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ON GLOBAL COHOMOLOGICAL WIDTH OF ARTIN ALGEBRAS

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Abstract. We study the global cohomological width of artin algebras. Using the construction of indecomposable objects in the triangulated category via taking cones due to Happel and Zacharia (2008), we establish that global cohomological width coincides with strong global dimension. Moreover, an upper bound for the global cohomological width of piecewise hereditary algebras is obtained. As an application, we construct finite-dimensional piecewise hereditary algebras of type $\mathbb{A}$ and $\mathbb{D}$ with global cohomological width an arbitrary positive integer m. Finally, we find a relation between recollements and global cohomological width.

1. Introduction.

Let R be a commutative artin ring. Throughout this article, all algebras are connected associative artin R-algebras with identity unless stated otherwise. In representation theory of algebras, an important homological invariant is global dimension. The global dimension of an algebra measures to some extent the complexity of its homological properties. The best-understood algebras are hereditary algebras, whose global dimensions are at most one. The module category of hereditary algebras is a classical example of a hereditary abelian category, i.e., an abelian category such that the functor $\mathrm{Ext}^2(-,-)$ vanishes. The indecomposable objects in the bounded derived category of these algebras are stalk complexes (see [Hap88, Kr07] for details).

The *piecewise hereditary algebras* are those algebras derived equivalent to some hereditary abelian category [HRS96b]. According to the celebrated classification theorems due to Happel and Reiten [Hap01, HR02], up to derived equivalences, there are only two classes of piecewise hereditary algebras: the algebras derived equivalent to some hereditary algebra and the algebras derived equivalent to some canonical algebra. In the research on piecewise hereditary algebras, much emphasis has been placed on those which are finite-dimensional over fields. Another homological invariant, *strong global*

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dimension, was introduced to measure how far an algebra is from being hereditary [Sko87]. Roughly speaking, the strong global dimension of an algebra is defined by taking the supremum over the lengths of all indecomposable objects in its bounded homotopy category of finite generated projective modules. Ringel conjectured that an algebra is piecewise hereditary if and only if its strong global dimension is finite. Much effort has been put toward the proof of this conjecture, for example for algebras with radical square zero [KSYZ04]. The conjecture was completely proved by Happel and Zacharia [HZ08]. As a byproduct, they described the algebras of strong global dimension two as those quasitilted algebras which are not hereditary. Here, we say an algebra A is *quasitilted* if there exists a hereditary abelian category $\mathcal{H}$ with a tilting object T such that $A = \mathrm{End}_{\mathcal{H}}(T)^{\mathrm{op}}$, or equivalently, $\mathrm{gl.dim}\, A \leq 2$ and either $\mathrm{pd}\, X \leq 1$ or $\mathrm{id}\, X \leq 1$ for any $X \in \mathrm{mod}\, A$ (see [HRS96a]). Indeed, quasitilted algebras are generalizations of tilted algebras, and the typical examples of quasitilted algebras are tilted algebras and canonical algebras.

More recently, Han and the present author introduced another carrier of homological information of algebras, *global cohomological width*, in the study of derived Brauer–Thrall type theorems [HZ13]. The global cohomological width was defined on the level of the bounded derived category as the supremum of the cohomological widths of indecomposable complexes. As was first observed for finite-dimensional algebras over algebraically closed fields, global cohomological width coincides with strong global dimension, even though these two invariants are defined in different frameworks [HZ13, Proposition 3].

In the present paper, we generalize the description of [HZ13, Proposition 3] to the case of artin algebras, based on the construction of indecomposables in a triangulated category via taking cones, due to Happel and Zacharia [HZ08, Corollary 1.4]. We mainly study the global cohomological width of artin algebras, and try to convince the readers that it is much more convenient to deal with the indecomposables in terms of homology rather than length, even though strong global dimension coincides with global cohomological width.

First, we study the behavior of global cohomological width under derived equivalences, retrieving the inequality between strong global dimensions under tilting equivalences, established by Happel and Zacharia [HZ10, Theorem 4.2]. Moreover, we find an upper bound for the global cohomological width of piecewise hereditary artin algebras, which coincides with the upper bound provided in [HZ08, Proposition 3.1] for strong global dimension by a careful analysis of normalized equivalences associated to finite-dimensional piecewise hereditary algebras over fields. By [Z14, Proposition 1.5], piecewise hereditary algebras can be characterized as the algebras with finite global cohomological width. We shall describe a special class of piecewise hereditary algebras, quasitilted algebras that are not hereditary, as those algebras of global cohomological width two (see also [HZ08, Proposition 3.3]). As an application of the upper bound we provide, we construct finite-dimensional piecewise hereditary algebras of type $\mathbb{A}$ and $\mathbb{D}$ with global cohomological width an arbitrary positive integer m by analysing tilting complexes. Finally, the relation of global cohomological width and recollements is discussed and we obtain the result of [AKL12, Lemma 5.6] in terms of global cohomological width.

This paper is organized as follows: following the introductory section, we introduce some notation and definitions, and then analyse the behavior of global cohomological width under derived equivalences. The second section is mainly devoted to proving that global cohomological width coincides with strong global dimension. In Section 3, we establish an upper bound for the global cohomological width of piecewise hereditary algebras, and characterize the algebras of global cohomological width two. The fourth section mainly constructs finite-dimensional piecewise hereditary algebras of type $\mathbb{A}$ and $\mathbb{D}$ of global cohomological width an arbitrary positive integer m. In the last section, we describe the relation between global cohomological width and recollements.

2. Preliminaries. Let A be an artin R-algebra. Denote by $\operatorname{mod} A$ the category of all finitely generated right A-modules, and by $C(A)$ the category of A-module complexes. $C^b(A)$ (resp. $C^b(\operatorname{proj} A)$) is the category of all bounded complexes of A-modules (resp. projective A-modules), while $C^{-,b}(\operatorname{proj} A)$ is the right bounded complexes of projective A-modules with bounded cohomology. Denote by $K^b(\operatorname{proj} A)$ and $K^{-,b}(\operatorname{proj} A)$ the homotopy categories corresponding to $C^b(\operatorname{proj} A)$ and $C^{-,b}(\operatorname{proj} A)$ respectively. Moreover, $D^b(A)$ is the bounded derived category of $\operatorname{mod} A$ with $[-1]$ the shift functor.

Definition 2.1 ([HZ13, Z14]). The *cohomological width* of a complex $X^\bullet \in D^b(A)$ is

$$\operatorname{hw}(X^\bullet) := \max\{j - i + 1 \mid H^i(X^\bullet) \neq 0 \neq H^j(X^\bullet)\},$$

and the *global cohomological width* of A is

$$\operatorname{gl.hw} A := \sup\{\operatorname{hw}(X^\bullet) \mid X^\bullet \in D^b(A) \text{ is indecomposable}\}.$$

Obviously, cohomological width is invariant under shifts and isomorphisms. Let $\mathcal{H}$ be a hereditary abelian category, i.e. $\operatorname{Ext}^2_{\mathcal{H}}(-,-) = 0$. The following lemma implies that any indecomposable object in the bounded derived category $D^b(\mathcal{H})$ of $\mathcal{H}$ is a stalk complex. In particular, this holds for the module category of a hereditary artin R-algebra. Therefore, if A is a hereditary artin R-algebra then $\mathrm{gl.hw}\, A = 1$.

Lemma 2.2 (see Krause [Kr07, Section 1.6]). *Let $\mathcal{H}$ be a hereditary abelian category and X be an indecomposable object in $D^b(\mathcal{H})$. Then $X^\bullet$ is isomorphic to a stalk complex.*

Recall that a complex $X^\bullet = (X^i, d^i) \in C(A)$ is said to be *minimal* if $\mathrm{Im}\, d^i \subseteq \mathrm{rad}\, X^{i+1}$ for all $i \in \mathbb{Z}$. For any complex $P^\bullet = (P^i, d^i) \in K^b(\mathrm{proj}\, A)$, there is a minimal complex $\bar{P}^\bullet = (\bar{P}^i, \bar{d}^i) \in K^b(\mathrm{proj}\, A)$, which is unique up to isomorphism in $C(A)$, such that $P^\bullet \cong \bar{P}^\bullet$ in $K^b(\mathrm{proj}\, A)$. The *length* of $P^\bullet$ is

$$l(P^\bullet) := \max\{j - i \mid \bar{P}^i \neq 0 \neq \bar{P}^j\}.$$

Clearly, for any complex $P^\bullet = (P^i, d^i) \in K^b(\mathrm{proj}\, A)$, we have $\mathrm{hw}(P^\bullet) \leq l(P^\bullet) + 1$.

The following proposition and corollary describe the behavior of cohomological width under derived equivalences. Note that they were originally proved for finite-dimensional algebras over algebraically closed fields [HZ13, Proposition 1(1)], but the proof also works for general artin algebras. Here, we provide the proof for the convenience of the readers.

Proposition 2.3. *Let A and B be two algebras, let ${}_AT^\bullet_B$ be a two-sided tilting complex of length $l({}_AT^\bullet)$ as a complex in $K^b(\mathrm{proj}\, A)$, and let $F = - \otimes^L_A T^\bullet_B : D^b(A) \to D^b(B)$ be a derived equivalence. Then for all $X^\bullet$ in $D^b(A)$,*

$$\mathrm{hw}(F(X^\bullet)) \leq \mathrm{hw}(X^\bullet) + l({}_AT^\bullet).$$

In particular, $\mathrm{gl.hw}\, B \leq \mathrm{gl.hw}\, A + l({}_AT^\bullet)$.

Proof. Recall that the *width* of a complex $Y^\bullet \in C^b(A)$ is

$$\mathrm{w}(Y^\bullet) := \max\{j - i + 1 \mid Y^j \neq 0 \neq Y^i\}.$$

For any $X^\bullet \in D^b(A)$, there exists a complex $\bar{X}^\bullet \in D^b(A)$ which can be obtained from $X^\bullet$ by good truncations, such that $\bar{X}^\bullet \cong X^\bullet$ with $\mathrm{hw}(\bar{X}^\bullet) = \mathrm{w}(\bar{X}^\bullet)$. Since ${}_AT^\bullet_B$ is a two-sided tilting complex, there is a perfect complex ${}_A\bar{T}^\bullet \in C^b(\mathrm{proj}\, A^{\mathrm{op}})$ with ${}_AT^\bullet \cong {}_A\bar{T}^\bullet$ and $l({}_AT^\bullet) = l({}_A\bar{T}^\bullet) = \mathrm{w}({}_A\bar{T}^\bullet) - 1$. Thus, viewed as a complex of R-modules, $F(\bar{X}^\bullet) = \bar{X}^\bullet \otimes^L_A T^\bullet \cong \bar{X}^\bullet \otimes_A \bar{T}^\bullet$. Hence $\mathrm{hw}(F(X^\bullet)) = \mathrm{hw}(F(\bar{X}^\bullet)) = \mathrm{hw}(\bar{X}^\bullet \otimes_A \bar{T}^\bullet) \leq \mathrm{w}(\bar{X}^\bullet \otimes_A \bar{T}^\bullet) \leq \mathrm{w}(\bar{X}^\bullet_A) + \mathrm{w}({}_A\bar{T}^\bullet) - 1 = \mathrm{hw}(X^\bullet_A) + l({}_AT^\bullet)$. ∎

Corollary 2.4. *Let algebras A and B be derived equivalent. Then* $\mathrm{gl.hw}\, A < \infty$ *if and only if* $\mathrm{gl.hw}\, B < \infty$.

Proof. Since A and B are derived equivalent, there is a two-sided tilting complex ${}_AT^\bullet_B$ such that $- \otimes^L_A T^\bullet_B : D^b(A) \to D^b(B)$ is a derived equivalence [Ri91]. So the corollary follows from Proposition 2.3. ∎

3. Global cohomological width and strong global dimension. In this section, we mainly focus on the relation between the global cohomological width and the strong global dimension of artin R-algebras. The definition of strong global dimension was introduced in [Sko87] for finite-dimensional algebras over fields, but the definition makes sense for general artin algebras.

DEFINITION 3.1. Let A be an artin R-algebra. The *strong global dimension* of A is

$$\text{s.gl.dim}\, A := \sup\{l(P^\bullet) \mid P^\bullet \in K^b(\text{proj}\, A) \text{ is indecomposable}\}.$$

It is clear that for a module of finite projective dimension, the length of its minimal projective resolution equals its projective dimension. Hence, if $\text{gl.dim}\, A < \infty$ then $\text{s.gl.dim}\, A \geq \text{gl.dim}\, A$.

Recall that for a complex $P^\bullet$ in $C^{-,b}(\text{proj}\, A)$ of the form

$$P^\bullet = \cdots \to P^{-n-1} \xrightarrow{d^{-n-1}} P^{-n} \xrightarrow{d^{-n}} \cdots \to P^{m-1} \xrightarrow{d^{m-1}} P^m \to 0,$$

its *brutal truncation* $\sigma_{\geq -n}(P^\bullet)$ is

$$\sigma_{\geq -n}(P^\bullet) = 0 \to P^{-n} \xrightarrow{d^{-n}} \cdots \to P^{m-1} \xrightarrow{d^{m-1}} P^m \to 0.$$

The following lemma due to Happel and Zacharia [HZ08, Corollary 1.4] provides a method to construct indecomposables via taking cones for general triangulated categories.

LEMMA 3.2. *Let $(\mathcal{T}, \Sigma)$ be a triangulated category and $f : X \to Y$ be nonzero and not invertible with X, Y indecomposable in $\mathcal{T}$. Suppose*

$$X \xrightarrow{f} Y \xrightarrow{u} C_f \xrightarrow{v} \Sigma X$$

is a triangle. If $\text{Hom}_{\mathcal{T}}(Y, \Sigma X) = 0$, then C_f is indecomposable.

As an application, we obtain two corollaries on how to construct indecomposable objects in a bounded derived category.

COROLLARY 3.3. *Let A be an artin algebra and*

$$Y^\bullet = 0 \to P^{-n} \xrightarrow{d^{-n}} \cdots \to P^{-1} \xrightarrow{d^{-1}} P^0 \to 0$$

be a minimal indecomposable complex in $K^b(\text{proj}\, A)$. If $X^\bullet$ is an indecomposable complex in $D^b(A)$ of the form

$$X^\bullet = 0 \to X^{-m} \xrightarrow{d^{-m}} \cdots \to X^{-n-2} \xrightarrow{d^{-n-2}} X^{-n-1} \to 0$$

and $0 \neq f \in \text{Hom}_A(X^{-n-1}, P^{-n})$ satisfying $d^{-n} f = 0 = f d^{-n-2}$, then

$$C_f^\bullet = 0 \to X^{-m} \xrightarrow{d^{-m}} \cdots \xrightarrow{d^{-n-2}} X^{-n-1} \xrightarrow{f} P^{-n} \xrightarrow{d^{-n}} \cdots \to P^{-1} \xrightarrow{d^{-1}} P^0 \to 0$$

is also indecomposable in $D^b(A)$.

Proof. Assume $f^\bullet = (f^i)$ with $f^{-n} = f$ and zero otherwise. Then $f^\bullet$ is nonzero and not invertible viewed as an element in $\mathrm{Hom}_{D^b(A)}(X^\bullet, Y^\bullet)$. Moreover, it is clear that $\mathrm{Hom}_{D^b(A)}(Y^\bullet, X^\bullet[-1]) = 0$. Thus the cone $C^\bullet_{f^\bullet} = C^\bullet_f$ is indecomposable by Lemma 3.2. ∎

COROLLARY 3.4. *Let A be an artin algebra, and let*

$$X^\bullet = \cdots \to P^{-n} \xrightarrow{d^{-n}} \cdots \to P^{-1} \xrightarrow{d^{-1}} P^0 \to 0$$

be a minimal indecomposable complex in $K^{-,b}(\mathrm{proj}\, A)$ such that $H^i(X^\bullet) = 0$ for $i \le -n+1$. If there is another indecomposable object

$$Y^\bullet = 0 \to Y^{-m} \xrightarrow{d^{-m}} \cdots \to Y^{-n-1} \xrightarrow{d^{-n-1}} Y^{-n} \to 0$$

in $D^b(A)$ and $0 \ne f^\bullet \in \mathrm{Hom}_{D^b(A)}(X^\bullet, Y^\bullet)$, then the cone $C^\bullet_{f^\bullet}$ is indecomposable.

Proof. Since $H^i(X^\bullet) = 0$ for $i \le -n+1$, $X^\bullet$ is quasi-isomorphic to the complex

$$\bar{X}^\bullet = 0 \to P^{-n+2}/\operatorname{Im} d^{-n+1} \xrightarrow{d^{-n+2}} P^{-n+3} \xrightarrow{d^{-n+3}} \cdots \to P^{-1} \xrightarrow{d^{-1}} P^0 \to 0.$$

Then $\mathrm{Hom}_{D^b(A)}(Y^\bullet, X^\bullet[-1]) \cong \mathrm{Hom}_{D^b(A)}(Y^\bullet, \bar{X}^\bullet[-1]) = 0$. Therefore, $C^\bullet_{f^\bullet}$ is indecomposable by Lemma 3.2. ∎

The following proposition is deduced using the constructions in previous corollaries; it was also proved in [Z14, Proposition 1.4] with a different argument.

PROPOSITION 3.5. *Let A be an artin R-algebra. Then* $\mathrm{gl.dim}\, A \le \mathrm{gl.hw}\, A$.

Proof. Let M be an indecomposable A-module with $\mathrm{pd}\, M > 1$, and let a minimal projective resolution of M be

$$P^\bullet_M = \cdots \to P^{-n-1} \xrightarrow{d^{-n-1}} P^{-n} \xrightarrow{d^{-n}} \cdots \to P^{-1} \xrightarrow{d^{-1}} P^0 \to 0,$$

such that $P^{-n-1} \ne 0$. We will prove that there is an indecomposable object in $D^b(A)$ of cohomological width $n+1$. Indeed, since $P^\bullet_M$ is exact at P^i for $i < 0$ and $P^{-n-1} \ne 0$, $P^\bullet_M$ is isomorphic to

$$\bar{P}^\bullet_M = 0 \to \operatorname{Im} d^{-n-1} \xrightarrow{i} P^{-n} \xrightarrow{d^{-n}} \cdots \to P^{-1} \xrightarrow{d^{-1}} P^0 \to 0$$

with $\operatorname{Im} d^{-n-1} \ne 0$. Take an indecomposable direct summand K of $\operatorname{Im} d^{-n-1}$, and assume $Y^\bullet$ is the stalk complex with K in component $-n-1$. Now consider the canonical epimorphism $f : P^\bullet_M \to Y^\bullet$. Then $C^\bullet_f$ is indecomposable by Corollary 3.4. Moreover, $H^{-n-1}(C^\bullet_f) \ne 0 \ne H^{-1}(C^\bullet_f)$ and $\mathrm{hw}(C^\bullet_f) = n+1$. Thus for any indecomposable module of projective dimension $m > 1$, there is an indecomposable complex with cohomological width m. Therefore, if $\mathrm{gl.dim}\, A < \infty$, then $\mathrm{gl.dim}\, A \le \mathrm{gl.hw}\, A$. If $\mathrm{gl.dim}\, A = \infty$, then there is a simple A-module S of infinite projective dimension. By an argument as above, we can construct indecomposable objects in $D^b(A)$ of arbitrarily large cohomological width, thus $\mathrm{gl.hw}\, A = \infty$. ■

Strong global dimension is defined by taking the supremum of the lengths of all indecomposable perfect complexes, while global width is defined in a totally different way on the level of the bounded derived category using homological information. Now we can prove that the two different parameters coincide for artin R-algebras, based on the constructions of indecomposable objects in the bounded derived category in the previous corollaries. Note that this result was obtained for finite-dimensional algebras over algebraically closed fields in [HZ13, Proposition 3].

THEOREM 3.6. *Let A be an artin R-algebra. Then* $\mathrm{gl.hw}\, A = \mathrm{s.gl.dim}\, A$.

Proof. First we prove $\mathrm{gl.hw}\, A \geq \mathrm{s.gl.dim}\, A$. It suffices to prove that for any minimal indecomposable complex, without loss of generality of the form

$$(*) \qquad P^\bullet = 0 \to P^{-n} \xrightarrow{d^{-n}} P^{-n+1} \xrightarrow{d^{-n+1}} \cdots \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \to 0,$$

we can find an indecomposable complex $Q^\bullet$ such that $n = l(P^\bullet) \leq \mathrm{hw}(Q^\bullet)$. Note that $H^0(P^\bullet) \neq 0$. If $H^{-n+1}(P^\bullet) \neq 0$ or $H^{-n}(P^\bullet) \neq 0$, then $\mathrm{hw}(P^\bullet) \geq n$ and $l(P^\bullet) \leq \mathrm{hw}(P^\bullet)$. If $H^{-n+1}(P^\bullet) = H^{-n}(P^\bullet) = 0$, then with a similar argument to the proof of Proposition 3.5, we take $Y^\bullet$ to be the stalk complex with K in component $-n$ such that K is an indecomposable direct summand of P^{-n}. Then the mapping cone of $f : P^\bullet \to Y^\bullet$, say $Q^\bullet$, is indecomposable by Corollary 3.4 and of cohomological width n.

It remains to prove $\mathrm{gl.hw}\, A \leq \mathrm{s.gl.dim}\, A$. Now we assume $\mathrm{s.gl.dim}\, A < \infty$; then we first claim $\mathrm{gl.dim}\, A < \infty$. Indeed, if there is a simple A-module S of infinite projective dimension with a minimal projective resolution $P^\bullet$, then it is easy to check that the brutal truncation $\sigma_{\geq -m}(P^\bullet) \in K^b(\mathrm{proj}\, A)$ is indecomposable for any $m > 0$, which is a contradiction. Thus $D^b(A) \simeq K^b(\mathrm{proj}\, A)$. Hence, for any indecomposable object $X^\bullet \in D^b(A)$, we can choose a minimal complex $P^\bullet \in K^b(\mathrm{proj}\, A)$ such that $X^\bullet \cong P^\bullet$ in $D^b(A)$. Now it is enough to show that each indecomposable complex $P^\bullet \in K^b(\mathrm{proj}\, A)$ has $\mathrm{hw}(P^\bullet) \leq l(Q^\bullet)$ for some indecomposable complex $Q^\bullet \in K^b(\mathrm{proj}\, A)$. Let $P^\bullet$ be a complex of the form $(*)$. Then $\mathrm{hw}(P^\bullet) \leq l(P^\bullet)$ if $H^{-n}(P^\bullet) = 0$. Assume $M = H^{-n}(P^\bullet) \neq 0$. Let $X^\bullet$ be the stalk complex with K in component $-n$ such that K is an indecomposable direct summand of M and $f : K \to P^{-n}$ is the canonical inclusion satisfying $df^{-n} = 0$. Then $Q^\bullet = C^\bullet_f$ is indecomposable by Corollary 3.3. Moreover, since $\mathrm{gl.dim}\, A < \infty$, $Q^\bullet_f$ is isomorphic to a perfect complex of length $n + \mathrm{pd}\, K + 1$. ■

REMARK 3.7. Let artin R-algebras A and B be derived equivalent by means of a tilting A-module T, i.e. a finitely generated A-module satisfying

(1) $\mathrm{pd}\, T = r$;

(2) $\mathrm{Ext}^i(T,T)=0$ for any $i>0$;
(3) there exists a sequence of the form

$$0\to A_A\to T_1\to T_2\to\cdots\to T_r\to 0$$

with $T_i\in\mathrm{add}(T)$.

Then by Proposition 2.3, we have $\mathrm{gl.hw}\,A\le \mathrm{gl.hw}\,B+l({}_AT^\bullet)=\mathrm{gl.hw}\,B+\mathrm{pd}\,T$. Thus, $\mathrm{gl.hw}\,B-\mathrm{pd}\,T\le \mathrm{gl.hw}\,A\le \mathrm{gl.hw}\,B+\mathrm{pd}\,T$ by tilting symmetry. Combining it with the above theorem, one can retrieve the inequality established by Happel and Zacharia [HZ10, Theorem 4.2]:

$$\mathrm{s.gl.dim}\,B-\mathrm{pd}\,T\le \mathrm{s.gl.dim}\,A\le \mathrm{s.gl.dim}\,B+\mathrm{pd}\,T.$$

4. Piecewise hereditary algebras and global cohomological width. Recall that an artin R-algebra A is said to be *piecewise hereditary of type* $\mathcal{H}$ if there is a triangle equivalence $D^b(A)\simeq D^b(\mathcal{H})$ for some hereditary abelian R-category $\mathcal{H}$ (see [HRS96b]). This section is devoted to exploring piecewise hereditary artin R-algebras in terms of global cohomological width.

Next we shall provide an upper bound for the global cohomological width of piecewise hereditary algebras. For this, we need some preparations.

Let $\mathcal{H}$ be a hereditary abelian category. Recall that a complex $T^\bullet$ in $D^b(\mathcal{H})$ is called a *tilting complex* if

(1) $\mathrm{Hom}_{D^b(\mathcal{H})}(T^\bullet,T^\bullet[i])=0$ for all $i\neq 0$;
(2) $\mathrm{add}\,T^\bullet$ generates $D^b(\mathcal{H})$ as a triangulated category

(see [HR98] for example). If $T^\bullet$ is a tilting complex in $D^b(\mathcal{H})$, then $\mathrm{Hom}_{D^b(\mathcal{H})}(T^\bullet,-):D^b(\mathcal{H})\to D^b(\mathrm{End}(T^\bullet))$ is an equivalence [Ri89]. The following is an important observation [HR98, Lemma 1.5].

LEMMA 4.1. *Let A be a piecewise hereditary algebra with $F:D^b(A)\to D^b(\mathcal{H})$ an equivalence. Then $F(A)$ is a tilting complex in $D^b(\mathcal{H})$.*

Let $\mathcal{H}$ be a hereditary abelian category with $F:D^b(A)\to D^b(\mathcal{H})$ an equivalence. Since an indecomposable object in $D^b(\mathcal{H})$ is isomorphic to a stalk complex by Lemma 2.2, the tilting object $F(A)$ has the form of $F(A)=\bigoplus_{i=0}^s T_i[r+i]$ such that $T_i\in\mathcal{H}$. Note that T_i need not be indecomposable for any $0\le i\le s$.

The following theorem establishes an upper bound for the global cohomological width of piecewise hereditary artin algebras, which coincides with the upper bound provided in [HZ08, Proposition 3.1] for the strong global dimension of finite-dimensional piecewise hereditary algebras over fields by a careful analysis of associated normalized equivalences.

THEOREM 4.2. *Let A be a connected piecewise hereditary algebra and $F:D^b(A)\to D^b(\mathcal{H})$ be an equivalence with $F(A)=\bigoplus_{i=0}^s T_i[r+i]$. Then* gl.hw $A \leq s+2$. *In particular,* gl.hw $A \leq \operatorname{rk} K_0(A)+1$, *where* $K_0(A)$ *is the Grothendieck group of* A.

Proof. For any $X^\bullet \in D^b(A)$, we have

$$H^j(X^\bullet) = \operatorname{Hom}_{D^b(A)}(A, X^\bullet[-j]) \cong \operatorname{Hom}_{D^b(\mathcal{H})}(FA, FX^\bullet[-j]).$$

Assume $X^\bullet$ is indecomposable and $FX^\bullet \in \mathcal{H}[l]$ for some $l \in \mathbb{Z}$. Note that $\operatorname{Hom}_{D(\mathcal{H})}(\mathcal{H}[i], \mathcal{H}[j]) \neq 0$ implies $j = i$ or $i-1$. If $H^j(X^\bullet) \neq 0$, then $r-1 \leq l-j \leq r+s$, and thus $r-l-1 \leq -j \leq r+s-l$. Therefore, $\operatorname{hw}(X^\bullet) \leq s+2$ for any indecomposable object $X^\bullet$ in $D^b(\mathcal{H})$. Consequently, gl.hw $A \leq s+2$.

We claim that the direct summands of $F(A)$ lie in continuous pieces of $\bigcup_{i\in\mathbb{Z}} \mathcal{H}[i]$, i.e., if there are direct summands T', T'' of $F(A)$ such that $T' \in \mathcal{H}[m]$ and $T'' \in \mathcal{H}[n]$, then for any integer $m < i_0 < n$, there exists another direct summand T of $F(A)$ with $T \in \mathcal{H}[i_0]$. Otherwise, we assume Q' is the direct sum of those direct summands in $\mathcal{H}[i]$ with $i < i_0$, and Q'' is the direct sum of those direct summands in $\mathcal{H}[i]$ with $i > i_0$. Then $\operatorname{Hom}_{D^b(\mathcal{H})}(Q', Q'') = 0 = \operatorname{Hom}_{D^b(\mathcal{H})}(Q'', Q')$. So $A \cong \operatorname{End}_{D^b(\mathcal{H})}(Q' \oplus Q'')^{\mathrm{op}}$ is not connected, contrary to assumption. Moreover, the number of indecomposable direct summands of $F(A)$ is precisely $\operatorname{rk} K_0(A)$. Thus, we have $\operatorname{rk} K_0(A) \geq s+1$ by the claim. Therefore, gl.hw $A \leq \operatorname{rk} K_0(A)+1$. ∎

Remark 4.3. Let the algebra A, the functor F and $F(A)$ be as defined in the above theorem. For any $X^\bullet \in D^b(A)$,

$$H^j(X^\bullet) = \operatorname{Hom}_{D^b(A)}(A, X^\bullet[-j]) \cong \operatorname{Hom}_{D^b(\mathcal{H})}(FA, FX^\bullet[-j]).$$

Therefore, if we write $\operatorname{Hom}_{\mathcal{H}}(-,-)$ as $\operatorname{Ext}^0_{\mathcal{H}}(-,-)$ and assume $FX^\bullet \cong M[l]$ for some $M \in \mathcal{H}$ and $l \in \mathbb{Z}$, then we can rewrite the cohomological width of $X^\bullet$ as

$$\begin{aligned}\operatorname{hw}(X^\bullet) &= \mathrm{w}_M \\ &:= \max\{i-t \mid \operatorname{Ext}^t_{\mathcal{H}}(T_i, M) \neq 0\} - \min\{i-t \mid \operatorname{Ext}^t_{\mathcal{H}}(T_i, M) \neq 0\} + 1,\end{aligned}$$

where the only possible values of t are 0 and 1. Thus, the global cohomological width of A can be described as

$$\text{gl.hw}\, A = \sup\{\mathrm{w}_M \mid M \text{ is indecomposable in } \mathcal{H}\}.$$

By the previous theorem, if an artin algebra A is piecewise hereditary, then the global cohomological width of A is finite. The converse also holds true. The following proposition from [Z14, Proposition 1.5] characterizes the piecewise hereditary algebras as those algebras with finite global cohomological width.

Proposition 4.4. *Let A be an artin R-algebra. Then A is piecewise hereditary if and only if* gl.hw $A < \infty$.

Now we consider artin algebras of global cohomological width n. By Lemma 2.2 and Proposition 3.5, $\mathrm{gl.hw}\, A = 1$ if and only if A is hereditary. Moreover, Happel and Zacharia established that for a finite-dimensional algebra A over a field, $\mathrm{s.gl.dim}\, A = 2$ if and only if A is a quasitilted algebra which is not hereditary [HZ08, Proposition 3.3]. Next, we study quasitilted artin R-algebras in terms of global cohomological width.

Recall from [HRS96a] that an algebra A is a *quasitilted algebra* if there exists a hereditary abelian category $\mathcal{H}$ with a tilting object T such that $A = \mathrm{End}_{\mathcal{H}}(T)^{\mathrm{op}}$, or equivalently, if $\mathrm{gl.dim}\, A \leq 2$, and $\mathrm{pd}\, X \leq 1$ or $\mathrm{id}\, X \leq 1$ for any $X \in \mathrm{mod}\, A$. The following proposition describes quasitilted algebras which are not hereditary as those algebras of global cohomological width two (see also [HZ08, Proposition 3.3]).

PROPOSITION 4.5. *Let A be an artin R-algebra. Then* $\mathrm{gl.hw}\, A = 2$ *if and only if A is a quasitilted algebra which is not hereditary.*

Proof. If A is a quasitilted algebra, then $A = \mathrm{End}_{\mathcal{H}}(T)^{\mathrm{op}}$ with T a tilting object in a hereditary abelian category $\mathcal{H}$. Thus $\mathrm{Hom}_{D^b(\mathcal{H})}(T, -) : D^b(\mathcal{H}) \to D^b(A)$ is an equivalence. Moreover, we assume its quasi-inverse is F. Note that $F(A) \cong T$. By Theorem 4.2, we have $\mathrm{gl.hw}\, A \leq 2$. If additionally, A is not hereditary, then $\mathrm{gl.hw}\, A = 2$.

Conversely, assume $\mathrm{gl.hw}\, A = 2$. Then $\mathrm{gl.dim}\, A \leq 2$ by Proposition 3.5 and it suffices to show that for any $M \in \mathrm{mod}\, A$, either $\mathrm{pd}\, M \leq 1$ or $\mathrm{id}\, M \leq 1$. Assume that on the contrary there exists an indecomposable A-module M with $\mathrm{pd}\, M = 2$ and $\mathrm{id}\, M = 2$. The condition $\mathrm{id}\, M = 2$ implies that there exists an indecomposable A-module X such that $\mathrm{Ext}^2(X, M) \neq 0$. Thus $\mathrm{pd}\, X = 2$. Take a minimal projective resolution of X,

$$P^\bullet = 0 \to P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \to 0.$$

Since $\mathrm{Ext}^2(X, M) \neq 0$, we have $\mathrm{Hom}_A(P^{-2}, M) \neq 0$. Let

$$Y^\bullet = 0 \to Q^{-1} \xrightarrow{\delta^{-1}} Q^0 \to 0$$

be a minimal projective presentation of M. Then $\mathrm{Hom}_A(P^{-2}, M) \neq 0$ implies $\mathrm{Hom}_A(P^{-2}, Q^0) \neq 0$. Take a nonzero element $f \in \mathrm{Hom}_A(P^{-2}, Q^0)$. Then f is naturally nonzero and noninvertible in $\mathrm{Hom}_{D^b(A)}(P^\bullet, Y^\bullet[-2])$. Consider the triangle

$$P^\bullet \xrightarrow{f} Y^\bullet[-2] \to C_f^\bullet \to P^\bullet[-1].$$

Since $Y^\bullet[-2]$ is indecomposable and $\mathrm{Hom}_{D^b(A)}(Y^\bullet[-2], P^\bullet[-1]) = 0$, by Lemma 3.2, $C_f^\bullet$ is indecomposable. Moreover, $H^{-3}(C_f^\bullet) \neq 0$ since $\mathrm{pd}\, M = 2$, and $H^{-1}(C_f^\bullet) = X \neq 0$. Thus $\mathrm{hw}(C_f^\bullet) = 3$, which contradicts the assumption. Therefore, A is a quasitilted algebra which is not hereditary. ∎

REMARK 4.6. By the above proof, if $\mathrm{gl.hw}\, A \le n$, then $\mathrm{pd}\, X + \mathrm{id}\, X \le n+1$ for any indecomposable finitely generated A-module X.

5. Applications. Throughout this section, we assume all algebras are finite-dimensional over a field k. We will construct piecewise hereditary algebras of type $\mathbb{A}$ and $\mathbb{D}$ with global cohomological width an arbitrary positive integer m. For this, we need the following lemma due to Happel and Reiten [HR98, Lemma 1.6].

LEMMA 5.1. *Let A be a finite-dimensional hereditary algebra. Then a complex $T^\bullet = \bigoplus_{i=1}^r T_i[i]$ satisfies $\mathrm{Hom}_{D^b(A)}(T^\bullet, T^\bullet[i]) = 0$ for any $i \neq 0$ if and only if $\mathrm{Hom}_A(T_s, T_t) = 0$ if $t \neq s$, and $\mathrm{Ext}^1_A(T_s, T_t) = 0$ if $t \neq s-1$.*

Proof. If $\mathrm{Hom}_{D^b(A)}(T^\bullet, T^\bullet[i]) = 0$ for any $i \neq 0$, then

$$\mathrm{Hom}_A(T_s, T_t) \cong \mathrm{Hom}_{D^b(A)}(T_s[s], T_t[s]),$$

which is a direct summand of $\mathrm{Hom}_{D^b(A)}(T^\bullet, T^\bullet[s-t])$, and is zero if $t \neq s$. Similarly, $\mathrm{Ext}^1_A(T_s, T_t) = 0$ if $t \neq s-1$. Conversely, since

$$\mathrm{Hom}_{D^b(A)}(T^\bullet, T^\bullet[i]) \cong \bigoplus_{1\le s,t\le r} \mathrm{Hom}_{D^b(A)}(T_s, T_t[t+i-s]),$$

and A is hereditary, the only nonzero direct summands are the ones such that $t+i-s = 0$ or 1. If $i \neq 0$, then $\mathrm{Hom}_{D^b(A)}(T_s, T_t[t+i-s]) = 0$ for any $1 \le s, t \le r$ by assumption, and thus $\mathrm{Hom}_{D^b(A)}(T^\bullet, T^\bullet[i]) = 0$. ■

THEOREM 5.2. *There are finite-dimensional piecewise hereditary k-algebras of type $\mathbb{A}$ and $\mathbb{D}$ of global cohomological width an arbitrary positive integer m.*

Proof. Since the global cohomological width of every hereditary algebra is precisely 1, we only need to consider the cases $m \ge 2$. Let $A = kQ_n$ with Q_n the quiver

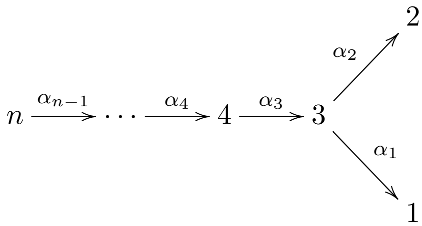
$$n \xrightarrow{\alpha_{n-1}} n-1 \xrightarrow{\alpha_{n-2}} \cdots \xrightarrow{\alpha_2} 2 \xrightarrow{\alpha_1} 1.$$

Assume the complex $T^\bullet = \bigoplus_{i=1}^n S_i[i]$ is in $D^b(A)$, where S_i is the simple A-module corresponding to vertex i. Since $\mathrm{Hom}_A(S_i, S_j) = 0$ if $j \neq i$, and $\mathrm{Ext}^1_A(S_i, S_j) = 0$ if $j \neq i-1$, we have $\mathrm{Hom}_{D^b(A)}(T^\bullet, T^\bullet[i]) = 0$ for any $i \neq 0$ by Lemma 5.1. Moreover, it is clear that $\mathrm{add}\, T^\bullet$ generates $D^b(A)$ as a triangulated category. Thus $T^\bullet$ is a tilting complex in $D^b(A)$. By a straightforward check, $B = \mathrm{End}_{D^b(A)}(T^\bullet) = kQ_n/I$ with I the admissible ideal generated by all paths of length two. Therefore, B is a piecewise hereditary algebra of type $\mathbb{A}$.

Next we will show $\mathrm{gl.hw}\, B = n-1$. Indeed, by Proposition 3.5, $\mathrm{gl.hw}\, B \ge \mathrm{gl.dim}\, B = n-1$. Moreover, Theorem 4.2 implies $\mathrm{gl.hw}\, B \le n+1$. If gl.hw $B = n+1$, then by Remark 4.3, there is an indecomposable A-module M such that $\mathrm{Ext}^1_A(S_1, M) \neq 0$ and $\mathrm{Hom}_A(S_n, M) \neq 0$. Note that $\mathrm{Ext}^1_A(S_1, M) = 0$ for any $M \in \mathrm{mod}\, A$ since S_1 is projective. Thus gl.hw $A \leq n$. Assume gl.hw $B = n$; then there exists an indecomposable A-module N such that $\mathrm{Hom}_A(S_1, N) \neq 0 \neq \mathrm{Hom}_A(S_n, N)$ by Remark 4.3, which is impossible. Therefore, gl.hw $B = n - 1$.

Let $F : D^b(B) \to D^b(A)$ be an equivalence, and $X^\bullet$ be a complex in $D^b(B)$ such that $FX^\bullet \cong P_{n-1}$, where P_{n-1} is the indecomposable projective A-module corresponding to vertex $n-1$. Then $X^\bullet$ is an object with the largest cohomological width by Remark 4.3 since $\mathrm{Hom}_A(S_1, P_{n-1}) \neq 0 \neq \mathrm{Ext}^1_A(S_n, P_{n-1})$. So kQ_{m+1}/I is piecewise hereditary of type $\mathbb{A}$ with global cohomological width m as required.

Now we consider the piecewise hereditary algebras of type $\mathbb{D}$. Let $A' = kQ'_n$ with Q'_n the quiver



such that $n \geq 4$. Similarly, $T^\bullet = (S_1 \oplus S_2)[2] \oplus (\bigoplus_{i=3}^n S_i[i]) \in D^b(A')$ is a tilting complex with $B' = \mathrm{End}_{D^b(A')}(T^\bullet) = kQ'_n/I'$, where S_i is the simple A'-module corresponding to vertex i, and I' is the ideal generated by all paths of length two. Moreover,

$$n - 2 = \mathrm{gl.dim}\, B' \leq \mathrm{gl.hw}\, B' \leq n$$

by Proposition 3.5 and Theorem 4.2. Since $\mathrm{Ext}^1_{A'}(S_1 \oplus S_2, -) = 0$, we have gl.hw $B' < n$ by Remark 4.3. Moreover, gl.hw $B' \neq n-1$ since there is no indecomposable A-module M such that $\mathrm{Hom}_A(S_1 \oplus S_2, M) \neq 0 \neq \mathrm{Hom}_A(S_n, M)$. Therefore, gl.hw $B' = n-2$. Thus kQ'_{m+2}/I' is piecewise hereditary of type $\mathbb{D}$ with global cohomological width m. ∎

Remark 5.3. In the proof of the above theorem, for the piecewise hereditary algebras of global cohomological width m we construct, their global cohomological width coincides with their global dimension. In general, there exists a piecewise hereditary algebra whose cohomological width is strictly greater than its global dimension. Let $A = kQ_5$ be the hereditary algebra, where Q_n is the quiver defined as above, and let

$$T^\bullet = S_2[1] \oplus (S_3 \oplus P_3)[2] \oplus S_4[3] \oplus S_5[4].$$

Then $T^\bullet$ is a tilting complex in $D^b(A)$ by Lemma 5.1 and the fact that add $T^\bullet$ generates $D^b(A)$ as a triangulated category. Moreover, $B = \mathrm{End}_{D^b(A)}(T^\bullet) \cong$

kQ_5/I, where I is the ideal generated by the relations $\alpha_4\alpha_3 = 0 = \alpha_2\alpha_1$. We have $\mathrm{gl.dim}\, B = 2$.

We claim that

$$\mathrm{gl.hw}\, B = 3 > \mathrm{gl.dim}\, B.$$

Indeed, $\mathrm{gl.hw}\, B \leq 5$ by Theorem 4.2. Since S_1 is the unique indecomposable A-module such that $\mathrm{Ext}^1_A(S_2, S_1) \neq 0$, but $\mathrm{Hom}_A(S_4, S_1) = 0 = \mathrm{Hom}_A(S_5, S_1)$, we have $\mathrm{gl.hw}\, B < 5$ by Remark 4.3. If $\mathrm{gl.hw}\, B = 4$ then there is an indecomposable A-module M satisfying $\mathrm{Hom}_A(S_2, M) \neq 0 \neq \mathrm{Hom}_A(S_5, M)$, which is impossible. Thus $\mathrm{gl.hw}\, B \leq 3$. Moreover, the indecomposable A-module P_4/S_1 satisfies

$$\mathrm{Hom}_A(S_2, P_4/S_1) \neq 0 \neq \mathrm{Ext}^1_A(S_5, P_4/S_1),$$

and hence $\mathrm{gl.hw}\, B = 3$ is strictly greater than the global dimension, as claimed.

6. Recollements and global cohomological width. Recollements of triangulated categories were originally introduced by Beilinson, Bernstein and Deligne [BBD82] and play an important role in the study of homological invariants of algebras, for example global dimension [K91, Wi91], finitistic dimension [CX14, Hap93], Hochschild dimension [Han14] and so on. Moreover, the relation of strong global dimension and recollements is described for finite-dimensional algebras over fields in [AKL12]. In this section, we analyse the behavior of the global cohomological width of artin algebras under recollements.

Let $\mathcal{T}_1$, $\mathcal{T}$ and $\mathcal{T}_2$ be triangulated categories. Recall from [BBD82] that a *recollement* of $\mathcal{T}$ relative to $\mathcal{T}_1$ and $\mathcal{T}_2$ is given by

$$\mathcal{T}_1 \underset{\xleftarrow{\;i^!\;}}{\overset{\xleftarrow{\;i^*\;}}{\xrightarrow{\;i_*=i_!\;}}} \mathcal{T} \underset{\xleftarrow{\;j_*\;}}{\overset{\xleftarrow{\;j_!\;}}{\xrightarrow{\;j^!=j^*\;}}} \mathcal{T}_2$$

such that

- (R1) (i^*, i_*), $(i_!, i^!)$, $(j_!, j^!)$ and (j^*, j_*) are adjoint pairs of triangle functors;
- (R2) i_*, $j_!$ and j_* are full embeddings;
- (R3) $j^! i_* = 0$ (and thus also $i^! j_* = 0$ and $i^* j_! = 0$);
- (R4) for each $X \in \mathcal{T}$, there are canonical triangles

$$j_! j^! X \to X \to i_* i^* X \to \Sigma j_! j^! X,$$
$$i_! i^! X \to X \to j_* j^* X \to \Sigma i_! i^! X,$$

where the maps are the counits and the units of the adjoint pairs respectively.

Proposition 6.1. *Let A be an artin algebra. Suppose there is a recollement*

$$D^b(B) \underset{\overset{\longleftarrow}{i^!}}{\overset{\overset{i^*}{\longleftarrow}}{\xrightarrow{i_*=i_!}}} D^b(A) \underset{\overset{\longleftarrow}{j_*}}{\overset{\overset{j_!}{\longleftarrow}}{\xrightarrow{j^!=j^*}}} D^b(C)\ .$$

Then

(1) $\mathrm{gl.hw}\, B \leq \mathrm{gl.hw}\, A + l(i_!B)$,
(2) $\mathrm{gl.hw}\, C \leq \mathrm{gl.hw}\, A + l(j_!C)$,

where $l(i_!B)$ and $l(j_!C)$ are the lengths of $i_!B$ and $j_!C$ respectively, viewed as objects in $K^b(\mathrm{proj}\, A)$. In particular, if A is a piecewise hereditary algebra, then so are B and C.

Proof. We only prove statement (1); the proof of (2) is similar. If $\mathrm{gl.hw}\, A = \infty$, then the inequality holds. Now suppose $\mathrm{gl.hw}\, A < \infty$. Then by Proposition 3.5, $\mathrm{gl.dim}\, A < \infty$ and thus $D^b(A) \simeq K^b(\mathrm{proj}\, A)$. Since $i_!$ is a full embedding, for any indecomposable object $X^\bullet \in D^b(B)$,

$$H^t(X^\bullet) = \mathrm{Hom}_{D^b(B)}(B, X^\bullet[-t]) \cong \mathrm{Hom}_{D^b(A)}(i_!B, i_!X^\bullet[-t]).$$

Note that $i_!(B) \in K^b(\mathrm{proj}\, A)$ with $n = l(i_!B)$, and we assume it is quasi-isomorphic to the minimal complex

$$P^\bullet = 0 \to P^{-n} \xrightarrow{d^{-n}} P^{-n+1} \xrightarrow{d^{-n+1}} \cdots \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \to 0.$$

Since $i_!X^\bullet$ is indecomposable, $h = \mathrm{hw}(i_!X^\bullet) \leq \mathrm{gl.hw}\, A$. By truncations, we can assume $i_!X^\bullet$ is quasi-isomorphic to the complex

$$\bar{X}^\bullet = 0 \to X^{s-h+1} \to X^{s-h+2} \to \cdots \to X^{s-1} \to X^s \to 0$$

such that $H^s(\bar{X}^\bullet) \neq 0 \neq H^{s-h+1}(\bar{X}^\bullet)$. Thus $H^t(X^\bullet) \neq 0$ implies $-n-s \leq -t \leq h-s-1$, and then $\mathrm{hw}(X^\bullet) \leq h+n \leq \mathrm{gl.hw}\, A + l(i_1B)$. Therefore, $\mathrm{gl.hw}\, B \leq \mathrm{gl.hw}\, A + l(i_!B)$. The remaining statement follows from Proposition 4.4. ∎

Remark 6.2. The proof above actually provides an alternative proof of Proposition 2.3 by viewing derived equivalences as trivial recollements. In general, the converse of the statement is not ture, i.e., if there is a recollement as in the above proposition such that B and C are piecewise hereditary, then A is not necessarily piecewise hereditary: see [AKL12, Example 4.3] for a counterexample.

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