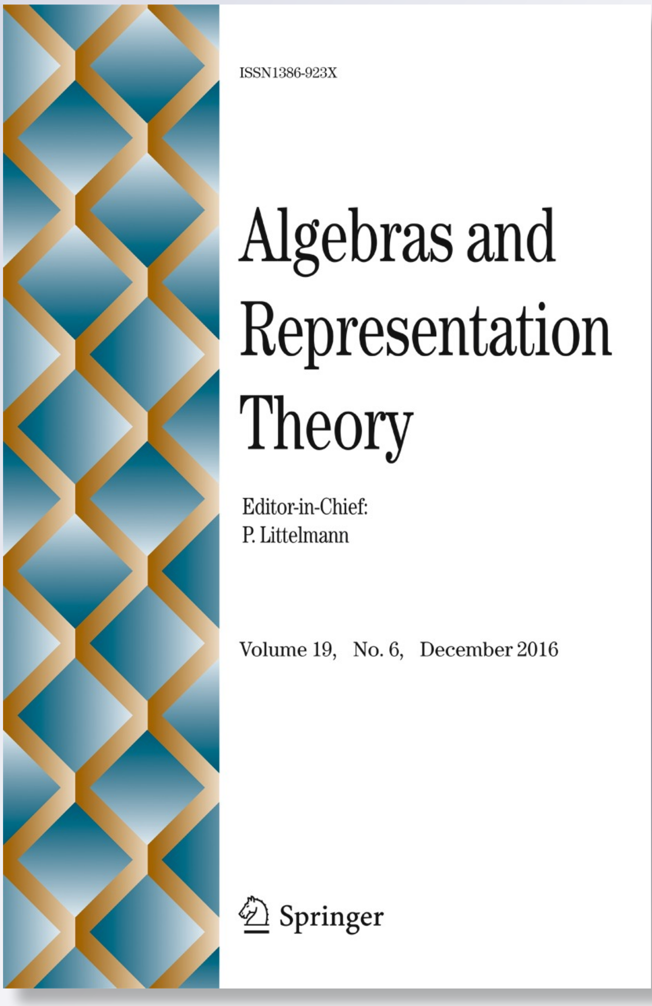
Brauer-Thrall Type Theorems for Derived Module Categories

Chao Zhang & Yang Han

Algebras and Representation Theory

ISSN 1386-923X
Volume 19
Number 6

Algebr Represent Theor (2016)
19:1369-1386
DOI 10.1007/s10468-016-9622-7



Algebr Represent Theor (2016) 19:1369–1386
DOI 10.1007/s10468-016-9622-7

Brauer-Thrall Type Theorems for Derived Module Categories

Chao Zhang[1] · **Yang Han**[2]

Received: 19 August 2015 / Accepted: 23 May 2016 / Published online: 3 June 2016

Abstract The numerical invariants (global) cohomological length, (global) cohomological width, and (global) cohomological range of a complex (an algebra) are introduced. Cohomological range leads to the concepts of derived bounded algebra and strongly derived unbounded algebra naturally. The first and second Brauer-Thrall type theorems for the bounded derived category of a finite-dimensional algebra over an algebraically closed field are obtained. The first Brauer-Thrall type theorem says that derived bounded algebras are just derived finite algebras. The second Brauer-Thrall type theorem says that an algebra is either derived discrete or strongly derived unbounded, but not both. Moreover, piecewise hereditary algebras and derived discrete algebras are characterized as the algebras of finite global cohomological width and the algebras of finite global cohomological length respectively.

Keywords Derived category · Indecomposable object · Derived finite algebra · Derived discrete algebra · Piecewise hereditary algebra

Mathematics Subject Classification (2010) 16E35 · 16G60 · 16E05 · 16G20

Presented by Yuri Drozd.

Dedicated to Professor Yingbo Zhang on the occasion of her 70th birthday

✉ Yang Han
hany@iss.ac.cn

[1] Department of Mathematics, Guizhou University, Guiyang 550025, People's Republic of China

[2] KLMM, ISS, AMSS, Chinese Academy of Sciences, Beijing 100190, People's Republic of China

1 Introduction

Throughout this paper, k is an algebraically closed field, all algebras are connected basic finite-dimensional associative k-algebras with identity, and all modules are finite-dimensional right modules, unless stated otherwise. One of the main topics in representation theory of algebras is to study the classification and distribution of indecomposable modules. In this aspect two famous problems are Brauer-Thrall conjectures I and II:

Brauer-Thrall Conjecture I The algebras of bounded representation type are of finite representation type.

Brauer-Thrall Conjecture II The algebras of unbounded representation type are of strongly unbounded representation type.

Here, we say an algebra is *of finite representation type* or *representation-finite* if there are only finitely many isomorphism classes of indecomposable modules. An algebra is said to be *of bounded representation type* if the dimensions of all indecomposable modules have a common upper bound, and *of unbounded representation type* otherwise. We say an algebra is *of strongly unbounded representation type* if there are infinitely many $d \in \mathbb{N}$ such that for each d, there exist infinitely many isomorphism classes of indecomposable modules of dimension d. Brauer-Thrall conjectures I and II were formulated by Jans [24]. Brauer-Thrall conjecture I was proved for finite-dimensional algebras over an arbitrary field by Roiter [29], and artin algebras by Auslander [5]. Brauer-Thrall conjecture II was proved for finite-dimensional algebras over an infinite perfect field by Nazarova and Roiter using matrix method [26, 30], and an algebraically closed field by Bautista using geometric method [6]. Refer to [28] for more on Brauer-Thrall conjectures.

Since Happel [17, 18], the bounded derived categories of finite-dimensional algebras have been studied widely. The study of the classification and distribution of indecomposable objects in the bounded derived category of an algebra is still an important theme in representation theory of algebras. It is natural to consider the derived versions of Brauer-Thrall conjectures. For this, one needs to find an invariant of a complex analogous to the dimension of a module. On this topic, Vossieck is undoubtedly a pioneer. He introduced and classified *derived discrete algebras*, i.e., the algebras whose bounded derived categories admit only finitely many isomorphism classes of indecomposable objects of arbitrarily given cohomology dimension vector, in his elegant paper [32]. Since a complex and its shifts are of different cohomology dimension vectors, for a non derived discrete algebra, there are always infinitely many $\mathbf{d} \in \mathbb{N}^{(\mathbb{Z})}$ such that for each $\mathbf{d}$, there exist infinitely many isomorphism classes of indecomposable objects of cohomology dimension vector $\mathbf{d}$ in its bounded derived category. Nevertheless, cohomology dimension vector is seemingly not a perfect invariant of complexes in the context of derived versions of Brauer-Thrall conjectures, because it is too fine to identify an indecomposable complex with its shifts and cannot be used to define the derived boundedness and strongly derived unboundedness of algebras.

In this paper, we shall introduce the cohomological range of a bounded complex which is a numerical invariant under shifts and isomorphisms. It leads to the concepts of derived bounded algebras and strongly derived unbounded algebras naturally. We shall prove the following two Brauer-Thrall type theorems for derived module categories:

Theorem I Derived bounded algebras are just derived finite algebras.

Theorem II An algebra is either derived discrete or strongly derived unbounded, but not both.

According to Theorem I and Theorem II, all algebras are divided into three disjoint classes: derived finite algebras, derived discrete but not derived finite algebras, and strongly derived unbounded algebras. In particular, Theorem II excludes the existence of such an algebra for which there are only (nonempty) finitely many $r \in \mathbb{N}$ such that for each r, up to shift and isomorphism, there exist infinitely many indecomposable objects of cohomological range r in its bounded derived category.

The paper is organized as follows: in Section 2, we shall introduce some numerical invariants of complexes (algebras) including (global) cohomological length, (global) cohomological width, and (global) cohomological range, and observe their behaviors under derived equivalences. Global cohomological width provides an alternative definition of strong global dimension on the level of bounded derived category, and piecewise hereditary algebras are characterized as the algebras of finite global cohomological width. Furthermore, we shall prove Theorem I. In Section 3, we shall show that strongly derived unboundedness is invariant under derived equivalences, and observe its relation with cleaving functors. Furthermore, we shall prove Theorem II for simply connected algebras, gentle algebras, and finally all algebras by using cleaving functors and covering theory. Moreover, derived discrete algebras are characterized as the algebras of finite global cohomological length.

2 The First Brauer-Thrall Type Theorem

2.1 Some Numerical Invariants of Complexes and Algebras

Let A be a (finite-dimensional) k-algebra. Denote by mod$A$ the category of all (finite-dimensional) right A-modules, and by proj$A$ its full subcategory consisting of all finite-dimensional projective right A-modules. Denote by $C(A)$ the category of all complexes of finite-dimensional right A-modules, and by $C^b(A)$ and $C^{-,b}(A)$ its full subcategories consisting of all bounded complexes and right bounded complexes with bounded cohomology respectively. Denote by $C^b(\text{proj}A)$ and $C^{-,b}(\text{proj}A)$ the full subcategories of $C^b(A)$ and $C^{-,b}(A)$ respectively consisting of all complexes of finite-dimensional projective modules. Denote by $K(A)$, $K^b(\text{proj}A)$ and $K^{-,b}(\text{proj}A)$ the homotopy categories of $C(A)$, $C^b(\text{proj}A)$ and $C^{-,b}(\text{proj}A)$ respectively. Denote by $D^b(A)$ the bounded derived category of mod$A$. Moreover, $\dim := \dim_k$, the dimension of a k-vector space.

Now we introduce some numerical invariants of complexes and algebras.

Definition 1 The *cohomological length* of a complex $X^\bullet \in D^b(A)$ is

$$\text{hl}(X^\bullet) := \max\{\dim H^i(X^\bullet) \mid i \in \mathbb{Z}\},$$

and the *global cohomological length* of A is

$$\text{gl.hl}A := \sup\{\text{hl}(X^\bullet) \mid X^\bullet \in D^b(A) \text{ is indecomposable}\}.$$

Obviously, the dimension of an A-module M is equal to the cohomological length of the stalk complex M. Note that there is a full embedding of mod$A$ into $D^b(A)$ which sends a module to the corresponding stalk complex. If gl.hl$A < \infty$ then A is representation-finite due to the truth of Brauer-Thrall conjecture I.

Definition 2 The *cohomological width* of a complex $X^\bullet \in D^b(A)$ is

$$\mathrm{hw}(X^\bullet) := \max\{j - i + 1 \mid H^i(X^\bullet) \neq 0 \neq H^j(X^\bullet)\},$$

and the *global cohomological width* of A is

$$\mathrm{gl.hw}A := \sup\{\mathrm{hw}(X^\bullet) \mid X^\bullet \in D^b(A) \text{ is indecomposable}\}.$$

Clearly, the cohomological width of a stalk complex is 1. If A is a hereditary algebra then every indecomposable complex $X^\bullet \in D^b(A)$ is isomorphic to a stalk complex by [18, I.5.2 Lemma]. Thus gl.hw$A = 1$.

Definition 3 The *cohomological range* of a complex $X^\bullet \in D^b(A)$ is

$$\mathrm{hr}(X^\bullet) := \mathrm{hl}(X^\bullet) \cdot \mathrm{hw}(X^\bullet),$$

and the *global cohomological range* of A is

$$\mathrm{gl.hr}\ A := \sup\{\mathrm{hr}(X^\bullet) \mid X^\bullet \in D^b(A) \text{ is indecomposable}\}.$$

The cohomological range of a complex will play a role similar to the dimension of a module. It is invariant under shifts and isomorphisms, since both cohomological length and cohomological width are.

Next we observe the behaviors of these invariants under derived equivalences. For this, we need do some preparations.

Let $\mathcal{T}$ be a triangulated k-category with [1] the shift functor. For $T \in \mathcal{T}$, we define $\langle T\rangle_n$ inductively by

$$\langle T\rangle_0 := \{X \in \mathcal{T} \mid X \text{ is a direct summand of } T[i] \text{ for some } i \in \mathbb{Z}\},$$

and

$$\langle T\rangle_n := \left\{ X \in \mathcal{T} \;\middle|\; \begin{array}{l} Y' \to X \oplus Y \to Y'' \to Y'[1] \text{ is a triangle in } \mathcal{T} \\ \text{with } Y', Y'' \in \langle T\rangle_{n-1} \text{ and } Y \in \mathcal{T} \end{array} \right\}.$$

Clearly, $\langle T\rangle_{n-1} \subseteq \langle T\rangle_n$ and $\langle T\rangle := \bigcup_{n\geq 0}\langle T\rangle_n$ is the smallest thick subcategory of $\mathcal{T}$ containing T. For $X \in \langle T\rangle$, the *distance* of X from T is

$$d(X, T) := \min\{n \in \mathbb{N} \mid X \in \langle T\rangle_n\}.$$

Lemma 1 *(See Geiss and Krause* [16, Lemma 4.1]*) Let $\mathcal{T}$ be a triangulated k-category, $T \in \mathcal{T}$ and $X \in \langle T\rangle$. Then for all $Y \in \mathcal{T}$,*

$$\dim \mathrm{Hom}_{\mathcal{T}}(X, Y) \leq 2^{d(X,T)} \sup_{i\in\mathbb{Z}} \dim \mathrm{Hom}_{\mathcal{T}}(T[i], Y).$$

Proposition 1 *Let A and B be two algebras, ${}_AT^\bullet_B$ a two-sided tilting complex, and $F = -\otimes^L_A T^\bullet_B : D^b(A) \to D^b(B)$ a derived equivalence. Then there are $N_1, N_2, N_3 \in \mathbb{N}$ such that for all $X^\bullet \in D^b(A)$,*

(1) $\mathrm{hw}(F(X^\bullet)) \leq \mathrm{hw}(X^\bullet) + N_1$,

(2) $\mathrm{hl}(F(X^\bullet)) \leq N_2 \cdot \mathrm{hl}(X^\bullet)$,
(3) $\mathrm{hr}(F(X^\bullet)) \leq N_3 \cdot \mathrm{hr}(X^\bullet)$.

Proof

(1) Recall that the *width* of a complex $Y^\bullet \in C^b(A)$ is

$$\mathrm{w}(Y^\bullet) := \max\{j - i + 1 \mid Y^j \neq 0 \neq Y^i\}.$$

For any $X^\bullet \in D^b(A)$, there exists a complex $\tilde{X}^\bullet \in D^b(A)$ which can be obtained from $X^\bullet$ by good truncations, such that $\mathrm{hw}(\tilde{X}^\bullet) = \mathrm{w}(\tilde{X}^\bullet)$ and $\tilde{X}^\bullet \cong X^\bullet$ in $D^b(A)$. Since ${}_AT^\bullet_B$ is a two-sided tilting complex, there is a perfect complex ${}_A\tilde{T}^\bullet \in C^b(\mathrm{proj}A^{\mathrm{op}})$ such that ${}_AT^\bullet \cong {}_A\tilde{T}^\bullet$ in $D^b(A^{\mathrm{op}})$. Thus $F(\tilde{X}^\bullet) = \tilde{X}^\bullet \otimes^L_A T^\bullet \cong \tilde{X}^\bullet \otimes_A \tilde{T}^\bullet$ in $D^b(k)$. Hence $\mathrm{hw}(F(X^\bullet)) = \mathrm{hw}(F(\tilde{X}^\bullet)) = \mathrm{hw}(\tilde{X}^\bullet \otimes_A \tilde{T}^\bullet) \leq \mathrm{w}(\tilde{X}^\bullet \otimes_A \tilde{T}^\bullet) \leq \mathrm{w}(\tilde{X}^\bullet_A) + \mathrm{w}({}_A\tilde{T}^\bullet) - 1 = \mathrm{hw}(\tilde{X}^\bullet_A) + \mathrm{w}({}_A\tilde{T}^\bullet) - 1 = \mathrm{hw}(X^\bullet_A) + \mathrm{w}({}_A\tilde{T}^\bullet) - 1$. So $N_1 := \mathrm{w}({}_A\tilde{T}^\bullet) - 1$ is as required.

(2) Since F is a derived equivalence, we have $B \in K^b(\mathrm{proj}B) = \langle F(A)\rangle$. By Lemma 1, we get

$$\begin{aligned}\dim H^i(F(X^\bullet)) &= \dim \mathrm{Hom}_{D^b(B)}(B, F(X^\bullet)[i]) \\ &\leq 2^{d(B,F(A))} \sup_{j\in\mathbb{Z}} \dim \mathrm{Hom}_{D^b(B)}(F(A), F(X^\bullet)[j]) \\ &= 2^{d(B,F(A))} \sup_{j\in\mathbb{Z}} \dim \mathrm{Hom}_{D^b(A)}(A, X^\bullet[j]) \\ &= 2^{d(B,F(A))} \sup_{j\in\mathbb{Z}} \dim H^j(X^\bullet) \\ &= 2^{d(B,F(A))}\, \mathrm{hl}(X^\bullet).\end{aligned}$$

Thus $N_2 := 2^{d(B,F(A))}$ is as required.

(3) It follows from (1) and (2) that $\mathrm{hr}(F(X^\bullet)) = \mathrm{hl}(F(X^\bullet)) \cdot \mathrm{hw}(F(X^\bullet)) \leq N_2 \cdot \mathrm{hl}(X^\bullet) \cdot (\mathrm{hw}(X^\bullet) + N_1) \leq N_2(N_1 + 1) \cdot \mathrm{hr}(X^\bullet)$. Thus $N_3 := N_2(N_1 + 1)$ is as required.

□

Corollary 1 *Let two algebras A and B be derived equivalent. Then gl.hw$A < \infty$ (resp. gl.hl$A < \infty$, gl.hr$A < \infty$) if and only if gl.hw $B < \infty$ (resp. gl.hl $B < \infty$, gl.hr$B < \infty$).*

Proof Since A and B are derived equivalent, there is a two-sided tilting complex ${}_AT^\bullet_B$ such that $-\otimes^L_A T^\bullet_B : D^b(A) \to D^b(B)$ is a derived equivalence [27]. So the corollary follows from Proposition 1. □

2.2 Strong Global Dimension

Strong global dimension was introduced by Skowroński in [31]. Happel and Zacharia characterized piecewise hereditary algebras as the algebras of finite strong global dimension [23]. Here, we adopt the definition of strong global dimension in [23], which is slightly different from that in [31].

Recall that a complex $X^\bullet = (X^i, d^i) \in C(A)$ is said to be *minimal* if $\mathrm{Im}d^i \subseteq \mathrm{rad}X^{i+1}$ for all $i \in \mathbb{Z}$. For any complex $P^\bullet = (P^i, d^i) \in K^b(\mathrm{proj}A)$, there is a minimal complex

$\bar{P}^\bullet = (\bar{P}^i, \bar{d}^i) \in K^b(\text{proj}A)$, which is unique up to isomorphism in $C^b(A)$, such that $P^\bullet \cong \bar{P}^\bullet$ in $K^b(\text{proj}A)$. The *length* of $P^\bullet$ is

$$l(P^\bullet) := \max\{j - i \mid \bar{P}^i \neq 0 \neq \bar{P}^j\}.$$

The *strong global dimension* of A is

$$\text{s.gl.dim}\, A := \sup\{l(P^\bullet) \mid P^\bullet \in K^b(\text{proj}A) \text{ is indecomposable}\}.$$

Obviously, for a module of finite projective dimension, the length of its minimal projective resolution equals to its projective dimension. Furthermore, if gl.dim $A < \infty$ then s.gl.dim $A \geq$ gl.dim A.

The following result sets up the connection between the indecomposable objects in $K^b(\text{proj}A)$ and those in $K^{-,b}(\text{proj}A)$.

Proposition 2 *Let* $P^\bullet \in K^{-,b}(\text{proj}A)$ *be a minimal complex and* $n := \min\{i \in \mathbb{Z} \mid H^i(P^\bullet) \neq 0\}$. *Then* $P^\bullet$ *is indecomposable if and only if so is the brutal truncation* $\sigma_{\geq j}(P^\bullet) \in K^b(\text{proj}A)$ *for some (resp. all)* $j < n$.

Proof Since $K^{-,b}(\text{proj}A) \simeq D^b(A)$ is a Krull-Schmidt category, a complex $X^\bullet \in K^{-,b}(\text{proj}A)$ is indecomposable if and only if its endomorphism algebra $\text{End}_{K(A)}(X^\bullet)$ is local, if and only if $\text{End}_{K(A)}(X^\bullet)/\text{radEnd}_{K(A)}(X^\bullet) \cong k$. Hence, it suffices to show

$$\text{End}_{K(A)}(P^\bullet)/\text{radEnd}_{K(A)}(P^\bullet) \cong \text{End}_{K(A)}(\sigma_{\geq j}(P^\bullet))/\text{radEnd}_{K(A)}(\sigma_{\geq j}(P^\bullet)).$$

Since $P^\bullet$ is minimal, all null homotopies in $\text{End}_{C(A)}(\sigma_{\geq j}(P^\bullet))$ form a nilpotent ideal of $\text{End}_{C(A)}(\sigma_{\geq j}(P^\bullet))$, thus are in $\text{radEnd}_{C(A)}(\sigma_{\geq j}(P^\bullet))$. Hence we have

$$\text{End}_{K(A)}(\sigma_{\geq j}(P^\bullet))/\text{radEnd}_{K(A)}(\sigma_{\geq j}(P^\bullet)) \cong \text{End}_{C(A)}(\sigma_{\geq j}(P^\bullet))/\text{radEnd}_{C(A)}(\sigma_{\geq j}(P^\bullet)).$$

Now it is enough to show

$$\text{End}_{K(A)}(P^\bullet)/\text{radEnd}_{K(A)}(P^\bullet) \cong \text{End}_{C(A)}(\sigma_{\geq j}(P^\bullet))/\text{radEnd}_{C(A)}(\sigma_{\geq j}(P^\bullet)).$$

Consider the composition of homomorphisms of algebras

$$\text{End}_{C(A)}(P^\bullet) \xrightarrow{\phi} \text{End}_{C(A)}(\sigma_{\geq j}(P^\bullet)) \xrightarrow{\psi} \text{End}_{C(A)}(\sigma_{\geq j}(P^\bullet))/\text{radEnd}_{C(A)}(\sigma_{\geq j}(P^\bullet)),$$

where ϕ is the natural restriction and ψ is the canonical epimorphism. Since $\sigma_{\leq j-1}(P^\bullet)$ is a minimal projective resolution of $\text{Ker}d^j$, every cochain map in $\text{End}_{C(A)}(\sigma_{\geq j}(P^\bullet))$ can be lifted to a cochain map in $\text{End}_{C(A)}(P^\bullet)$, i.e., ϕ is surjective. Thus the composition $\varphi := \psi\phi$ is surjective. Since $P^\bullet$ is a minimal complex, all null homotopies in $\text{End}_{C(A)}(P^\bullet)$ form a nilpotent ideal of $\text{End}_{C(A)}(P^\bullet)$, thus are in $\text{radEnd}_{C(A)}(P^\bullet)$. Furthermore, ϕ maps all null homotopies in $\text{End}_{C(A)}(P^\bullet)$ into $\text{radEnd}_{C(A)}(\sigma_{\geq j}(P^\bullet))$. Hence φ induces a surjective homomorphism of algebras

$$\bar{\varphi} : \text{End}_{K(A)}(P^\bullet) \twoheadrightarrow \text{End}_{C(A)}(\sigma_{\geq j}(P^\bullet))/\text{radEnd}_{C(A)}(\sigma_{\geq j}(P^\bullet)).$$

Now it is sufficient to show that $\text{Ker}\bar{\varphi} = \text{radEnd}_{K(A)}(P^\bullet)$. Clearly, $\text{Ker}\bar{\varphi} \supseteq \text{radEnd}_{K(A)}(P^\bullet)$. Conversely, for any $\bar{f}^\bullet \in \text{Ker}\bar{\varphi}$ with $f^\bullet \in \text{End}_{C(A)}(P^\bullet)$, we have $\psi(\phi(f^\bullet)) = \bar{\varphi}(\bar{f}^\bullet) = 0$. Thus $\phi(f^\bullet)$ is nilpotent, i.e., there exists $t \in \mathbb{N}$ such that $(f^i)^t = 0$ for all $i \geq j$. Since $\sigma_{\leq j-1}(P^\bullet)$ is a minimal projective resolution of $\text{Ker}d^j$, the restriction $\sigma_{\leq j-1}(f^\bullet) \in \text{End}_{C(A)}(\sigma_{\leq j-1}(P^\bullet))$ of $f^\bullet$ is a lift of the restriction of f^j on $\text{Ker}d^j$. Thus $(\sigma_{\leq j-1}(f^\bullet))^t$ is a lift of the restriction of $(f^j)^t$ on $\text{Ker}d^j$, i.e., a lift of zero morphism. Hence $(\sigma_{\leq j-1}(f^\bullet))^t$ is a null homotopy. Therefore, $\bar{f^\bullet}^t = 0$, i.e., $\bar{f^\bullet}$ is nilpotent, in $\mathrm{End}_{K(A)}(P^\bullet)$. So $\mathrm{Ker}\,\bar{\varphi}$ is a nilpotent ideal of $\mathrm{End}_{K(A)}(P^\bullet)$. Consequently, $\mathrm{Ker}\,\bar{\varphi} \subseteq \mathrm{rad}\mathrm{End}_{K(A)}(P^\bullet)$. □

Corollary 2 *Let A be an algebra. Then gl.dim$A \leq$ s.gl.dim$A$.*

Proof We have known gl.dim$A \leq$ s.gl.dim$A$ if gl.dim$A < \infty$. If gl.dim$A = \infty$ then there is a simple A-module S of infinite projective dimension. Thus S admits an infinite minimal projective resolution. By Proposition 2, there are indecomposable objects in $K^b(\mathrm{proj}A)$ of arbitrarily large length, which implies s.gl.dim$A = \infty$. □

Remark 1 It is possible gl.dim$A <$ s.gl.dim$A$. Indeed, since piecewise hereditary algebras are factors of finite-dimensional hereditary algebras [21, Theorem 1.1], all algebras of finite global dimension and with oriented cycles in their quivers are not piecewise hereditary, thus of infinite strong global dimension by [23, Theorem 3.2].

As an additional corollary, we give a characterization of global cohomological width on the level of bounded homotopy categories of finite-dimensional projective modules.

Corollary 3 *Let A be an algebra. Then*

$$\textit{gl.hw } A = \sup\{hw(P^\bullet) \mid P^\bullet \textit{ is (minimal) indecomposable in } K^b(\mathrm{proj}A)\}.$$

Proof Clearly, the value of the right hand side of the equation is invariant no matter whether we assume that the indecomposable complex $P^\bullet \in K^b(\mathrm{proj}A)$ is minimal or not. Since $K^b(\mathrm{proj}A) \subseteq D^b(A)$, the right hand side is not larger than gl.hw$A$. Conversely, by Proposition 2, any minimal indecomposable complex $P^\bullet \in K^{-,b}(\mathrm{proj}A) \simeq D^b(A)$ has the property that $\mathrm{hw}(\sigma_{\geq j}(P^\bullet)) \geq \mathrm{hw}(P^\bullet)$ and $\sigma_{\geq j}(P^\bullet) \in K^b(\mathrm{proj}A)$ is indecomposable for $j \ll 0$. Thus the right hand side is not smaller than gl.hw$A$. □

The following result implies that the global cohomological width can provide an alternative definition of strong global dimension on the level of bounded derived category.

Proposition 3 *Let A be an algebra. Then gl.hw$A =$ s.gl.dim$A$.*

Proof First we show gl.hw$A \geq$ s.gl.dim$A$. It suffices to prove that for any minimal indecomposable complex $P^\bullet \in K^b(\mathrm{proj}A)$ with $l(P^\bullet) = n$, there is an indecomposable complex $Q^\bullet \in K^b(\mathrm{proj}A)$ such that $\mathrm{hw}(Q^\bullet) \geq n$. Without loss of generality, we assume

$$P^\bullet = 0 \longrightarrow P^{-n} \xrightarrow{d^{-n}} P^{-n+1} \xrightarrow{d^{-n+1}} \cdots \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \longrightarrow 0.$$

Since $P^\bullet$ is minimal, we have $H^0(P^\bullet) \neq 0$. If $H^{-n}(P^\bullet) \neq 0$ or $H^{-n+1}(P^\bullet) \neq 0$ then $\mathrm{hw}(P^\bullet) \geq n$. Thus $Q^\bullet := P^\bullet$ is as required. If $H^{-n}(P^\bullet) = 0 = H^{-n+1}(P^\bullet)$ then $Q^\bullet := \sigma_{\geq -n+1}(P^\bullet)$ is as required, since it is indecomposable by Proposition 2 and $\mathrm{hw}(Q^\bullet) = n$.

Next we show s.gl.dim$A \geq$ gl.hw$A$. By Corollary 3, it is enough to show that for any minimal indecomposable complex $P^\bullet \in K^b(\mathrm{proj}A)$, there is a minimal indecomposable complex $Q^\bullet \in K^b(\text{proj}A)$ such that $l(Q^\bullet) \geq \text{hw}(P^\bullet)$. Without loss of generality, we still assume that $P^\bullet$ is of the above form. If $H^{-n}(P^\bullet) = 0$ then $l(P^\bullet) \geq \text{hw}(P^\bullet)$. Thus $Q^\bullet := P^\bullet$ is as required. If $H^{-n}(P^\bullet) \cong \text{Ker}d^{-n} \neq 0$, we take a minimal projective resolution of $\text{Ker}d^{-n}$, say

$$P'^\bullet = \cdots \longrightarrow P^{-n-2} \xrightarrow{d^{-n-2}} P^{-n-1} \longrightarrow 0.$$

Gluing $P'^\bullet$ and $P^\bullet$ together, we get a minimal complex

$$P''^\bullet = \cdots \longrightarrow P^{-n-2} \xrightarrow{d^{-n-2}} P^{-n-1} \xrightarrow{d^{-n-1}} P^{-n} \xrightarrow{d^{-n}} \cdots \xrightarrow{d^{-1}} P^0 \longrightarrow 0,$$

where d^{-n-1} is the composition $P^{-n-1} \twoheadrightarrow \text{Ker}d^{-n} \hookrightarrow P^{-n}$. Since $P^\bullet = \sigma_{\geq -n}(P''^\bullet)$ is indecomposable and $H^i(P''^\bullet) = 0$ for all $i \leq -n$, by Proposition 2, $P''^\bullet$ is indecomposable. Also by Proposition 2, we have $Q^\bullet := \sigma_{\geq -n-1}(P''^\bullet)$ is indecomposable with $l(Q^\bullet) = n + 1 = \text{hw}(P^\bullet)$. □

Recall that an algebra A is said to be *piecewise hereditary* if there is a triangle equivalence $D^b(A) \simeq D^b(\mathcal{H})$ for some hereditary abelian k-category $\mathcal{H}$ (Ref. [21]). In this case, $\mathcal{H}$ must have a tilting object [20]. Thus there are exactly two classes of piecewise hereditary algebras whose derived categories are triangle equivalent to either $D^b(kQ)$ for some finite connected quiver Q without oriented cycles, or $D^b(\text{coh}\mathbb{X})$ for some weighted projective line $\mathbb{X}$ (Ref. [19]).

As a corollary of Proposition 3, piecewise hereditary algebras can be characterized as the algebras of finite global cohomological width.

Corollary 4 *An algebra A is piecewise hereditary if and only if gl.hw$A < \infty$.*

Proof It follows from [23, Theorem 3.2] and Proposition 3. □

2.3 The First Brauer-Thrall Type Theorem

Definition 4 An algebra A is said to be *derived bounded* if gl.hr$A < \infty$, i.e., the cohomological ranges of all indecomposable objects in $D^b(A)$ have a common upper bound.

Recall that an algebra A is said to be *derived finite* if up to shift and isomorphism there are only finitely many indecomposable objects in $D^b(A)$ (Ref. [8]). Now we can prove Theorem I, another proof of which will be given at the end of this paper (Remark 2).

Theorem 1 *Let A be an algebra. Then the following assertions are equivalent:*

(1) *A is derived bounded;*
(2) *A is derived finite;*
(3) *A is piecewise hereditary of Dynkin type.*

Proof (1) ⇒ (3): By assumption, gl.hr$A < \infty$. Thus gl.hw$A < \infty$. It follows from Corollary 4 that A is piecewise hereditary. By [19, Theorem 3.1], $D^b(A) \simeq D^b(kQ)$ for some finite connected quiver Q without oriented cycles, or $D^b(A) \simeq D^b(\text{coh}\mathbb{X})$ for some weighted projective line $\mathbb{X}$. In the first case, by Corollary 1 we have gl.hr $kQ < \infty$. Hence

Q is a Dynkin quiver. In the second case, by [15, Theorem 3.2], $D^b(A)$ is triangle equivalent to $D^b(C)$ for a canonical algebra C. Since C is representation-infinite, gl.hr$C = \infty$. By Corollary 1, we have gl.hr$A = \infty$, which is a contradiction.

(3) $\Rightarrow$ (2): This is well-known [18].

(2) $\Rightarrow$ (1): Trivial. □

3 The Second Brauer-Thrall Type Theorem

3.1 Strongly Derived Unbounded Algebras

Recall that the *cohomology dimension vector* of a complex $X^\bullet \in D^b(A)$ is $\mathbf{d}(X^\bullet) := (\dim H^n(X^\bullet))_{n\in\mathbb{Z}} \in \mathbb{N}^{(\mathbb{Z})}$. An algebra A is said to be *derived discrete* if for any $\mathbf{d} \in \mathbb{N}^{(\mathbb{Z})}$, up to isomorphism, there are only finitely many indecomposable objects in $D^b(A)$ of cohomology dimension vector $\mathbf{d}$ (Ref. [32]). It is easy to see that an algebra A is derived discrete if and only if for any $r \in \mathbb{N}$, up to shift and isomorphism, there are only finitely many indecomposable objects in $D^b(A)$ of cohomological range r.

Definition 5 An algebra A is said to be *strongly derived unbounded* if there is an (strictly) increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ such that for each r_i, up to shift and isomorphism, there are infinitely many indecomposable objects in $D^b(A)$ of cohomological range r_i.

Note that all representation-infinite algebras are strongly unbounded due to the truth of Brauer-Thrall conjecture II, thus strongly derived unbounded. Moreover, it is impossible that an algebra is both derived discrete and strongly derived unbounded.

Now we show that derived equivalences preserve strongly derived unboundedness.

Proposition 4 *Let two algebras A and B be derived equivalent. Then A is strongly derived unbounded if and only if so is B.*

Proof Let ${}_AT^\bullet_B$ be a two-sided tilting complex such that $F = -\otimes^L_A T^\bullet_B : D^b(A) \to D^b(B)$ is a derived equivalence. Assume that A is strongly derived unbounded. Then there exist an increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and infinitely many indecomposable objects $\{X^\bullet_{ij} \in D^b(A) \mid i, j \in \mathbb{N}\}$ which are pairwise different up to shift and isomorphism such that $\mathrm{hr}(X^\bullet_{ij}) = r_i$ for all $j \in \mathbb{N}$. It follows from Proposition 1 (3) that there exist two positive integers N and N', such that for any $X^\bullet_{ij}$,

$$\frac{1}{N'} \cdot \mathrm{hr}(X^\bullet_{ij}) \le \mathrm{hr}(F(X^\bullet_{ij})) \le N \cdot \mathrm{hr}(X^\bullet_{ij}).$$

In order to show that B is strongly derived unbounded, we shall find inductively an increasing sequence $\{r'_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and infinitely many indecomposable objects $\{Y^\bullet_{ij} \in D^b(B) \mid i, j \in \mathbb{N}\}$ which are pairwise different up to shift and isomorphism such that $\mathrm{hr}(Y^\bullet_{ij}) = r'_i$ for all $j \in \mathbb{N}$. For $i = 1$, we have $0 < \mathrm{hr}(F(X^\bullet_{1j})) \le N \cdot \mathrm{hr}(X^\bullet_{1j}) = N \cdot r_1$. Since $F(X^\bullet_{1j})$, $j \in \mathbb{N}$, are also pairwise different indecomposable objects up to shift and isomorphism, we can choose $0 < r'_1 \le Nr_1$ and infinitely many indecomposable objects $\{Y^\bullet_{1j} \mid j \in \mathbb{N}\} \subseteq \{F(X^\bullet_{1j}) \mid j \in \mathbb{N}\}$ which are pairwise different up to shift and isomorphism such that $\mathrm{hr}(Y^\bullet_{1j}) = r'_1$ for all $j \in \mathbb{N}$. Assume that we have found r'_i. We choose some r_l with $r_l > N' \cdot r'_i$. Since

$$r'_i < \frac{1}{N'} \cdot r_l = \frac{1}{N'} \cdot \mathrm{hr}(X^\bullet_{lj}) \le \mathrm{hr}(F(X^\bullet_{lj})) \le N \cdot \mathrm{hr}(X^\bullet_{lj}) = N \cdot r_l,$$

we can choose $r'_i < r'_{i+1} \le N \cdot r_l$ and infinitely many indecomposable objects $\{Y^\bullet_{i+1,j} \mid j \in \mathbb{N}\} \subseteq \{F(X^\bullet_{lj}) \mid j \in \mathbb{N}\}$ which are pairwise different up to shift and isomorphism such that $\mathrm{hr}(Y^\bullet_{i+1,j}) = r'_{i+1}$ for all $j \in \mathbb{N}$. □

3.2 Cleaving Functors

Cleaving functors were introduced in [7] as a tool for proving that certain algebras are representation-infinite. In this part, we will observe the relations between cleaving functors and strongly derived unboundedness of algebras.

In order to use cleaving functors, one needs to view basic finite-dimensional algebras or bound quiver algebras as finite spectroids. Recall that a *locally bounded spectroid* [14] (=locally bounded category [11]) is a small k-linear category A satisfying:

(1) different objects in A are not isomorphic;
(2) the endomorphism algebra $A(a, a)$ is local for all $a \in A$;
(3) $\sum_{x \in A} \dim A(a, x) < \infty$ and $\sum_{x \in A} \dim A(x, a) < \infty$ for all $a \in A$.

A *finite spectroid* is a locally bounded spectroid with finitely many objects. Let A be a finite spectroid. A *right A-module M* is just a covariant k-functor from A to the category of k-vector spaces. The *dimension* of M is $\dim M := \sum_{a \in A} \dim M(a)$. Denote by $\mathrm{mod} A$ the category of finite-dimensional right A-modules. The indecomposable projective A-modules are $P_a = A(a, -)$ and indecomposable injective A-modules are $I_a = DA(-, a)$ for all $a \in A$, where $D = \mathrm{Hom}_k(-, k)$. A bound quiver algebra kQ/I with Q a finite quiver and I an admissible ideal can be viewed as a finite spectroid A by taking the vertices in Q_0 as objects and the k-linear combinations of paths in kQ/I as morphisms. Conversely, a finite spectroid A admits a presentation $kQ/I \xrightarrow{\sim} A$ for a finite quiver Q and an admissible ideal I (Ref. [14, Chapter 8]). In these cases, kQ/I and A have equivalent (finite-dimensional) module categories. Throughout this section, we do not differentiate the terminologies "(basic finite-dimensional) algebra", "bound quiver algebra" and "finite spectroid". In particular, all the concepts and notations concerning module category defined for a bound quiver algebra make sense for a finite spectroid.

To a k-functor $F : B \to A$ between finite spectroids, we associates a *restriction functor* $F_* : \mathrm{mod} A \to \mathrm{mod} B$, which is given by $F_*(M) = M \circ F$ and exact. The restriction functor F_* admits a left adjoint functor F^*, called the *extension functor*, which sends a projective B-module $B(b, -)$ to a projective A-module $A(Fb, -)$. If $\mathrm{gl.dim} B < \infty$ then F_* extends naturally to a derived functor $F_* : D^b(A) \to D^b(B)$, which has a left adjoint $\mathbf{L}F^* : D^b(B) \to D^b(A)$. Note that $\mathbf{L}F^*$ is the left derived functor associated with F^* and maps $K^b(\mathrm{proj} B)$ into $K^b(\mathrm{proj} A)$. We refer to [33] for the definition of derived functors.

A k-functor $F : B \to A$ between finite spectroids with $\mathrm{gl.dim} B < \infty$ is called a *cleaving functor* [7, 32] if it satisfies the following equivalent conditions:

(1) The linear map $B(b, b') \to A(Fb, Fb')$ associated with F admits a natural retraction for all $b, b' \in B$;

(2) The adjunction morphism $\phi_M : M \to (F_* \circ F^*)(M)$ admits a natural retraction for all $M \in \mathrm{mod}B$;
(3) The adjunction morphism $\Phi_{X^\bullet} : X^\bullet \to (F_* \circ \mathbf{L}F^*)(X^\bullet)$ admits a natural retraction for all $X^\bullet \in D^b(B)$.

Proposition 5 *Let $F : B \to A$ be a cleaving functor between finite spectroids with $gl.dimB < \infty$. Then the following assertions hold:*

(1) *If B is strongly derived unbounded then so is A.*
(2) *If $gl.hlA < \infty$ (resp. $gl.hwA < \infty$, $gl.hrA < \infty$), then $gl.hlB < \infty$ (resp. $gl.hwB < \infty$, $gl.hrB < \infty$).*

Proof

(1) Assume that there exist an increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and indecomposable objects $\{X_{ij}^\bullet \in D^b(B) \mid i, j \in \mathbb{N}\}$ which are pairwise different up to shift and isomorphism such that $\mathrm{hr}(X_{ij}^\bullet) = r_i$ for all $j \in \mathbb{N}$. Since F is a cleaving functor, $X_{ij}^\bullet$ is a direct summand of $(F_* \circ \mathbf{L}F^*)(X_{ij}^\bullet)$. Thus for any $X_{ij}^\bullet$, we can choose an indecomposable direct summand $Y_{ij}^\bullet$ of $\mathbf{L}F^*(X_{ij}^\bullet)$, such that $X_{ij}^\bullet$ is a direct summand of $F_*(Y_{ij}^\bullet)$. Clearly, for any $i \in \mathbb{N}$, the set $\{Y_{ij}^\bullet \mid j \in \mathbb{N}\}$ contains infinitely many elements which are pairwise different up to shift and isomorphism. To prove A is strongly derived unbounded, by the proof of Proposition 4, it is enough to show that there exist N', $N \in \mathbb{N}$ such that for any $X_{ij}^\bullet$, the inequalities $\frac{1}{N'} \cdot \mathrm{hr}(X_{ij}^\bullet) \le \mathrm{hr}(Y_{ij}^\bullet) \le N \cdot \mathrm{hr}(X_{ij}^\bullet)$ hold.
For any $a \in A$, we have

$$\begin{aligned} H^m(\mathbf{L}F^*(X_{ij}^\bullet))(a) &\cong \mathrm{Hom}_{D^b(A)}(\mathbf{L}F^*(X_{ij}^\bullet), I_a[m]) \\ &\cong \mathrm{Hom}_{D^b(B)}(X_{ij}^\bullet, F_*(I_a)[m]) \\ &\cong H^m(\mathrm{RHom}_B(X_{ij}^\bullet, F_*(I_a))). \end{aligned}$$

Since $\mathrm{gl.dim}B < \infty$, the B-module $F_*(I_a)$ admits a minimal injective resolution

$$0 \to F_*(I_a) \to E_a^0 \to E_a^1 \to \cdots \to E_a^{r_a} \to 0,$$

and there is a bounded converging spectral sequence

$$\mathrm{Ext}_B^p(H^{-q}(X_{ij}^\bullet), F_*(I_a)) \Rightarrow H^{p+q}(\mathrm{RHom}_B(X_{ij}^\bullet, F_*(I_a))).$$

Thus $\mathrm{hw}(Y_{ij}^\bullet) \le \mathrm{hw}(\mathbf{L}F^*(X_{ij}^\bullet)) \le \mathrm{hw}(X_{ij}^\bullet) + \mathrm{gl.dim}B$, and

$$\begin{aligned} \dim H^m(Y_{ij}^\bullet) &= \sum_{a \in A} \dim H^m(Y_{ij}^\bullet)(a) \\ &\le \sum_{a \in A} \dim H^m(\mathbf{L}F^*(X_{ij}^\bullet))(a) \\ &= \sum_{a \in A} \dim H^m(\mathrm{RHom}_B(X_{ij}^\bullet, F_*(I_a))) \\ &\le \sum_{a \in A} \sum_{p+q=m} \dim \mathrm{Ext}_B^p(H^{-q}(X_{ij}^\bullet), F_*(I_a)) \\ &\le \sum_{a \in A} \sum_{p=0}^{r_a} \dim H^{p-m}(X_{ij}^\bullet) \cdot \dim E_a^p \\ &\le \sum_{a \in A} \mathrm{hl}(X_{ij}^\bullet) \cdot (r_a + 1) \cdot \max_{0 \le p \le r_a} \left\{\dim E_a^p\right\} \\ &\le n_0(A) \cdot \mathrm{hl}(X_{ij}^\bullet) \cdot (\mathrm{gl.dim}B + 1) \cdot \max_{a \in A,\, 0 \le p \le r_a} \left\{\dim E_a^p\right\}, \end{aligned}$$

where $n_0(A)$ denotes the number of objects in A.

Set $N_0 = n_0(A) \cdot (\text{gl.dim}B + 1) \cdot \max_{a \in A,\ 1 \le p \le r_a} \{\dim E_a^p\}$. Then $\text{hl}(Y_{ij}^\bullet) \le N_0 \cdot \text{hl}(X_{ij}^\bullet)$ and

$$\begin{aligned} \text{hr}(Y_{ij}^\bullet) &= \text{hw}(Y_{ij}^\bullet) \cdot \text{hl}(Y_{ij}^\bullet) \\ &\le (\text{hw}(X_{ij}^\bullet) + \text{gl.dim}B) \cdot N_0 \cdot \text{hl}(X_{ij}^\bullet) \\ &\le N_0 \cdot (\text{gl.dim}B + 1) \cdot \text{hr}(X_{ij}^\bullet). \end{aligned}$$

So $N := N_0 \cdot (\text{gl.dim}B + 1)$ is as required.

Assume the indecomposable projective B-module $Q_b = B(b, -)$ and indecomposable projective A-module $P_a = A(a, -)$ for all $b \in B$ and $a \in A$. Then

$$\begin{aligned} \dim H^m(X_{ij}^\bullet) &\le \dim H^m(F_*(Y_{ij}^\bullet)) \\ &= \sum_{b \in B} \dim \text{Hom}_{D^b(B)}(Q_b, F_*(Y_{ij}^\bullet)[m]) \\ &= \sum_{b \in B} \dim \text{Hom}_{D^b(A)}(\mathbf{L}F^*(Q_b), Y_{ij}^\bullet[m]) \\ &= \sum_{b \in B} \dim \text{Hom}_{D^b(A)}(F^*(Q_b), Y_{ij}^\bullet[m]) \\ &= \sum_{b \in B} \dim \text{Hom}_{D^b(A)}(P_{F(b)}, Y_{ij}^\bullet[m]) \\ &\le n_0(B) \cdot \dim \text{Hom}_{D^b(A)}(A, Y_{ij}^\bullet[m]) \\ &= n_0(B) \cdot \dim H^m(Y_{ij}^\bullet) \end{aligned}$$

for all $m \in \mathbb{Z}$, where $n_0(B)$ denotes the number of objects in B. Thus $\text{hl}(X_{ij}^\bullet) \le n_0(B) \cdot \text{hl}(Y_{ij}^\bullet)$, $\text{hw}(X_{ij}^\bullet) \le \text{hw}(Y_{ij}^\bullet)$, and $\text{hr}(Y_{ij}^\bullet) \ge \frac{1}{n_0(B)} \cdot \text{hr}(X_{ij}^\bullet)$. So $N' := n_0(B)$ is as required.

(2) It can be read off from the proof of (1) that for any indecomposable object $X^\bullet \in D^b(B)$, there exists an indecomposable object $Y^\bullet \in D^b(A)$ such that $\dim H^m(X^\bullet) \le n_0(B) \cdot \dim H^m(Y^\bullet)$ for all $m \in \mathbb{Z}$. Then the statement follows.

□

3.3 Simply Connected Algebras

To a tilting A-module T_A, one can associate a torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in mod$A$, and a torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in modEnd$_A(T)$. The Brenner-Butler theorem in classical tilting theory establishes the equivalence between $\mathcal{F}(T)$ and $\mathcal{X}(T)$ under the restriction of functor $F = \text{Ext}_A^1(T_A, -)$, and the equivalence between $\mathcal{T}(T)$ and $\mathcal{Y}(T)$ under the restriction of functor $G = \text{Hom}_A(T_A, -)$ (Ref. [22, Theorem (2.1)]). We say a torsion pair $(\mathcal{T}, \mathcal{F})$ in mod$A$ *splits* if any indecomposable M in mod$A$ is either in $\mathcal{T}$ or in $\mathcal{F}$. A tilting A-module T is said to be *separating* if the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ splits. A tilting A-module T is said to be *splitting* if the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ splits. Refer to [3, Chapter VI].

Recall that an algebra A is said to be *triangular* if its quiver Q_A has no oriented cycles. A triangular algebra A is said to be *simply connected* if for any presentation $A \cong kQ/I$, the fundamental group $\Pi_1(Q, I)$ of (Q, I) is trivial [25]. Now we prove Theorem II for simply connected algebras.

Lemma 2 *A simply connected algebra A is either derived discrete or strongly derived unbounded. Moreover, a simply connected algebra A is derived discrete if and only if it is piecewise hereditary of Dynkin type, if and only if gl.hl$A < \infty$.*

Proof According to Corollary 1 and Proposition 4, it is enough to show that a simply connected algebra A is tilting equivalent (thus derived equivalent) to either a hereditary algebra or a representation-infinite algebra. If A is itself hereditary or representation-infinite then we have nothing to do. If A is representation-finite but not hereditary then, by [1, Theorem], there exists a separating but not splitting basic tilting A-module T. Put $A_1 = \mathrm{End}_A(T)$. Then there are more indecomposable modules in modA_1 than in mod$A$, in particular A and A_1 are not isomorphic as algebras. Moreover, A_1 is still simply connected by [4, Corollary] and thus triangular. Since A_1 is a tilted algebra of A, they have the same number of simple modules [22, Corollary (3.1)]. If A_1 is hereditary or representation-infinite then we have nothing to do. If A_1 is representation-finite but not hereditary then there exists a separating but not splitting basic tilting A_1-module, and we can proceed as above repetitively. We claim this process must stop in finite steps, and thus A is tilting equivalent to either a hereditary algebra or a representation-infinite algebra. Indeed, for any $n \in \mathbb{N}$, there are only finitely many (unnecessarily connected) basic representation-finite triangular algebras having n simple modules up to isomorphism (compare with [18, Chapter IV, Lemma 7.3]). We can prove this by induction on n. If $n = 1$, then there exists only one basic triangular algebra up to isomorphism. Assume that it is true for $n - 1$ and B is a basic representation-finite triangular algebra having n simple modules. Then B is a one-point extension of a basic representation-finite triangular algebra with $n-1$ simples, say C, by some C-module $M = \oplus_{i=1}^{r} M_i$ with M_i being indecomposable. Since C is representation-finite, we have $r \leq 3$. Indeed, if $r \geq 4$ then dim $e(\mathrm{rad}B/\mathrm{rad}^2 B)(1-e) =$ dim $M/\mathrm{rad}M \geq 4$, where e is the idempotent of B corresponding to the extension vertex. Thus in the quiver of B there will be at least four arrows starting from the extension vertex, which implies that B is representation-infinite. It is a contradiction. Therefore, the number of the isomorphism classes of basic representation-finite triangular algebras having n simple modules is finite. Furthermore, the tilting process above must stop in finite steps, since representation-finite simply connectedness and the number of simples are invariant under this process. □

3.4 The second Brauer-Thrall Type Theorem

Bekkert and Merklen have classified the indecomposable objects in the derived category of a gentle algebra [8, Theorem 3]. Now we apply their results to prove Theorem II for gentle algebras.

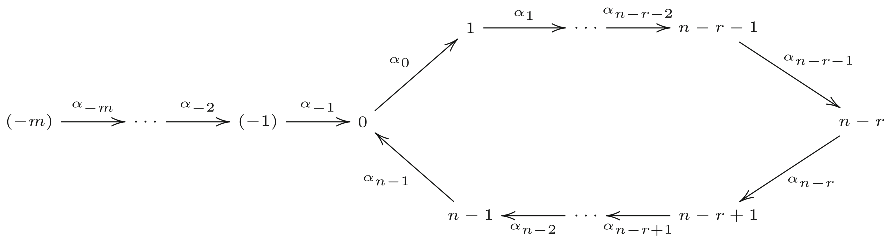
Lemma 3 *A gentle algebra A is either derived discrete or strongly derived unbounded. Moreover, A is derived discrete if and only if gl.hl$A < \infty$.*

Proof It follows from [8, Theorem 4] that a gentle algebra A is derived discrete if and only if A does not contain generalized bands.

If A contains a generalized band w, then one can construct indecomposable complexes $\{P^\bullet_{w,f} \mid f = (x-\lambda)^d \in k[x],\ \lambda \in k^*,\ d > 0\}$ which are pairwise different up to shift and isomorphism such that $P^\bullet_{w,f}$ and $P^\bullet_{w,f'}$ are of the same cohomological range (resp. cohomological length) if and only if $\deg(f) = \deg(f')$ (Ref. [8, Definition 3]). Thus A is strongly derived unbounded and gl.hl$A = \infty$.

If $A = kQ/I$ does not contain generalized bands we shall prove gl.hl$A < \infty$. By Bobiński, Geiss and Skowroński's classification of derived discrete algebras [9, Theorem

A], we know that A is derived equivalent to a gentle algebra $\Lambda(r,n,m)$ with $n \geq r \geq 1$ and $m \geq 0$, which is given by the quiver



with the relations $\alpha_{n-1}\alpha_0, \alpha_{n-2}\alpha_{n-1}, \cdots, \alpha_{n-r}\alpha_{n-r+1}$. According to Corollary 1, it suffices to show that $\mathrm{gl.hl}\Lambda(r,n,m) \leq \dim\Lambda(r,n,m) < \infty$. Note that any generalized string w of $\Lambda(r,n,m)$ must be a sub-generalized string of the following generalized strings or their inverses:

$$(\alpha_i \cdots \alpha_{n-r})[(\alpha_{n-r+1}) \cdots (\alpha_{n-1})(\alpha_0 \cdots \alpha_{n-r})]^p\,(\alpha_{n-r+1}) \cdots (\alpha_{n-1})(\alpha_j \cdots \alpha_{-1})^{-1},$$

with $-m \leq i \leq n-r$, $-m \leq j \leq -1$ and $p \geq 0$. By Bekkert and Merklen's construction of the indecomposable objects in the bounded derived category of a gentle algebra [8, Definition 2 and Theorem 3], every indecomposable projective direct summand of each component of the indecomposable object $P_w^\bullet \in K^b(\mathrm{proj}\Lambda(r,n,m))$ is multiplicity-free, and hence $\mathrm{gl.hl}\Lambda(r,n,m) \leq \dim\Lambda(r,n,m) < \infty$. □

Let A_n^m be the finite spectroid defined by the quiver

$$n \xrightarrow{\alpha_{n-1}} n-1 \xrightarrow{\alpha_{n-2}} \cdots \xrightarrow{\alpha_2} 2 \xrightarrow{\alpha_1} 1\ ,$$

and the admissible ideal generated by all paths of length m.

Lemma 4 *The finite spectroid A_{3m}^m with $m \geq 3$ is strongly derived unbounded and gl.hl$A_{3m}^m = \infty$.*

Proof Let $B = A_{3m}^m$, $w_1 = \alpha_{3m-1}$, $w_2 = \alpha_{3m-2} \cdots \alpha_{2m}$, $w_3 = \alpha_{2m-1} \cdots \alpha_{m+1}$, $w_4 = \alpha_m \cdots \alpha_2$, $w_5 = \alpha_1$, $w_1' = \alpha_{3m-1} \cdots \alpha_{2m+1}$, $w_2' = \alpha_{2m}$, $w_3' = w_3$, $w_4' = \alpha_m$, and $w_5' = \alpha_{m-1} \cdots \alpha_1$. Then we construct a family of complexes $\{P_{\lambda,d}^\bullet \mid \lambda \in k, d \geq 1\}$ by

$$\begin{aligned} P_{\lambda,d}^\bullet := 0 \to P_1^d \xrightarrow{\delta^0} P_m^d \oplus P_2^d \xrightarrow{\delta^1} P_{m+1}^d \oplus P_{m+1}^d \xrightarrow{\delta^2} P_{2m}^d \oplus P_{2m}^d \\ \xrightarrow{\delta^3} P_{2m+1}^d \oplus P_{3m-1}^d \xrightarrow{\delta^4} P_{3m}^d \to 0 \end{aligned}$$

with the differential maps

$$\delta^0 := \begin{pmatrix} P(w_5')\mathbf{I}_d \\ P(w_5)\mathbf{J}_{\lambda,d} \end{pmatrix}, \quad \delta^i := \begin{pmatrix} P(w_{5-i}')\mathbf{I}_d & 0 \\ 0 & P(w_{5-i})\mathbf{I}_d \end{pmatrix}, \quad \text{for } i = 1, 2, 3,$$

and $\delta^4 := (P(w_1')\mathbf{I}_d, P(w_1)\mathbf{I}_d)$. Here $\mathbf{J}_{\lambda,d}$ denotes the upper triangular $d \times d$ Jordan block with eigenvalue $\lambda \in k$, and the map $P(u)$ from $P_{t(u)}$ to $P_{o(u)}$ is the left multiplication by the path u with origin $o(u)$ and terminus $t(u)$. In fact, the complex $P^{\bullet}_{\lambda,d}$ can be illustrated as follows

$$\begin{array}{ccccccccccc}
P_1^d & \xrightarrow{P(w_5')\mathbf{I}_d} & P_m^d & \xrightarrow{P(w_4')\mathbf{I}_d} & P_{m+1}^d & \xrightarrow{P(w_3')\mathbf{I}_d} & P_{2m}^d & \xrightarrow{P(w_2')\mathbf{I}_d} & P_{2m+1}^d & \xrightarrow{P(w_1')\mathbf{I}_d} & P_{3m}^d \\
& \searrow^{P(w_5)\mathbf{J}_{\lambda,d}} & & & & & & & & \nearrow_{P(w_1)\mathbf{I}_d} & \\
& & P_2^d & \xrightarrow{P(w_4)\mathbf{I}_d} & P_{m+1}^d & \xrightarrow[P(w_3)\mathbf{I}_d]{} & P_{2m}^d & \xrightarrow[P(w_2)\mathbf{I}_d]{} & P_{3m-1}^d & &
\end{array}$$

where P_1^d lies in the 0-th component of $P^{\bullet}_{\lambda,d}$.

It is elementary to show that $\mathrm{End}_{C^b(B)}(P^{\bullet}_{\lambda,d})$ is local, i.e., the complex $P^{\bullet}_{\lambda,d}$ is indecomposable, for all $\lambda \in k$ and $d \geq 1$. Indeed, if $f^{\bullet} = (f^i) \in \mathrm{End}_{C^b(B)}(P^{\bullet}_{\lambda,d})$ then $f^i = 0$ unless $0 \leq i \leq 5$. As a k-vector space,

$$\mathrm{Hom}_B(P_i, P_j) \cong e_j B e_i \cong \begin{cases} k, & \text{if } i \leq j < i + m; \\ 0, & \text{otherwise.} \end{cases}$$

Thus each f^i can be expressed as a matrix and by the construction of $P^{\bullet}_{\lambda,d}$, f^2 and f^3 can be written as the same block matrix of form

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}, \; f_{ij} \in k^{d\times d}.$$

Since $m > 2$, we have

$$f^1 = \begin{pmatrix} f^1_{11} & f^1_{12} \\ 0 & f^1_{22} \end{pmatrix}, \; f^4 = \begin{pmatrix} f^4_{11} & 0 \\ f^4_{21} & f^4_{22} \end{pmatrix}, \; f^h_{ij} \in k^{d\times d}.$$

The commutativity of cochain map forces

$$f^1 = f^2 = f^3 = f^4 = \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} \end{pmatrix}.$$

Furthermore, $f^0, f^5 \in k^{d\times d}$ satisfy

$$\begin{pmatrix} \mathbf{I}_d \\ \mathbf{J}_{\lambda,d} \end{pmatrix} f^0 = \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} \end{pmatrix} \begin{pmatrix} \mathbf{I}_d \\ \mathbf{J}_{\lambda,d} \end{pmatrix}$$

and

$$f^5 \begin{pmatrix} \mathbf{I}_d & \mathbf{I}_d \end{pmatrix} = \begin{pmatrix} \mathbf{I}_d & \mathbf{I}_d \end{pmatrix} \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} \end{pmatrix}.$$

Therefore, $f^0 = f_{11} = f_{22} = f^5$ and $\mathbf{J}_{\lambda,d} f^0 = f^0 \mathbf{J}_{\lambda,d}$, and thus f^0 is of the form

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_{d-1} & x_d \\ 0 & x_1 & \cdots & x_{d-2} & x_{d-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x_1 & x_2 \\ 0 & 0 & \cdots & 0 & x_1 \end{pmatrix}, \; x_i \in k.$$

Hence $\mathrm{End}_{C^b(B)}(P^{\bullet}_{\lambda,d})$ is local. Moreover, the complexes $\{P^{\bullet}_{\lambda,d} \mid \lambda \in k, d \geq 1\}$ are pairwise different up to shift and isomorphism by a similar argument on the morphisms between these $P^{\bullet}_{\lambda,d}$'s.

Now it suffices to show that $\mathrm{hr}(P^{\bullet}_{\lambda,d}) = \mathrm{hr}(P^{\bullet}_{\lambda',d'})$ and $\mathrm{hl}(P^{\bullet}_{\lambda,d}) = \mathrm{hl}(P^{\bullet}_{\lambda',d'})$ if and only if $d = d'$, which implies B is strongly derived unbounded and gl.hl$B = \infty$. Indeed, it is clear that $H^i(P^\bullet_{\lambda,d})$ is independent of λ except $i = 0, 1$. Moreover, $H^0(P^\bullet_{\lambda,d}) = 0$ and dim$H^1(P^\bullet_{\lambda,d})$ is independent of λ since δ^0 is injective. Hence, $P^\bullet_{\lambda,d}$'s are of the same cohomological range and cohomological length for a fixed d. Conversely, we have $\mathrm{hw}(P^\bullet_{\lambda,d}) = 5$ due to $H^1(P^\bullet_{\lambda,d}) \neq 0 \neq H^5(P^\bullet_{\lambda,d})$ and $\mathrm{hl}(P^\bullet_{\lambda,d}) = d \cdot \mathrm{hl}(P^\bullet_{\lambda,1})$. Thus, $P^\bullet_{\lambda,d}$'s are of distinct cohomological ranges and cohomological lengths for different d. □

Lemma 5 *If a finite spectroid A is not strongly derived unbounded (resp. A is of finite global cohomological length) then the endomorphism algebra $A(a, a)$ is isomorphic to either k or $k[x]/(x^2)$ for all $a \in A$.*

Proof If A is not strongly derived unbounded (resp. A is of finite global cohomological length) then A is representation-finite. Thus for any $a \in A$, $A(a, a)$ is a uniserial local algebra, and then $A(a, a) \cong k$ or $A(a, a) \cong k[x]/(x^m)$ with $m \geq 2$. Note that the functor $F : A^m_n \to A$ given by $F(i) = a$ and $F(\alpha_j) = x$ is a cleaving functor. If $m \geq 3$ then, by Lemma 4, A^m_{3m} is strongly derived unbounded and gl.hl$A^m_{3m} = \infty$. It follows from Proposition 5 that A is strongly derived unbounded and gl.hl$A = \infty$, which is a contradiction. □

Now we can prove Theorem II for all algebras.

Theorem 2 *A finite spectroid is either derived discrete or strongly derived unbounded.*

Proof Assume that a finite spectroid A is not strongly derived unbounded. Then A is representation-finite. It follows from Lemma 5 that the endomorphism algebra $A(a, a)$ is isomorphic to either k or $k[x]/(x^2)$ for all $a \in A$. Thus A does not contain Riedtmann contours, and hence it is standard [7, Section 9].

If A is simply connected then A is derived discrete by Lemma 2. If A is not simply connected then A admits a Galois covering $\pi : \tilde{A} \to A$ with non-trivial free Galois group G such that $\tilde{A}$ is a simply connected locally bounded spectroid [12], hence the filtered union of its connected convex finite full subspectroids [12, 13]. Any connected convex finite full subspectroid B of $\tilde{A}$ is simply connected, thus gl.dim$B < \infty$. Note that the composition of the embedding functor $B \hookrightarrow \tilde{A}$ and the covering functor π is a cleaving functor. By Proposition 5, B is not strongly derived unbounded. It follows from Lemma 2 that B is piecewise hereditary of Dynkin type. By the same argument as that in the proof of [32, Lemma 4.4], we obtain B is piecewise hereditary of type $\mathbb{A}$. Thus $\tilde{A}$ admits a presentation given by a gentle quiver (Q, I) (Ref. [2, Theorem]), and so does A. Therefore, A is derived discrete by Lemma 3. □

Next we show that derived discrete algebras can be characterized as the algebras of finite global cohomological length. Moreover, we summarize in the following proposition all previous results on global finiteness of the homological invariants introduced in this paper.

Proposition 6 *Let A be a finite spectroid. The following assertions hold:*

(1) gl.hl$A < \infty$ *if and only if A is derived discrete;*
(2) gl.hw$A < \infty$ *if and only if A is piecewise hereditary;*
(3) gl.hr$A < \infty$ *if and only if A is piecewise hereditary of Dynkin type.*

Proof If A is derived discrete then by Vossieck's classification of derived discrete algebras [32, Theorem], A is either piecewise hereditary of Dynkin type or derived equivalent to some gentle algebras without generalized bands. In the case A is piecewise hereditary of Dynkin type, by Corollary 1, we have gl.hl$A < \infty$. In the other case, by Lemma 3, we have gl.hl$A < \infty$.

Conversely, it is enough to repeat the proof of Theorem 4 and replace the phrase "not strongly derived unbounded" with "of finite global cohomological length".

The statements (2) and (3) are actually Corollary 4 and Theorem 3 respectively. □

Remark 2 By Proposition 6, we know piecewise hereditary algebras and derived discrete algebras can be characterized as the algebras of finite global cohomological width and the algebras of finite global cohomological length respectively, which provides another proof of the first Brauer-Thrall type theorem for derived category. Indeed, an algebra A satisfies gl.hr$A < \infty$ if and only if both gl.hw$A < \infty$ and gl.hl$A < \infty$, if and only if A is both piecewise hereditary and derived discrete, i.e., piecewise hereditary of Dynkin type.

We conclude this paper with a question. In [10], Bongartz proved that for a finite-dimensional algebra A over an algebraically closed field k, there are no gaps in the sequence of dimensions of finite-dimensional indecomposable A-modules. More precisely, if there is an indecomposable A-module of dimension $n \geq 2$ then there is also one of dimension $n - 1$. It is natural to consider the derived version of the above Bongartz's theorem and ask whether there are no gaps in the sequence of cohomological ranges of indecomposable objects in $D^b(A)$.

Question Is there an indecomposable object in $D^b(A)$ of cohomological range $r - 1$ if there is one of cohomological range $r \geq 2$?

Acknowledgments The authors thank Xiao He and Yongyun Qin for many helpful discussions on this topic. They are supported by the National Natural Science Foundation of China (Grant No. 11171325 and 11571341).

References

1. Assem, I.: Separating splitting tilting modules and hereditary algebras. Canad. Math. Bull. **30**, 177–181 (1987)
2. Assem, I., Happel, D.: Generalized tilted algebras of type $\mathbb{A}_n$. Comm. Algebra **9**, 2101–2125 (1981)
3. Assem, I., Simson, D., Skowroński, A.: Elements of the representation theory of associative algebras, 1 : Techniques of representation theory, London Mathematical Society Student Texts, vol. 65. Cambridge University Press, Cambridge (2006)
4. Assem, I., Skowroński, A.: Tilting simply connected algebras. Comm. Algebra **22**, 4611–4619 (1994)
5. Auslander, M.: Representation theory of artin algebras II. Comm. Algebra **2**, 269–310 (1974)
6. Bautista, R.: On algebras of strongly unbounded representation type. Comment. Math. Helv. **60**, 392–399 (1985)
7. Bautista, R., Gabriel, P., Roiter, A.V., Salmerín, L.: Representation-finite algebras and multiplicative bases. Invent. Math. **81**, 217–285 (1985)
8. Bekkert, V., Merklen, H.: Indecomposables in derived categories of gentle algebras. Alg. Rep. Theory **6**, 285–302 (2003)
9. Bobiński, G., Geiss, C., Skowroński, A.: Classification of discrete derived categories. Cent. Eur. J. Math. **2**, 19–49 (2004)
10. Bongartz, K.: Indecomposables live in all smaller lengths. Represent. Theory **17**, 199–225 (2013)
11. Bongartz, K., Gabriel, P.: Covering spaces in representation theory. Invent. Math. **65**, 331–378 (1982)

12. Bretscher, O., Gabriel, P.: The standard form of a representation-finite algebra. Bull. Soc. Math. France **111**, 21–40 (1983)
13. Gabriel, P.: The universal cover of a representation-finite algebra. Lect. Notes Math. **903**, 68–105 (1981)
14. Gabriel, P., Roiter, A.V.: Representations of finite-dimensional algebras. Springer-Verlag, Berlin, New York (1997)
15. Geigle, W., Lenzing, H.: A class of weighted projective curves arising in representation theory of fnite-dimensional algebras. Lect. Notes Math. **1273**, 265–297 (1987)
16. Geiss, C., Krause, H.: On the notion of derived tameness. J. Algebra Appl. **1**, 133–157 (2002)
17. Happel, D.: On the derived category of a finite-dimensional algebra. Comment. Math. Helv. **62**, 339–389 (1987)
18. Happel, D.: Triangulated categories in the representation theory of finite dimensional algebras. London Math. Soc. Lect. Notes Ser., 119 (1988)
19. Happel, D.: A characterization of hereditary categories with tilting object. Invent. Math. **144**, 381–398 (2001)
20. Happel, D., Reiten, I.: Directing objects in hereditary categories. Contem. Math. **229**, 169–179 (1998)
21. Happel, D., Reiten, I., Smalø, S.: Piecewise hereditary algebras. Arch. Math. **66**, 182–186 (1996)
22. Happel, D., Ringel, C.M.: Tilted algebras. Trans. Amer. Math. Soc. **274**, 399–443 (1982)
23. Happel, D., Zacharia, D.: A homological characterization of piecewise hereditary algebras. Math. Z. **260**, 177–185 (2008)
24. Jans, J.P.: On the indecomposable representations of algebras. Ann. Math. **66**, 418–429 (1957)
25. Martínez-Villa, R., de la Peña, J.A.: The universal cover of a quiver with relations. J. Pure Appl. Algebra **30**, 277–292 (1983)
26. Nazarova, L.A., Roiter, A.V.: Kategorielle matrizen-probleme und die Brauer-Thrall-vermutung. Mitt. Math. Sem. Giessen Heft **115**, 1–153 (1975)
27. Rickard, J.: Derived equivalences as derived functors. J. London Math. Soc. **43**, 37–48 (1991)
28. Ringel, C.M.: Report on the Brauer-Thrall conjectures: Roiter's theorem and the theorem of Nazarova and Roiter. Lect. Notes Math. **831**, 104–136 (1980)
29. Roiter, A.V.: The unboundedness of the dimensions of the indecomposable representations of algebras that have an infinite number of indecomposable representations. Izv. Akad. Nauk SSSR Ser. Math. **32**, 1275–1282 (1968). English transl.: Math. USSR, Izv. **2**, 1223–1230 (1968)
30. Roiter, A.V.: Matrix problems. Proceedings of ICM Helsinki, 319–322 (1978)
31. Skowroński, A.: On algebras with finite strong global dimension. Bull. Polish Acad. Sci. **35**, 539–547 (1987)
32. Vossieck, D.: The algebras with discrete derived category. J. Algebra **243**, 168–176 (2001)
33. Weibel, C.A.: An introduction to homological algebra, Cambridge Studies in Advance Mathematics, vol. 38. Cambridge University Press, Cambridge (1994)