}_{\mathcal{C}_6}\\&\quad+\underbrace{\varepsilon^7J\Big\langle(\partial_t+\tilde{\mathbf{v}}\cdot\nabla)\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\Delta(\mathbf{v}_R\cdot\nabla Q_R)\Big\rangle}_{\mathbf{S}_3}\\&\quad\underbrace{-\varepsilon^7J\Big\langle\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),(\Delta\tilde{\mathbf{v}}\cdot\nabla+2\partial_i\tilde{\mathbf{v}}\cdot\nabla\partial_i)(\mathbf{v}_R\cdot\nabla Q_R)\Big\rangle}_{\mathcal{C}_7}\\&\quad+\underbrace{\varepsilon^{10}J\Big\langle\partial_i\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\partial_i(\mathbf{v}_R\cdot\nabla(\mathbf{v}_R\cdot\nabla Q_R))\Big\rangle}_{\mathcal{C}_8}.\end{aligned}$$

According to Lemma 3.1, we obtain

$$\begin{aligned}&\mathcal{C}_6\le C\varepsilon^7\|\mathbf{v}_R\|_{H^3}\|\mathbf{v}_R\|_{H^2}\|\nabla\dot{Q}_R\|_{H^1}\le C\big(\varepsilon\mathfrak{E}^{\frac32}+\varepsilon\mathfrak{E}\mathfrak{F}^{\frac12}\big),\\&\mathcal{C}_7\le C\varepsilon^7\|\mathbf{v}_R\|^2_{H^2}\|\nabla Q_R\|_{H^2}\le C\varepsilon\mathfrak{E}^{\frac32},\\&\mathcal{C}_8\le C\varepsilon^{10}\|\mathbf{v}_R\|_{H^3}\|\mathbf{v}_R\|^2_{H^2}\|\nabla Q_R\|_{H^2}\le C\big(\varepsilon^2\mathfrak{E}^2+\varepsilon^2\mathfrak{E}^{\frac32}\mathfrak{F}^{\frac12}\big).\end{aligned}$$

Then we have

$$\begin{aligned}&\varepsilon^4\Big\langle\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\Delta\widetilde{\mathbf{F}}_R\Big\rangle\\&\quad\le-\varepsilon^4J\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q})\|^2_{L^2}-\varepsilon^7J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle\Delta(\mathbf{v}_R\cdot\nabla\widetilde{Q}),\Delta(\mathbf{v}_R\cdot\nabla Q_R)\Big\rangle\end{aligned}$$
$$+\mathbf{S}_3+C\big(\mathfrak{E}+\varepsilon\mathfrak{E}^{\frac32}+\varepsilon^2\mathfrak{E}^2+\varepsilon\mathfrak{E}\mathfrak{F}^{\frac12}+\varepsilon^2\mathfrak{E}^{\frac32}\mathfrak{F}^{\frac12}\big). \tag{C.2}$$

Finally, it remains to estimate $\varepsilon^4\langle\Delta\widetilde{\mathbf{F}}_2,\Delta\dot{Q}_R\rangle$. By integration by parts, we have

$$\begin{aligned}&\varepsilon^4\Big\langle\Delta\widetilde{\mathbf{F}}_2,\Delta\dot{Q}_R\Big\rangle\\&\quad=-\varepsilon^7J\Big\langle\Delta(\partial_t+\tilde{\mathbf{v}}\cdot\nabla)(\mathbf{v}_R\cdot\nabla Q_R),\Delta\dot{Q}_R\Big\rangle-\varepsilon^7J\Big\langle\Delta(\mathbf{v}_R\cdot\nabla\dot{Q}_R),\Delta\dot{Q}_R\Big\rangle\\&\quad\quad-\varepsilon^{10}J\Big\langle\Delta(\mathbf{v}_R\cdot\nabla(\mathbf{v}_R\cdot\nabla Q_R)),\Delta\dot{Q}_R\Big\rangle\end{aligned}$$

$$
\begin{aligned}
&= -\varepsilon^7 J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle \Delta(\mathbf{v}_R\cdot\nabla Q_R), \Delta\dot{Q}_R\Big\rangle + \varepsilon^7 J\Big\langle \Delta(\mathbf{v}_R\cdot\nabla Q_R), \Delta\ddot{Q}_R\Big\rangle \\
&\quad \underbrace{-\varepsilon^7 J\Big\langle \Delta\tilde{\mathbf{v}}\cdot\nabla(\mathbf{v}_R\cdot\nabla Q_R), \Delta\dot{Q}_R\Big\rangle}_{\mathcal{D}_1} \underbrace{-\varepsilon^7 J\Big\langle 2\partial_i\tilde{\mathbf{v}}\cdot\nabla\partial_i(\mathbf{v}_R\cdot\nabla Q_R), \Delta\dot{Q}_R\Big\rangle}_{\mathcal{D}_2} \\
&\quad \underbrace{-\varepsilon^7 J\Big\langle \Delta(\mathbf{v}_R\cdot\nabla Q_R), (\Delta\tilde{\mathbf{v}}\cdot\nabla + 2\partial_i\tilde{\mathbf{v}}\cdot\nabla\partial_i)\dot{Q}_R\Big\rangle}_{\mathcal{D}_3} \\
&\quad \underbrace{-\varepsilon^7 J\Big\langle \Delta\mathbf{v}_R\cdot\nabla\dot{Q}_R + 2\partial_i\mathbf{v}_R\cdot\nabla\partial_i\dot{Q}_R, \Delta\dot{Q}_R\Big\rangle}_{\mathcal{D}_4} \\
&\quad \underbrace{-\varepsilon^{10} J\Big\langle \Delta(\mathbf{v}_R\cdot\nabla(\mathbf{v}_R\cdot\nabla Q_R)), \Delta\dot{Q}_R\Big\rangle}_{\mathcal{D}_5}.
\end{aligned}
$$

Using Lemma 3.1, we get

$$
\begin{aligned}
&\mathcal{D}_1 \le C\varepsilon^7\|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{H^1}\|\Delta\dot{Q}_R\|_{L^2} \le C\varepsilon^2\mathfrak{E}^{\frac{3}{2}}, \\
&\mathcal{D}_2 \le C\varepsilon^7\|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{H^2}\|\Delta\dot{Q}_R\|_{L^2} \le C\varepsilon\mathfrak{E}^{\frac{3}{2}}, \\
&\mathcal{D}_3 \le C\varepsilon^7\|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{H^2}\|\dot{Q}_R\|_{H^2} \le C\varepsilon\mathfrak{E}^{\frac{3}{2}}, \\
&\mathcal{D}_4 \le C\varepsilon^7\|\mathbf{v}_R\|_{H^3}\|\dot{Q}_R\|_{H^2}\|\Delta\dot{Q}_R\|_{L^2} + C\varepsilon^7\|\partial_i\mathbf{v}_R\|_{H^2}\|\nabla\partial_i\dot{Q}_R\|_{L^2}\|\Delta\dot{Q}_R\|_{L^2} \\
&\qquad \le C\big(\varepsilon\mathfrak{E}^{\frac{3}{2}} + \varepsilon\mathfrak{E}\mathfrak{F}^{\frac{1}{2}}\big)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{D}_5 =& -\varepsilon^{10} J\Big\langle \Delta\mathbf{v}_R\cdot\nabla(\mathbf{v}_R\cdot\nabla Q_R), \Delta\dot{Q}_R\Big\rangle - \varepsilon^{10} J\Big\langle 2\partial_i\mathbf{v}_R\cdot\nabla\partial_i(\mathbf{v}_R\cdot\nabla Q_R), \Delta\dot{Q}_R\Big\rangle \\
&- \underbrace{\varepsilon^{10} J\Big\langle (\mathbf{v}_R\cdot\nabla)\Delta(\mathbf{v}_R\cdot\nabla Q_R), \Delta\dot{Q}_R\Big\rangle}_{\mathbf{W}_3} \\
\le& -\mathbf{W}_3 + C\varepsilon^{10}\|\mathbf{v}_R\|_{H^3}\|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{H^2}\|\Delta\dot{Q}_R\|_{L^2} \\
\le& -\mathbf{W}_3 + C\big(\varepsilon^2\mathfrak{E}^2 + \varepsilon^2\mathfrak{E}^{\frac{3}{2}}\mathfrak{F}^{\frac{1}{2}}\big).
\end{aligned}
$$

From (3.2), we obtain

$$
\begin{aligned}
&\varepsilon^7 J\Big\langle \Delta(\mathbf{v}_R\cdot\nabla Q_R), \Delta\ddot{Q}_R\Big\rangle \\
&\quad = \underbrace{-\varepsilon^7\mu_1\Big\langle \Delta(\mathbf{v}_R\cdot\nabla Q_R), \Delta\dot{Q}_R\Big\rangle}_{\mathcal{D}_6} \underbrace{-\varepsilon^7\Big\langle \Delta(\mathbf{v}_R\cdot\nabla Q_R), \frac{1}{\varepsilon}\Delta\mathscr{H}^\varepsilon_{\mathbf{n}}(Q_R)\Big\rangle}_{\mathcal{D}_7} \\
&\qquad \underbrace{+\varepsilon^7\Big\langle \Delta(\mathbf{v}_R\cdot\nabla Q_R), -\frac{\mu_2}{2}\Delta\mathbf{D}_R + \mu_1\Delta[\mathbf{\Omega}_R, Q_0]\Big\rangle}_{\mathcal{D}_8} + \underbrace{\varepsilon^7\Big\langle \Delta(\mathbf{v}_R\cdot\nabla Q_R), \Delta\mathbf{F}_R\Big\rangle}_{\mathcal{D}_9} \\
&\qquad + \varepsilon^7\Big\langle \Delta(\mathbf{v}_R\cdot\nabla Q_R), \Delta\widetilde{\mathbf{F}}_R\Big\rangle.
\end{aligned}
$$

Likewise, applying Lemma 3.1 leads to

$$
\begin{aligned}
&\mathcal{D}_6 \le C\varepsilon^7\|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{H^2}\|\Delta\dot{Q}_R\|_{L^2} \le C\varepsilon\mathfrak{E}^{\frac{3}{2}}, \\
&\mathcal{D}_8 \le C\varepsilon^7\|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{H^2}\|\nabla\Delta\mathbf{v}_R\|_{L^2} \le C\varepsilon\mathfrak{E}\mathfrak{F}^{\frac{1}{2}}, \\
&\mathcal{D}_9 \le C\varepsilon^7\|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{H^2}\|\Delta\mathbf{F}_R\|_{L^2} \le C\varepsilon\mathfrak{E}\|\varepsilon^2\Delta\mathbf{F}_R\|_{L^2}.
\end{aligned}
$$

Notice that if we replace $\mathbf{v}_0$ and Q with $\mathbf{v}_R$ and ΔQ_R in (2.36), respectively, then it follows that

$$-\varepsilon^7\Big\langle(\mathbf{v}_R\cdot\nabla)\Delta Q_R,\mathscr{L}(\Delta Q_R)\Big\rangle\le C\varepsilon^7\|\nabla\mathbf{v}_R\|_{H^2}\|\Delta Q_R\|_{H^1}^2\le C\big(\varepsilon\mathfrak{E}^{\frac32}+\varepsilon\mathfrak{E}\mathfrak{F}^{\frac12}\big).$$

Then we have

$$\begin{aligned}\mathcal{D}_7=&-\varepsilon^6\Big\langle\big(\Delta\mathbf{v}_R\cdot\nabla+2\partial_i\mathbf{v}_R\cdot\nabla\partial_i\big)Q_R,\Delta\mathscr{H}_{\mathbf{n}}(Q_R)\Big\rangle\\&-\varepsilon^6\Big\langle(\mathbf{v}_R\cdot\nabla)\Delta Q_R,\Delta\mathscr{H}_{\mathbf{n}}(Q_R)\Big\rangle\\&+\varepsilon^7\Big\langle\partial_j\big(\Delta\mathbf{v}_R\cdot\nabla+2\partial_i\mathbf{v}_R\cdot\nabla\partial_i\big)Q_R,\partial_j\mathscr{L}(Q_R)\Big\rangle\\&-\varepsilon^7\Big\langle(\mathbf{v}_R\cdot\nabla)\Delta Q_R,\mathscr{L}(\Delta Q_R)\Big\rangle\\\le&\, C\varepsilon^6\|\mathbf{v}_R\|_{H^3}\|Q\|_{H^2}^2+C\varepsilon^6\|\mathbf{v}_R\|_{H^2}\|\nabla\Delta Q_R\|_{L^2}\|Q_R\|_{H^2}\\&+C\varepsilon^7\|\mathbf{v}_R\|_{H^3}\|Q_R\|_{H^3}^2+C\big(\varepsilon\mathfrak{E}^{\frac32}+\varepsilon\mathfrak{E}\mathfrak{F}^{\frac12}\big)\\\le&\, C\big(\varepsilon\mathfrak{E}^{\frac32}+\varepsilon\mathfrak{E}\mathfrak{F}^{\frac12}\big).\end{aligned}$$

Thus the following bound holds:

$$\begin{aligned}\varepsilon^4\Big\langle\Delta\widetilde{\mathbf{F}}_2,\Delta\dot{Q}_R\Big\rangle\le&-\varepsilon^7J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle\Delta(\mathbf{v}_R\cdot\nabla Q_R),\Delta\dot{Q}_R\Big\rangle+\varepsilon^7\Big\langle\Delta(\mathbf{v}_R\cdot\nabla Q_R),\Delta\widetilde{\mathbf{F}}\Big\rangle-\mathbf{W}_3\\&+C\big(\varepsilon\mathfrak{E}^{\frac32}+\varepsilon^2\mathfrak{E}^2+\varepsilon\mathfrak{E}\mathfrak{F}^{\frac12}+\varepsilon^2\mathfrak{E}^{\frac32}\mathfrak{F}^{\frac12}\big)+C\varepsilon\mathfrak{E}\|\varepsilon^2\Delta\mathbf{F}_R\|_{L^2}.\end{aligned}\tag{C.3}$$

We next deal with the term $\varepsilon^7\langle\Delta(\mathbf{v}_R\cdot\nabla Q_R),\Delta\widetilde{\mathbf{F}}_R\rangle$. It is easy to see that

$$\begin{aligned}\varepsilon^7\Big\langle\Delta(\mathbf{v}_R\cdot\nabla Q_R),\Delta\widetilde{\mathbf{F}}_1\Big\rangle&=-\mathbf{S}_3-\varepsilon^7J\Big\langle\Delta(\mathbf{v}_R\cdot\nabla Q_R),(\Delta\tilde{\mathbf{v}}\cdot\nabla+2\partial_i\tilde{\mathbf{v}}\cdot\nabla\partial_i)(\mathbf{v}_R\cdot\nabla\widetilde{Q})\Big\rangle\\&\le-\mathbf{S}_3+C\varepsilon^7\|\mathbf{v}_R\|_{H^2}^2\|\nabla Q_R\|_{H^2}\\&\le-\mathbf{S}_3+C\varepsilon\mathfrak{E}^{\frac32}.\end{aligned}$$

By a straightforward computation, one checks that

$$\begin{aligned}&\varepsilon^7\Big\langle\Delta(\mathbf{v}_R\cdot\nabla Q_R),\Delta\widetilde{\mathbf{F}}_2\Big\rangle\\&=-\varepsilon^{10}\frac{J}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta(\mathbf{v}_R\cdot\nabla Q_R)\|_{L^2}^2\underbrace{-\varepsilon^{10}J\Big\langle\Delta(\mathbf{v}_R\cdot\nabla Q_R),\Delta(\mathbf{v}_R\cdot\nabla\dot{Q}_R)\Big\rangle}_{\mathcal{D}_{10}}\\&\quad\underbrace{-\varepsilon^{10}J\Big\langle\Delta(\mathbf{v}_R\cdot\nabla Q_R),(\Delta\tilde{\mathbf{v}}\cdot\nabla+2\partial_i\tilde{\mathbf{v}}\cdot\nabla\partial_i)(\mathbf{v}_R\cdot\nabla Q_R)\Big\rangle}_{\mathcal{D}_{11}}\\&\quad\underbrace{-\varepsilon^{13}J\Big\langle\Delta(\mathbf{v}_R\cdot\nabla Q_R),\Delta(\mathbf{v}_R\cdot\nabla(\mathbf{v}_R\cdot\nabla Q_R))\Big\rangle}_{\mathcal{D}_{12}}.\end{aligned}$$

Similarly, by Lemma 3.1 we have

$$\begin{aligned}\mathcal{D}_{10}&=\mathbf{W}_3-\varepsilon^{10}J\Big\langle\Delta(\mathbf{v}_R\cdot\nabla Q_R),(\Delta\mathbf{v}_R\cdot\nabla+2\partial_i\mathbf{v}_R\cdot\nabla\partial_i)\dot{Q}_R\Big\rangle\\&\le\mathbf{W}_3+C\varepsilon^{10}\|\mathbf{v}_R\|_{H^2}\|\nabla Q_R\|_{H^2}\|\mathbf{v}_R\|_{H^3}\|\dot{Q}_R\|_{H^2}\\&\le\mathbf{W}_3+C\big(\varepsilon^2\mathfrak{E}^2+\varepsilon^2\mathfrak{E}^{\frac32}\mathfrak{F}^{\frac12}\big),\end{aligned}$$

$$\mathcal{D}_{11} \le C\varepsilon^{10}\|\mathbf{v}_R\|_{H^2}^2\|\nabla Q_R\|_{H^2}^2 \le C\varepsilon^2\mathfrak{E}^2,$$
$$\mathcal{D}_{12} = -\varepsilon^{13}J\Big\langle \Delta(\mathbf{v}_R\cdot\nabla Q_R), (\Delta\mathbf{v}_R\cdot\nabla + 2\partial_i\mathbf{v}_R\cdot\nabla\partial_i)(\mathbf{v}_R\cdot\nabla Q_R)\Big\rangle$$
$$\le \varepsilon^{13}\|\mathbf{v}_R\|_{H^2}^2\|\nabla Q_R\|_{H^2}^2\|\mathbf{v}_R\|_{H^3} \le C\big(\varepsilon^4\mathfrak{E}^{\frac52} + \varepsilon^3\mathfrak{E}^2\mathfrak{F}^{\frac12}\big).$$

Thus we get

$$\varepsilon^7\Big\langle \Delta(\mathbf{v}_R\cdot\nabla Q_R), \Delta\widetilde{\mathbf{F}}_R\Big\rangle \le -\varepsilon^{10}\frac{J}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta(\mathbf{v}_R\cdot\nabla Q_R)\|_{L^2}^2 - \mathbf{S}_3 + \mathbf{W}_3 + C\big(\varepsilon\mathfrak{E}^{\frac32} + \varepsilon^2\mathfrak{E}^2 + \varepsilon^4\mathfrak{E}^{\frac52} + \varepsilon^2\mathfrak{E}^{\frac32}\mathfrak{F}^{\frac12} + \varepsilon^3\mathfrak{E}^2\mathfrak{F}^{\frac12}\big). \tag{C.4}$$

In conclusion, putting together these estimates (C.1)–(C.4) and discarding the cancellation terms, we obtain the following estimate:

$$\varepsilon^4\left\langle \Delta\widetilde{\mathbf{F}}_R, \Delta\dot{Q}_R\right\rangle \le -\varepsilon^4\frac{J}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta(\mathbf{v}_R\cdot\nabla Q^\varepsilon)\|_{L^2}^2 - \varepsilon^4 J\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle \Delta(\mathbf{v}_R\cdot\nabla Q^\varepsilon), \Delta\dot{Q}_R\Big\rangle + C\left(1+\mathfrak{E}+\varepsilon^2\mathfrak{E}^2+\varepsilon^8\mathfrak{E}^5\right) + (\delta + C\varepsilon^2\mathfrak{E})\mathfrak{F}.$$

Appendix D. Proof of (3.33). We first apply the derivative operator Δ on (3.2), then multiply $\Delta\dot{Q}_R$ and integrate the resulting identity on $\mathbb{R}^3$ with respect to $\mathbf{x}$. Again applying the operator Δ on (3.3) and taking the L^2-inner product with $\Delta\mathbf{v}_R$ enable us to derive the following equality:

$$\begin{aligned}
&\varepsilon^4\Big\langle \partial_t(\Delta\mathbf{v}_R), \Delta\mathbf{v}_R\Big\rangle + \varepsilon^4 J\Big\langle \partial_t(\Delta\dot{Q}_R), \Delta\dot{Q}_R\Big\rangle \\
&= \underbrace{-\varepsilon^4\Big\langle \Delta\Big(\beta_1 Q_0(Q_0:\mathbf{D}_R) + \beta_4\mathbf{D}_R + \beta_5\mathbf{D}_R\cdot Q_0 + \beta_6 Q_0\cdot\mathbf{D}_R\Big), \nabla\Delta\mathbf{v}_R\Big\rangle}_{\mathcal{K}_1} \\
&\quad \underbrace{-\varepsilon^4\beta_7\Big\langle \Delta(\mathbf{D}_R\cdot Q_0^2 + Q_0^2\cdot\mathbf{D}_R), \nabla\Delta\mathbf{v}_R\Big\rangle}_{\mathcal{K}_2} \\
&\quad \underbrace{-\varepsilon^4\frac{\mu_2}{2}\Big\langle \Delta(\dot{Q}_R - [\mathbf{\Omega}_R, Q_0]), \nabla\Delta\mathbf{v}_R\Big\rangle}_{\mathcal{K}_3} \underbrace{-\varepsilon^4\mu_1\Big\langle \Delta\big[Q_0, (\dot{Q}_R - [\mathbf{\Omega}_R, Q_0])\big], \nabla\Delta\mathbf{v}_R\Big\rangle}_{\mathcal{K}_4} \\
&\quad \underbrace{-\varepsilon^4\Big\langle \Delta\tilde{\mathbf{v}}\cdot\nabla\mathbf{v}_R + 2\partial_i\tilde{\mathbf{v}}\cdot\nabla\partial_i\mathbf{v}_R, \Delta\mathbf{v}_R\Big\rangle - \varepsilon^4\Big\langle \Delta\mathbf{G}_R, \nabla\Delta\mathbf{v}_R\Big\rangle + \varepsilon^4\Big\langle \Delta\mathbf{G}'_R, \Delta\mathbf{v}_R\Big\rangle}_{\mathcal{K}_5} \\
&\quad \underbrace{-\varepsilon^4\frac{\mu_2}{2}\Big\langle \Delta\mathbf{D}_R, \Delta\dot{Q}_R\Big\rangle}_{\mathcal{K}_6} \underbrace{-\varepsilon^4\mu_1\Big\langle \Delta\dot{Q}_R - \Delta[\mathbf{\Omega}_R, Q_0], \Delta\dot{Q}_R\Big\rangle_{L^2}^2}_{\mathcal{K}_7} \underbrace{-\varepsilon^4\Big\langle \frac{1}{\varepsilon}\Delta\mathscr{H}_{\mathbf{n}}^{\varepsilon}(Q_R), \Delta\dot{Q}_R\Big\rangle}_{\mathcal{K}_8} \\
&\quad \underbrace{-\varepsilon^4\Big\langle \Delta\tilde{\mathbf{v}}\cdot\nabla\dot{Q}_R + 2\partial_i\tilde{\mathbf{v}}\cdot\nabla\partial_i\dot{Q}_R, \Delta\dot{Q}_R\Big\rangle}_{\mathcal{K}_9} + \varepsilon^4\Big\langle \Delta\mathbf{F}_R + \Delta\widetilde{\mathbf{F}}_R, \Delta\dot{Q}_R\Big\rangle.
\end{aligned}$$

The terms on the right-hand sides can be estimated as follows. By the analysis for the construction of the terms $\mathcal{K}_1$ and $\mathcal{K}_2$, we have

$$\begin{aligned}
\mathcal{K}_1 + \mathcal{K}_2 \le &-\varepsilon^4\Big\langle \beta_1 Q_0(Q_0:\Delta\mathbf{D}_R) + \beta_4\Delta\mathbf{D}_R + \beta_5\Delta\mathbf{D}_R\cdot Q_0 + \beta_6 Q_0\cdot\Delta\mathbf{D}_R, \nabla\Delta\mathbf{v}_R\Big\rangle \\
&-\varepsilon^4\Big\langle \beta_7(\Delta\mathbf{D}_R\cdot Q_0^2 + Q_0^2\cdot\Delta\mathbf{D}_R), \nabla\Delta\mathbf{v}_R\Big\rangle + C\|\varepsilon^2\nabla\mathbf{v}_R\|_{H^1}\|\varepsilon^2\nabla\Delta\mathbf{v}_R\|_{L^2}
\end{aligned}$$

$$\leq -\varepsilon^4\Big\langle \beta_1 Q_0(Q_0:\Delta\mathbf{D}_R)+\beta_4\Delta\mathbf{D}_R+\frac{\beta_5+\beta_6}{2}(Q_0\cdot\Delta\mathbf{D}_R+\Delta\mathbf{D}_R\cdot Q_0),\Delta\mathbf{D}_R\Big\rangle$$
$$-\varepsilon^4\Big\langle \beta_7(\Delta\mathbf{D}_R\cdot Q_0^2+Q_0^2\cdot\Delta\mathbf{D}_R),\Delta\mathbf{D}_R\Big\rangle+\underbrace{\varepsilon^4\frac{\mu_2}{2}\Big\langle[\Delta\mathbf{D}_R,Q_0],\nabla\Delta\mathbf{v}_R\Big\rangle}_{\mathcal{K}_1'}+C\mathfrak{E}^{\frac12}\mathfrak{F}^{\frac12}.$$

It can be easy to observe that

$$\begin{aligned}\mathcal{K}_1'+\mathcal{K}_3+\mathcal{K}_6 &\leq \varepsilon^4\frac{\mu_2}{2}\langle[\Delta\mathbf{D}_R,Q_0],\Delta\mathbf{\Omega}_R\rangle-\varepsilon^4\mu_2\langle\Delta\mathbf{D}_R,\Delta\dot{Q}_R\rangle\\&\quad+\varepsilon^4\frac{\mu_2}{2}\langle[\Delta\mathbf{\Omega}_R,Q_0],\Delta\mathbf{D}_R\rangle+C\|\varepsilon^2\nabla\mathbf{v}_R\|_{H^1}\|\varepsilon^2\nabla\Delta\mathbf{v}_R\|_{L^2}\\&\leq -\varepsilon^4\mu_2\langle\Delta\dot{Q}_R-[\Delta\mathbf{\Omega}_R,Q_0],\Delta\mathbf{D}_R\rangle+C\mathfrak{E}^{\frac12}\mathfrak{F}^{\frac12}.\end{aligned}$$

The terms $\mathcal{K}_4$ and $\mathcal{K}_7$ can be estimated as

$$\begin{aligned}\mathcal{K}_4+\mathcal{K}_7 \leq &-\varepsilon^4\mu_1\Big\langle\big[Q_0,(\Delta\dot{Q}_R-[\Delta\mathbf{\Omega}_R,Q_0])\big],\nabla\Delta\mathbf{v}_R\Big\rangle\\&-\varepsilon^4\mu_1\Big\langle\Delta\dot{Q}_R-[\Delta\mathbf{\Omega}_R,Q_0],\Delta\dot{Q}_R\Big\rangle\\&+C\Big(\varepsilon\|\varepsilon\dot{Q}_R\|_{H^1}+\|\varepsilon^2\nabla\mathbf{v}_R\|_{H^1}\Big)\|\varepsilon^2\nabla\Delta\mathbf{v}_R\|_{L^2}\\&+C\|\varepsilon^2\nabla\mathbf{v}_R\|_{H^1}\|\varepsilon^2\Delta\dot{Q}_R\|_{L^2}\\\leq &-\varepsilon^4\mu_1\big\|\Delta\dot{Q}_R-[\Delta\mathbf{\Omega}_R,Q_0]\big\|_{L^2}^2+C\big(\mathfrak{E}^{\frac12}\mathfrak{F}^{\frac12}+\mathfrak{E}\big).\end{aligned}$$

Combining with the equality

$$\begin{aligned}&-\varepsilon^4\mu_1\big\|\Delta\dot{Q}_R-[\Delta\mathbf{\Omega}_R,Q_0]\big\|_{L^2}^2-\varepsilon^4\mu_2\langle\Delta\dot{Q}_R-[\Delta\mathbf{\Omega}_R,Q_0],\Delta\mathbf{D}_R\rangle\\&=-\varepsilon^4\mu_1\Big\|\Delta\dot{Q}_R-[\Delta\mathbf{\Omega}_R,Q_0]+\frac{\mu_2}{2\mu_1}\Delta\mathbf{D}_R\Big\|_{L^2}^2+\frac{\mu_2^2}{4\mu_1}\|\Delta\mathbf{D}_R\|_{L^2}^2,\end{aligned}$$

and by using the dissipation relation (2.22), we have the following estimate:

$$\begin{aligned}&\mathcal{K}_1+\mathcal{K}_2+\mathcal{K}_3+\mathcal{K}_4+\mathcal{K}_6+\mathcal{K}_7\\&\leq -\varepsilon^4\beta_1 s^2\|\mathbf{nn}:\Delta\mathbf{D}_R\|_{L^2}^2-\varepsilon^4\left(\beta_4-\frac{s(\beta_5+\beta_6)}{3}+\frac29\beta_7 s^2\right)\|\Delta\mathbf{D}_R\|_{L^2}^2\\&\quad-\varepsilon^4\left(s(\beta_5+\beta_6)+\frac23\beta_7 s^2\right)\|\mathbf{n}\cdot\Delta\mathbf{D}_R\|_{L^2}^2-\varepsilon^4\mu_2\langle\Delta\dot{Q}_R-[\Delta\mathbf{\Omega}_R,Q_0],\Delta\mathbf{D}_R\rangle\\&\quad-\varepsilon^4\mu_1\big\|\Delta\dot{Q}_R-[\Delta\mathbf{\Omega}_R,Q_0]\big\|_{L^2}^2+C(\mathfrak{E}^{\frac12}\mathfrak{F}^{\frac12}+\mathfrak{E})\\&\leq -\varepsilon^4\tilde\beta_1\|\mathbf{nn}:\Delta\mathbf{D}_R\|_{L^2}^2-\varepsilon^4\tilde\beta_2\|\Delta\mathbf{D}_R\|_{L^2}^2-\varepsilon^4\tilde\beta_3\|\mathbf{n}\cdot\Delta\mathbf{D}_R\|_{L^2}^2\\&\quad-4\varepsilon^4\delta\|\Delta\mathbf{D}_R\|_{L^2}^2+C\big(\mathfrak{E}^{\frac12}\mathfrak{F}^{\frac12}+\mathfrak{E}\big)\\&\leq -4\varepsilon^4\delta\|\nabla\Delta\mathbf{v}_R\|_{L^2}^2+C\mathfrak{E}+\delta\mathfrak{F},\end{aligned}$$

where $\delta>0$ is small enough, such that the coefficients $\tilde\beta_i(i=1,2,3)$ given by (2.39) satisfy the relation (2.23). As for the estimates of the terms $\mathcal{K}_5$ and $\mathcal{K}_9$, it is easy to obtain

$$\begin{aligned}\mathcal{K}_5+\mathcal{K}_9&\leq C\Big(\|\varepsilon^2\nabla\mathbf{v}_R\|_{H^1}\|\varepsilon^2\Delta\mathbf{v}_R\|_{L^2}+\|\varepsilon^2\Delta\mathbf{G}_R\|_{L^2}\|\varepsilon^2\nabla\Delta\mathbf{v}_R\|_{L^2}\\&\qquad+\|\varepsilon^2\Delta\mathbf{G}_R'\|_{L^2}\|\varepsilon^2\Delta\mathbf{v}_R\|_{L^2}+\|\varepsilon^2\nabla\dot{Q}_R\|_{H^1}\|\varepsilon^2\Delta\dot{Q}_R\|_{L^2}\Big)\\&\leq C\mathfrak{E}+C\left(\|\varepsilon^2\Delta\mathbf{G}_R\|_{L^2}\mathfrak{F}^{\frac12}+\|\varepsilon^2\Delta\mathbf{G}_R'\|_{L^2}\mathfrak{E}^{\frac12}\right).\end{aligned}$$

Similar to the derivation of (3.31), the term $\mathcal{K}_8$ can be handled as

$$\mathcal{K}_8 \leq -\frac{\varepsilon^3}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathscr{H}^{\varepsilon}_{\mathbf{n}}(\Delta Q_R), \Delta Q_R\rangle + \delta\mathfrak{F} + C\mathfrak{E}.$$

As a consequence, from the above estimates, we can conclude that

$$\begin{aligned}
&\varepsilon^4\Big\langle \partial_t(\Delta \mathbf{v}_R), \Delta\mathbf{v}_R\Big\rangle + \varepsilon^4 J\Big\langle \partial_t(\Delta\dot{Q}_R), \Delta\dot{Q}_R\Big\rangle \\
&\quad + \frac{\varepsilon^3}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathscr{H}^{\varepsilon}_{\mathbf{n}}(\Delta Q_R), \Delta Q_R\rangle + 4\varepsilon^4\delta\|\nabla\Delta\mathbf{v}_R\|^2_{L^2} \\
&\quad \leq C\Big(\|\varepsilon^2\Delta\mathbf{G}_R\|_{L^2}\mathfrak{F}^{\frac{1}{2}} + \|\varepsilon^2\Delta\mathbf{G}'_R\|_{L^2}\mathfrak{E}^{\frac{1}{2}} + \|\varepsilon^2\Delta\mathbf{F}_R\|_{L^2}\mathfrak{E}^{\frac{1}{2}}\Big) \\
&\quad + \varepsilon^4\langle\Delta\widetilde{\mathbf{F}}_R, \Delta\dot{Q}_R\rangle + C\mathfrak{E} + \delta\mathfrak{F}.
\end{aligned}$$

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