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The Geometric Model of Gentle One-Cycle Algebras

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Abstract

In this paper, we mainly study the geometric model of the derived category of gentle one-cycle algebras provided by Opper, Plamondon and Schroll. We provide a realization of AAG-invariant on the surface, which is slightly different from the realization in their paper, and deduce a standard form of marked surfaces of gentle one-cycle algebras under derived equivalences. As an application, we classify those derived-unique gentle one-cycle algebras.

Keywords Derived equivalence · Derived standard form · Marked ribbon surface · AAG-invariant · Derived-unique algebras

Mathematics Subject Classification 16E35 · 16G60 · 16E05 · 16G20

1 Introduction

Gentle algebras, introduced in 1980's by Assem-Skowroński [7], are a class of important algebras in the representation theory, whose derived categories have been extensively studied in recent years. From the homological aspect, there are many interesting results related to the indecomposables, morphisms, derived equivalences and so on [3,8,11,21,22]. To be more precise, the indecomposable objects in derived categories of gentle algebras and the morphisms between objects have been explicitly described by Bekkert and Merklen [8] and Arnesen et al. [3], respectively. The derived

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equivalence is also an important theme since Richard's work [21], for the reason that many homological invariants preserve under the derived equivalence, such as the rank of Grothendieck group, the finiteness of global dimension and so on, see [14]. Schröer and Zimmermann show that the class of gentle algebras is closed under derived equivalences [22]. Avella-Alaminos and Geiss constructed a combinatorial function ϕ (we call it AAG-invariant) for each gentle algebra and proved that ϕ is a derived invariant. Moreover, ϕ is a perfect invariant to judge the derived equivalence for those gentle one-cycle algebras, i.e., $D^b(A) \simeq D^b(B)$ if and only if $\phi(A) = \phi(B)$, see [1]. A classification of graded gentle algebras with one cycle was established in [20] by constructing the graded tilting complex and using the dg version of Richard's theorem on derived equivalence.

Recently, geometric models for gentle algebras are extensively studied [5,10,12]. In [9,15], a connection between graded gentle algebras and Fukaya categories was established; they proved that collections of formal generators in (partially wrapped) Fukaya categories define graded gentle algebras. Conversely, in [16,17], given a homologically smooth graded gentle algebra A, a graded surface with stops (S_A, M_A, η_A) is constructed, where S_A is an oriented smooth surface with non-empty boundary, M_A is a set of stops on the boundary of A and η_A is a line field on A, such that the partially wrapped Fukaya category $\mathcal{W}(S_A, M_A)$ and derived category $D(A)$ are equivalent. Moreover, in [17], the indecomposable objects and the basis of morphisms between objects in the derived category $D^b(A)$ are described by the curves and the intersection points of the curves, respectively, and also the AAG-invariant. A complete classification of gentle algebras is established by Amiot et al. [4] with a geometric method via winding numbers and Arf invariants, which perfected the classification work of the derived equivalence of gentle algebras. The classification work was also obtained by Opper independently [18]. Moreover, in [18], the authors also provided a new proof of well-known results; namely, gentle algebras are closed under derived equivalences [22] and gentle algebras are Gorenstein algebras [13].

In this paper, we mainly study the gentle one-cycle algebras in terms of the geometric model. To be more precise, we provide a standard form of marked surfaces of gentle one-cycle algebras using the realization of AAG-invariant, and then, we prove that a gentle one-cycle algebra A is derived-unique if and only if it is Kronecker algebra, or the quiver Q of A is an oriented cycle with n vertices and the number of relations equals $n-1$ or n, where *derived-unique algebras* are those algebras for which the notions of derived equivalence and Morita equivalence coincide [19]. The paper is organized as follows: in Sect. 2, we shall introduce some basic notions, and we recall the geometric model of gentle algebras. In Sect. 3, we provide some properties of marked ribbon surfaces, recall the definition of AAG-invariant and realize AAG-invariant by the marked ribbon surface of any gentle one-cycle algebra. In Sect. 4, we provide a standard form of gentle one-cycle algebras by the geometric model. Finally, we characterize the derived-unique gentle one-cycle algebras as an application in Sect. 5.

2 The Geometric Model of Gentle Algebras

Let $A = kQ/I$ be a k-algebra over an algebraically closed field k with $Q = (Q_0, Q_1, s, t)$ a finite quiver, where Q_0 is the set of all vertices, Q_1 is the set of arrows and $s, t : Q_0 \to Q_1$ are the source and target of an arrow in Q_1. For the multiplication $\alpha\beta$ of two arrows $\alpha, \beta \in Q_0$, we define it is the concatenation if $t(\alpha) = s(\beta)$ or zero otherwise, see [6].

We use Q_ℓ to denote the set of all paths of length ℓ. Thus, Q_0 and Q_1 are the sets of all trivial paths and all paths of length 1, respectively; for an arbitrary set X with finite elements, $\sharp X$ is the number of elements in X.

Definition 2.1 A finite-dimensional algebra is *gentle* if it is isomorphic to an algebra which admits a presentation $A = kQ/I$ where

(1) for any $\alpha \in Q_1$, $\sharp\{\beta \in Q_1 | s(\alpha) = t(\beta)\} \leq 2$, $\sharp\{\gamma \in Q_1 | t(\alpha) = s(\gamma)\} \leq 2$;
(2) for any arrow $\alpha \in Q_1$, there is at most one arrow $\gamma \in Q_1$ such that $s(\alpha) = t(\gamma)$ (resp. $t(\alpha) = s(\gamma)$) and $\gamma\alpha \notin I$ (resp. $\alpha\gamma \notin I$);
(3) for any arrow $\alpha \in Q_1$, there is at most one arrow $\beta \in Q_1$ such that $s(\alpha) = t(\beta)$ (resp. $t(\alpha) = s(\beta)$) and $\beta\alpha \in I$ (resp. $\alpha\beta \in I$);
(4) the ideal I of the path algebra kQ is an admissible ideal generated by paths of length 2.

The definition of permitted threads and forbidden threads is originally introduced by Avella–Alaminos and Geiss, which is essential in the definition of the AAG-invariant of gentle algebras [1].

Definition 2.2 Let $A = kQ/I$ be a gentle algebra.

A *non-trivial permitted path* of A is a path $p = \alpha_1 \cdots \alpha_s$ where $\alpha_i\alpha_{i+1} \notin I$ for each $i = 1, 2, \ldots, s-1$, and a *non-trivial permitted thread* of A is a maximal permitted path; a *trivial permitted thread* is a trivial path ε_v over the vertexes v of Q where v satisfies that $\sharp\{\alpha \in Q_1 | s(\alpha) = v\} \leq 1$, $\sharp\{\alpha \in Q_1 | t(\alpha) = v\} \leq 1$, and if $\beta, \gamma \in Q_1$ are arrows such that $t(\beta) = s(\gamma) = v$, then $\beta\gamma \notin I$.

A *non-trivial forbidden path* of A is a path $p = \alpha_1 \cdots \alpha_s$ where $\alpha_i\alpha_j \in I$ for each $i, j = 1, 2, \ldots, s$, and a *non-trivial forbidden thread* of A is a maximal permitted path; a *trivial permitted thread* is a trivial path ε_v over the vertexes v of Q where the v is such that $\sharp\{\alpha \in Q_1 | s(\alpha) = v\} \leq 1$, $\sharp\{\alpha \in Q_1 | t(\alpha) = v\} \leq 1$ and if $\beta, \gamma \in Q_1$ are such that $t(\beta) = s(\gamma)$, then $\beta\gamma \in I$.

Definition 2.3 [17] A *marked ribbon graph* is a sextuple $\Gamma = (V, E, \mathfrak{v}, \mathfrak{e}, \mathfrak{m}, \sigma)$ where

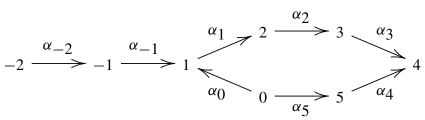
(1) V and E are two finite sets, all elements of V and E are called *vertices* and *half-edges* of Γ, respectively;
(2) $\mathfrak{v} : E \to V$ is such a function: for each half-edge $x \in E$, $\mathfrak{v}(x)$ is the endpoint of x; and $\mathfrak{e} : E \to E$ is such an involution function (i.e., $\mathfrak{e}^2 = 1_E$): for each half-edges $x \in E$, $\mathfrak{e}(x)$ is another half-edges which connect to the x;
(3) $\mathfrak{m} : V \to E$ is a function such that for every $y \in V$, $\mathfrak{m}(y) \in \mathfrak{v}^{-1}(y)$, that is, we choose exactly one half-edge $\mathfrak{m}(y)$ at each vertex.
(4) $\sigma : E \to E$ is a permutation whose orbit correspond to the sets $\mathfrak{v}^{-1}(v)$ for all $v \in V$.

The marked ribbon graphs of a gentle algebra can be defined as follows.

Definition 2.4 [17] Let $A = kQ/I$ be a gentle algebra. Then, the marked ribbon graph $\Gamma_A = (V, E, \mathfrak{v}, \mathfrak{e}, \mathfrak{m}, \sigma)$ of A is defined as follows.

(1) V is a set consisting of all permitted threads of A.
(2) For any permitted thread $w \in V$ and each vertex $v \in Q_0$ which w passes (the case that w passes v multiple times is permitted), there is a half-edge $x \in E$ attached to w (in the anticlockwise order), and the function $\mathfrak{v} : E \to V$ sends x to the $w \in V$. To emphasize the orientation, we define a cyclic permutation σ on the set $\mathfrak{v}^{-1}(w)$ of half-edges w attached by the anticlockwise orientation. For convenience, the half-edge x is denoted by $[v, w]$.
(3) For every vertex $v \in Q_0$, there are exactly two permitted threads passing through it (maybe the same one passes through it two times) and thus two half-edges labeled with v, and the involution function $\mathfrak{e} : E \to E$ sends one to the other.
(4) For each $w \in V$ of A, the vertices of Q which the permitted thread w passes through are ordered from the starting point to the endpoint.
(5) The map $\mathfrak{m} : V \to E$ sends every permitted thread in V to the half-edge labeled by its endpoint.

Example 2.5 Let $A = kQ/I$ be a gentle algebra with Q given by



and $I = \langle \alpha_0\alpha_1, \alpha_1\alpha_2 \rangle$, then its marked ribbon graph Γ_A is shown in Fig. 1. The set of permitted threads $V = \{\alpha_{-2}\alpha_{-1}\alpha_1, \alpha_2\alpha_3, e_5, \alpha_0, e_{-2}, e_{-1}, \alpha_5\alpha_4, e_3\}$. Let $\alpha_{-2}\alpha_{-1}\alpha_1 = y$, then there are four half-edges $[1, y]$, $[-2, y]$, $[-1, y]$ and $[2, y]$ attached to y, such that $\mathfrak{v}([1, y]) = \mathfrak{v}([-2, y]) = \mathfrak{v}([-1, y]) = \mathfrak{v}([2, y]) = y$. The function $\mathfrak{e}$ sends $[1, y]$, $[-2, y]$, $[-1, y]$, and $[2, y]$ to $[1, \alpha_0]$, $[-2, e_{-2}]$, $[-1, e_{-1}]$ and $[2, \alpha_2\alpha_3]$, respectively. The function $\mathfrak{m} : V \to E$ satisfies that $\mathfrak{m}(y) = [2, y]$.

Definition 2.6 [17] Let $\Gamma = (V, E, \mathfrak{v}, \mathfrak{e}, \mathfrak{m})$ be a connected marked ribbon graph. The *marked ribbon surface* of Γ, denoted by (S_Γ, M), is constructed as follows:

(1) For any $y \in V$, P_y is an $2d(y)$-gon with counterclockwise orientation, where $d(y)$ is the number of half-edges attached to y.
(2) Following the cyclic orientation of $y \in V$, label every side of P_y with the half-edge $x \in E$, such that $\mathfrak{v}(x) = y$.
(3) For each $x \in E$, identify the side of P_y labeled x and the side of $P_{\mathfrak{v}(\mathfrak{e}(x))}$ labeled $\mathfrak{e}(x)$, glue P_y and $P_{\mathfrak{v}(\mathfrak{e}(x))}$ to form a surface S_Γ, such that x and $\mathfrak{e}(x)$ are glued together respecting the orientation of the polygons.
(4) For any polygon P_y, we add a marked point on the boundary of the surface S_Γ between the edge labeled by $\mathfrak{m}(y)$ and the edge labeled by $\sigma(\mathfrak{m}(y))$ in counterclockwise orientation. We denote by M the set of all marked points on the surface.

Moreover, for a gentle algebra $A = kQ/I$, the *marked ribbon surface* S_A *of* A is the marked ribbon surface of Γ_A.

Remark 2.7 (1) By [17, Proposition 1.6], there is a unique marked embedding up to homotopy from the ribbon graph Γ_A into S_A, sending the vertices to the marked points. We denote by $\mathfrak{E}$ the set of all the edges of Γ_A in S_A under the embedding. Moreover, $\mathfrak{E}$ forms a *full formal arc system* [15, Section 3.4] of the marked surface S_A, i.e., $\mathfrak{E}$ satisfies:

- Each element in $\mathfrak{E}$ is an arc, namely a continuous function $\gamma : [0, 1] \to S_A$ satisfying
  - both $\gamma(0)$ and $\gamma(1)$ are in M;
  - for any $0 < t < 1$, $\gamma(t)$ is in $S_A \setminus \partial S_A$, where ∂S_A is the boundary of S_A.
- $\mathfrak{E}$ is a collection of pairwise disjoint and non-isotopic arcs, such that $\mathfrak{E}$ cut out S_A into polygons which have exactly one boundary arc not belonging to $\mathfrak{E}$.

(2) There is an equivalent construction of the marked surface S_A in the proof of [16, Theorem 3.2.2] from the ribbon graph Γ_A by replacing vertices of Γ_A with 2-disks and replacing half-edges of Γ_A with thin rectangles.

Example 2.8 Let $A = kQ/I$ be a gentle algebra in Example 2.5 and Γ_A of A be the marked ribbon graph in Fig. 1. Consider the permitted thread $y = \alpha_{-2}\alpha_{-1}\alpha_1 \in V$ of A, we have four half-edges $[i, y]$ for $i \in \{-2, -1, 1, 2\}$ and

$$\begin{aligned} P_{\mathfrak{v}(\mathfrak{e}([1,y]))} &= P_{\mathfrak{v}([1,\alpha_0])} = P_{\alpha_0};\\ P_{\mathfrak{v}(\mathfrak{e}([-2,y]))} &= P_{\mathfrak{v}([-2,e_{-2}])} = P_{e_{-2}};\\ P_{\mathfrak{v}(\mathfrak{e}([-1,y]))} &= P_{\mathfrak{v}([-1,e_{-1}])} = P_{e_{-1}};\\ P_{\mathfrak{v}(\mathfrak{e}([2,y]))} &= P_{\mathfrak{v}([2,\alpha_2\alpha_3])} = P_{\alpha_2\alpha_3}. \end{aligned}$$

All the polygons associated with permitted threads are illustrated as the first figure in Fig. 2. Moreover, gluing all polygons by Definition 2.6, we have the ribbon surface S_A of A, see the figure II in Fig. 2. Note that in these figures, the black curves connecting the marked points are precisely the marked embedding of the ribbon graph Γ_A

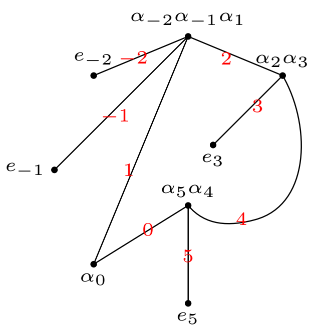


Fig. 1 Ribbon graph in Example 2.5

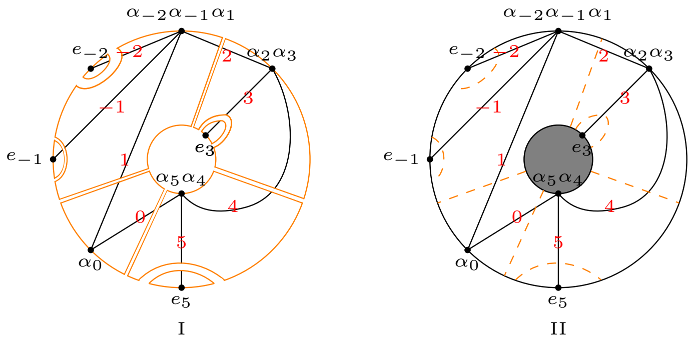


Fig. 2 Marked surface of the gentle algebra in Example 2.5

into the ribbon surface S_A. Here, the marked embedding is the orientation-preserving embedding provided in [17, Proposition 1.6].

By the above construction, for a given gentle algebra A, we have a ribbon graph Γ_A which can be embedded into a marked surface S_A. Conversely, for each marked ribbon surface (S, M) constructed from a ribbon graph Γ_A, one can recover the original gentle algebra A. One way is the definition from [17, Section 1.5] using the lamination. For convenience, we adopt the equivalent definition from [15] using a full formal arc system, see [17, Section 1.7].

Definition 2.9 Let $S_A = (S, M)$ be a marked surface from a gentle algebra A, Γ_A be the ribbon graph embedded into S_A and $\mathfrak{C}$ be the full formal arc system which cuts out S_A into polygons with exactly a single boundary arc not belonging to $\mathfrak{C}$. Then, we associate a quiver $Q = (Q_0, Q_1)$ and relation $I = \langle R \rangle$ to S_A as follows:

(1) the vertices in Q_0 correspond to the arcs in Γ_A;
(2) there is an arrow from i to j in Q_1 whenever there is a polygon Σ in S_A such that Σ has sides i and j with j following i in the such an orientation that the surface lies to the right;
(3) R is the set of such composition ab of arrows $a : i \to j$ and $b : j \to k$ that j follows i and k follows j at different endpoints of j.

Example 2.10 Let $A = kQ/I$ be a gentle algebra in Example 2.5 and Γ_A of A be the marked ribbon graph in Fig. 1. Then, the marked ribbon surface S_A is shown in I of Fig. 3. By the above definition, the set of all arcs of S_A is $\mathfrak{C} = \{Y_1Y_2, Y_1Y_3, Y_1Y_4, Y_4Y_8, Y_5Y_8, Y_6Y_8, Y_6Y_7, Y_1Y_6\}$, the quiver $Q = (Q_0, Q_1)$ of the marked surface is of the form as in II of Fig. 3, and the relation $I = \langle \alpha_0\alpha_1, \alpha_1\alpha_2 \rangle$.

More generally, the following theorem due to Opper et al. [17] shows that the construction as above provides us a way to recover the original gentle algebra.

Theorem 2.11 [17, Proposition 1.21] (Opper–Plamondon–Schroll) *Let A be a gentle algebra with Γ_A and S_A the associated ribbon graph and marked ribbon surface, and A_S be the algebra constructed from S_A. Then, $A \cong A_S$.*

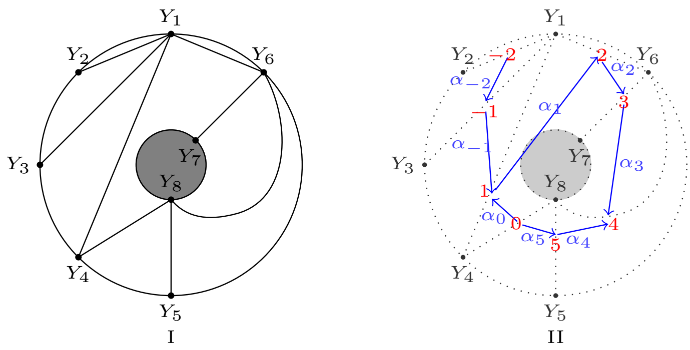


Fig. 3 Associated quiver of the marked surface in Example 2.5

Throughout this paper, a gentle algebra is called to be *gentle one-cycle* if its underlying graph has exactly one cycle. We also fix some notations: For a marked ribbon surface $S_\Gamma = (S_\Gamma, M)$ induced by marked ribbon graph Γ, the set $\mathfrak{E}$ of all arcs formed by the image under the embedding from Γ to S_Γ is a full formal arc system by Remark 2.7 (1) and thus $\mathfrak{E}$ cut S_Γ into polygons $\{P_i \mid i \in I\}$ with precise one boundary edge. We denote by ∂P the boundary edge $P \cap \partial S_\Gamma$ for each polygon P.

Remark 2.12 Let $A = kQ/I$ be a gentle one-cycle algebra, Γ_A be its ribbon graph, and S_A be their marked ribbon surface. Then, by [17, Corollary 1.24], S_A is an annulus. In this case, S_A has two boundary-components, and we can fix one of them as the *inner boundary-component* γ^{in} of S_A and the other one as the *outer boundary-component* γ^{out}. By Remark 2.7, all polygons $(P_i)_{i\in I}$ can be divided into two types

(I) the polygons P_i^{in} $(1 \le i \le m)$ whose boundary edge lie in γ^{in};
(II) the polygons P_j^{out} $(1 \le j \le m')$ whose boundary edge lie in γ^{out}.

We will refer polygons of type (I) or (II) according to the label of the item. Moreover, we denote by $\mathfrak{E}P_i$ the set $\{s \in \mathfrak{E} \mid s \text{ is an edge of } P_i\}$. In the next section, $(\sharp\mathfrak{E}P_i^{\text{in}})_{1\le i\le m}$ and $(\sharp\mathfrak{E}P_j^{\text{out}})_{1\le j\le m'}$ are important values for computing the AAG-invariants of gentle one-cycle algebras.

Example 2.13 Let $A = kQ/I$ be the gentle algebra in Example 2.5, its marked ribbon surface S_A is shown in Example 2.8. By Remark 2.12, we have two polygons P_1^{in} and P_2^{in} of the type (I), and others are of the type (II), see Fig. 4. Moreover, we have $\mathfrak{E}P_1^{\text{in}} = \{Y_1Y_4, Y_4Y_8, Y_7Y_6, Y_6Y_1\}$, $\mathfrak{E}P_2^{\text{in}} = \{Y_8Y_6, Y_6Y_7\}$ and $\sharp\mathfrak{E}P_1^{\text{in}} = 4$, $\sharp\mathfrak{E}P_2^{\text{in}} = 2$.

3 The AAG-Invariants of Gentle Algebras

In this section, we recall the definition of AAG-invariant for gentle algebras [1, Section 3] and then realize the AAG-invariant of gentle one-cycle algebras in its marked ribbon surface.

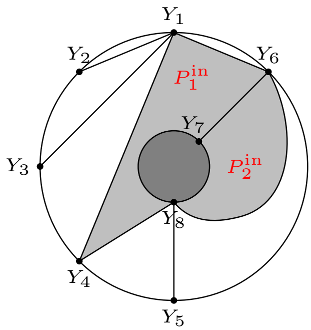


Fig. 4 Type of the polygons of the gentle algebra in Example 2.5

Definition 3.1 Let $A = kQ/I$ be a gentle algebra, its AAG-invariant is a function $\phi_A : \mathbb{N}^2 \to \mathbb{N}$, denoted by $[(b_1, c_1), \ldots, (b_m, c_m)]$, such that the set $\{(b_i, c_i) | 1 \le i \le m\}$ is the support of the function and (b_i, c_i) is written $\phi_A(b_i, c_i)$ times in the multiset $[(b_1, c_1), \ldots, (b_m, c_m)]$, where the sequence $\{(b_i, c_i)\}_{i=1}^m$ is defined as follows:

Step 1

(i) Let H_0 be a permitted thread of A.

(ii) If $H_i = \alpha_1 \cdots \alpha_r$ is defined, consider the forbidden thread $\Pi_i = \beta_1 \cdots \beta_s$ which ends at $e(H_i)$ and satisfies $\alpha_r \ne \beta_s$. (If $H_i = \varepsilon_q$ is a trivial permitted thread on the vertex $q \in Q_0$ which is sink or source, then consider the trivial forbidden thread $\Pi_i = \varepsilon_q$.)

(iii) Let $H_{i+1} = \alpha'_1 \cdots \alpha'_t$ $(t \in \mathbb{N})$ be the permitted thread which starts at $s(\Pi_i)$ and such that $\beta_1 \ne \alpha'_1$.

This process stops when $H_b = H_0$ for some $b \in \mathbb{N}$. Let c be the total length of $\Pi_1, \ldots, \Pi_b$, then we obtain a pair (b, c).

Step 2 Repeat the Step 1 until all permitted threads appear.

Step 3 If there are oriented cycles in which every two consecutive arrows form a relation, then we add a pair $(0, l)$ for each those cycles, where l is the length of the oriented cycle.

Moreover, we do not differentiate the function ϕ_A and the multiset $[(b_1, c_1), \ldots, (b_m, c_m)]$ as in [1].

The following proposition originally provided in [1] implies that the AAG-invariant provides us a perfect way to judge the derived equivalences of gentle one-cycle algebras.

Proposition 3.2 *The number of arrows, the number of cycles and the AAG-invariant are invariant under derived equivalences for gentle algebras. Moreover, if A and B are both gentle one-cycle algebras, then $D^b(A) \simeq D^b(A')$ if and only if $\phi_A = \phi_{A'}$.* □

For a gentle one-cycle algebra $A = kQ/I$, one can calculate its AAG-invariant by its marked ribbon surface by the following theorem which is essentially equivalent to [17, Theorem 6.1] but in a different form more or less.

Theorem 3.3 *Let $A = kQ/I$ be a gentle one-cycle algebra with $n = \sharp Q_0$, $\Gamma_A = (V, E, \mathfrak{v}, \mathfrak{e}, \mathfrak{m}, \sigma)$ and $S_A = (S_A, M)$ be its marked ribbon graph and marked ribbon surface, respectively, and there be m polygons $P_1, P_2, \dots, P_m$ of the type (I) in Remark 2.12. Then, $\phi_A = [(n-b, n-c), (b, c)]$, where b equals the number of marked points on the γ^{in} and $c = \sum_{i=1}^{m} \sharp\mathfrak{C}P_i - b$.*

Proof If A has a oriented cycle with full relations, i.e., any two consecutive arrows form a relation on the cycle. We assume that all arrows on the cycle are clockwise; then, there exists a unique polygon P in class (I) given by Remark 2.12 in S_A such that the second integer pair of ϕ_A is $(0, \sharp\mathfrak{C}P)$ obviously.

Now suppose that A has no full relations cycle, and we calculate the second integer pair of ϕ_A. Let Ψ be the one-to-one correspondence from Q_0 to the full formal arc system of S_A. We assume that H_0 is a permitted thread of A with $t(H_0) = v$, we can find a forbidden thread $\Pi_0 = \alpha_1 \cdots \alpha_t$ such that $s(\alpha_1) = u$, $t(\alpha_i) = w_i$, and $t(\alpha_t) = v$. Since A is a gentle one-cycle algebra, by Remark 2.12, the marked ribbon surface S_A is an annulus. Then, there is a unique polygon P_0 whose edges correspond to $\{u, w_1, \dots, w_{t-1}, v\}$ such that the ending points of $\Psi(w_i)$ and $\Psi(u)$ are the same boundary-component of S_A, and the starting point of $\Psi(u)$ and the ending point of $\Psi(v)$ are on the other boundary-components. Thus, $\sharp\mathfrak{C}P_0 = t + 1 = l(\Pi_0) + 1$, where $l(\Pi_0)$ is the length of Π_0. Without loss of generality, we assume that the polygon P_0 belongs to the class (I) in Remark 2.12; then, the ending points of $\Psi(w_i)$ and $\Psi(u)$ are on the outer boundary component γ^{out} of S_A. By Definition 3.1, we obtain two sequences $\{H_j\}_{0\le j\le m}$ and $\{\Pi_j\}_{0\le j\le m}$, such that each permitted thread H_j corresponds to a marked point on the inner boundary-component of S_A, and each Π_j corresponds to a polygon P_j which belong to class (I) with $\sharp\mathfrak{C}P_j = l(\Pi_j) + 1$. Therefore,

$$\sum_{j=1}^{m} \sharp\mathfrak{C}P_j = c + m,$$

where m is the number of inner polygons, which is precisely the number of marked points on γ^{in} and c is the total length of permitted threads $\{\Pi_j\}_{0\le j\le m}$. Thus, $(b, c) = (m, \sum_j \sharp\mathfrak{C}P_j - m)$ is the second integer pair of ϕ_A. With a similar argument, we can calculate the first integer pair $(b', c') = (m', \sum_j \sharp\mathfrak{C}P'_j - m')$, where any P'_j is a polygon of the type (II). Moreover, $b + b'$ is the number of total permitted threads of A, and $c + c'$ equals double the number of arcs minus $b + b'$ since each arc is an edge of an inner polygon and an outer one. Therefore, $b + b' = n$ and $c + c' = 2n - (b + b') = n$. □

Let $A = kQ/I$ be an arbitrary gentle one-cycle algebra, and $\phi_A = [(b_1, c_1), (b_2, c_2)]$ be its AAG-invariant. In this paper, we always fix the order of pairs of ϕ_A such that b_2 is the number of marked points in the inner boundary-component of S_A. For two gentle one-cycle algebras A and A', we can compute $\phi_A = [(b_1, c_1), (b_2, c_2)]$ and $\phi_{A'} = [(b'_1, c'_1), (b'_2, c'_2)]$ by Theorem 3.3, and define *the drop of AAG-invariant* $\Delta\phi_{A,A'} := \phi_{A'} - \phi_A = [(b'_1 - b_1, c'_1 - c_1), (b'_2 - b_2, c'_2 - c_2)]$. Moreover, for n gentle one-cycle algebras $A_1, A_2, \ldots, A_n$, we define the drop $\Delta\phi_{A_1,A_n} = \sum_{i=1}^{n-1} \Delta\phi_{A_i,A_{i+1}}$, and $\Delta\phi_{A_1,A_n} = [(0,0),(0,0)]$ yields $D^b(A_1) \simeq D^b(A_n)$ by Proposition 3.2.

Example 3.4 Let $A = kQ/I$ be the gentle algebra defined in Example 2.5. Then, its marked ribbon surface S_A is as shown in Example 2.8. Following the notation in Examples 2.10 and 2.13, we obtain the set of all polygons of the type (I) is $\{P_1^{\text{in}} = Y_1Y_4Y_8Y_7Y_6Y_1, P_2^{\text{in}} = Y_8Y_6Y_7Y_8\}$. The second pair (b, c) of AAG-invariant ϕ_A satisfies that $b = 2$ and $c = \sharp\mathfrak{E}P_1^{\text{in}} + \sharp\mathfrak{E}P_2^{\text{in}} - b = 4 + 2 - 2 = 4$; thus, $\phi_A = [(8-2, 8-4), (2,4)] = [(6,4),(2,4)]$. Indeed, the first pair $(6, 4)$ can also be calculated by the polygons P_j^{out} of the type (II). Since

$$\begin{aligned} P_1^{\text{out}} &= Y_1Y_2Y_1, \\ P_2^{\text{out}} &= Y_1Y_2Y_3Y_1, \\ P_3^{\text{out}} &= Y_1Y_3Y_4Y_1, \\ P_4^{\text{out}} &= Y_4Y_5Y_8Y_4, \\ P_5^{\text{out}} &= Y_5Y_6Y_8Y_5, \\ P_6^{\text{out}} &= Y_1Y_6Y_1. \end{aligned}$$

Thus, $b = 6$ and $c = \sum_{j=1}^{6} \sharp\mathfrak{E}P_j^{\text{out}} - b = 1+2+2+2+2+1-6 = 4$.

4 The Derived Standard Forms of Gentle One-Cycle Algebras

In this section, we provide a standard form of gentle one-cycle algebra under the derived equivalences. For convenience, we need the following definition.

Definition 4.1 (*Branches*) Let $A = kQ/I$ be a gentle one-cycle algebra. A connected subquiver $\hat{Q}$ is a *branch* of Q if

(1) each arrow $\alpha \in (\hat{Q})_1$ does not lie on the cycle of Q.
(2) $\hat{Q}$ is a maximal subquiver in the sense of (1), i.e., for any connected subquiver Q' such that $\hat{Q} \subsetneq Q' \subseteq Q$, then there is an arrow $\alpha \in Q'$ lying on the cycle.

Remark 4.2 (1) If a gentle one-cycle algebra has no branch, then it is an algebra of the type $\tilde{\mathbb{A}}_n$.
(2) Let $A = kQ/I$ be an arbitrary gentle one-cycle algebra with at least one branch and S_A be the marked ribbon surface of A. Then, there is an arrow α in this branch such that $s(\alpha)$ is a source of Q, or $t(\alpha)$ is a sink of Q, and we call this source or sink of Q is an *end of branch* of A. Moreover, either $s(\alpha)$ or $t(\alpha)$ corresponds to an edge $Y_1Y_2 \in \mathfrak{E}$ of S_A such that Y_1 and Y_2 are on the same boundary-component of S_A, and either the number of edges $\sharp\mathfrak{v}^{-1}(Y_1)$ in $\mathfrak{E}$ connecting to Y_1 or the number $\sharp\mathfrak{v}^{-1}(Y_2)$ connecting to Y_2 is one. If we remove α from Q, then we get an algebra $A' = kQ'/I'$, and the surface $S_{A'}$ can be obtained from S_A by removing the edge Y_1Y_2.

Lemma 4.3 *Let $A = kQ/I$ be a gentle one-cycle algebra with at least one branch. If we remove the arrow α at the end of a branch and obtain a new gentle algebra A', then $\Delta\phi_{A,A'} = [(0,0),(-1,-1)]$ or $[(-1,-1),(0,0)]$.*

Proof By remark 4.2 (2), if we remove such an arrow α, then the change on surface is deleting the edge Y_1Y_2 corresponding to the source or the sink of α. Without loss of generality, we assume that both Y_1 and Y_2 lie on the inner boundary-component γ^{in} of S_A; then, Y_1Y_2 and γ^{in} form a 2-gon P_1 of type (I) in Remark 2.12. Thus, $\sharp\mathfrak{E}P_1 = 1$ by Theorem 3.3, and the second integer pair $(b,c) = (b, 1 + \sum_{i=2}^{n} \sharp\mathfrak{E}P_i - b)$. If one deletes the edge Y_1Y_2 from the surface S_A, then polygon P_1 vanishes; thus, the number of marked points b is decreased by one, and $c' = \sum_{i=2}^{n} \sharp\mathfrak{E}(P_i') - (b-1)$, where $P_i' = P_i$ stays unchanged for all $i \geq 3$ and the number of inner edges of P_2, which is adjacent to P_1, is also decreased by one since the common edge Y_1Y_2 with P_1 vanishes. Then, $c' = c - 1$. The first integer pair of ϕ_A stays unchanged. Therefore, $\Delta\phi_{A,A'} = [(0,0),(-1,-1)]$. With a similar discussion, if both Y_1 and Y_2 are on the γ^{out}, then we have $\Delta\phi_{A,A'} = [(-1,-1),(0,0)]$. □

Lemma 4.4 *Let $A = kQ/I$ be a gentle algebra of the type $\tilde{\mathbb{A}}_n$ with $I \neq 0$, and $A' = kQ/I'$ be a gentle algebra obtained by removing a relation $\alpha\beta$ on the cycle of Q from I, then $\Delta\phi_{A,A'} = [(-1,0),(+1,0)]$ or $[(+1,0),(-1,0)]$.*

Proof Let Γ_A be the marked ribbon graph and S_A be the marked ribbon surface of A. By Definition 2.9, there is a one-to-one correspondence $\Psi : Q_0 \to \mathfrak{E}$. Then, for a relation $\alpha\beta$ on the cycle, the vertices $s(\alpha)$, $t(\alpha) = s(\beta)$ and $t(\beta)$ correspond to three edges Y_1Y_2, Y_2Y_3 and Y_3Y_4 of S_A, such that Y_2, Y_3 are marked points on the same boundary component of S_A, and the edge $\Psi(t(\alpha)) = Y_2Y_3$ forms a 2-gon P_1 with the boundary of S_A. See Fig. 5.

If we remove the relation $\alpha\beta$ from I, then the marked ribbon surface $S_{A'}$ of $A' = kQ/I'$ changes in the following way: one endpoint of $\Psi(t(\alpha))$, for example Y_2, stays on the previous boundary component such that both $\Psi(t(\beta))$ and $\Psi(s(\alpha))$ connect to Y_2, and the other endpoint Y_3 lies on the other boundary component, see Fig. 6.

Now we observe the value of ϕ. Let $\phi_A = [(b_1,c_1),(b_2,c_2)]$, $\phi_{A'} = [(b_1',c_1'),(b_2',c_2')]$. We assume that the marked points Y_2, Y_3 be on the inner boundary component γ^{in} of S_A. For convenience, we denote by P_1^{in} the inner polygon formed by Y_2Y_3. By Theorem 3.3, we have

$$b_2' = b_2 - 1; \quad c_2 = \sum_{i=1}^{m} \sharp\mathfrak{E}P_i^{\text{in}} - b_2; \quad \text{and} \quad c_2' = \sum_{i=2}^{n} \sharp\mathfrak{E}P_i^{\text{in}} - b_2'.$$

Since $c_2 - c_2' = \sharp\mathfrak{E}P_1^{\text{in}} - b_2 + b_2' = 1 + b_2' - b_2 = 0$, we have $\Delta\phi_{A,A'} = [(+1,0),(-1,0)]$. Similarly, if Y_2, Y_3 are on the outer boundary-component γ^{out}, then $\Delta\phi_{A,A'} = [(-1,0),(+1,0)]$. □

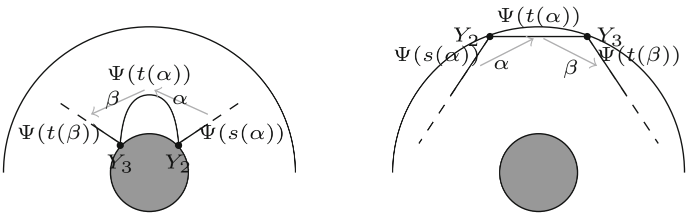


Fig. 5 Two cases with a relation on the cycle

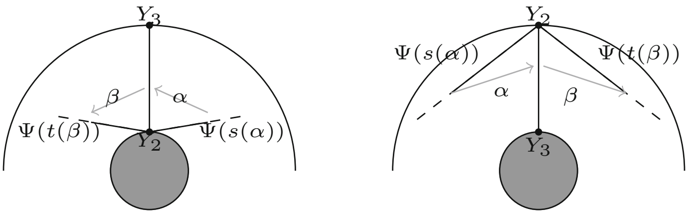


Fig. 6 Change of surfaces when removing the relation

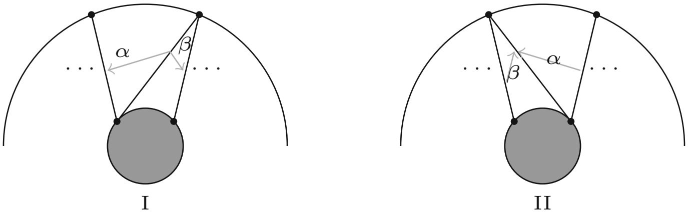
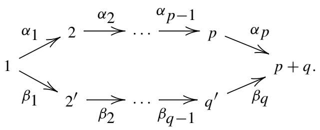


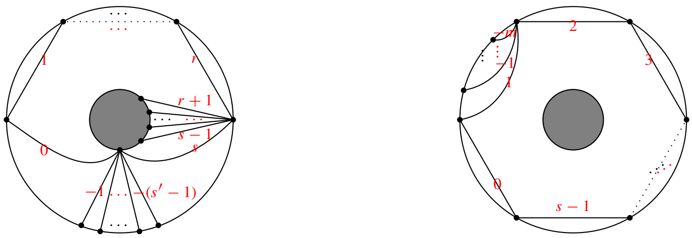
Fig. 7 Marked ribbon surfaces when exchanging the position of two arrows

Lemma 4.5 [7] *Let* $A = kQ$ *be a gentle algebra of type* $\tilde{\mathbb{A}}_n$. *Then,* $D^b(A) \simeq D^b(kQ_{p,q})$ *with* $p, q \geq 1$ *and* $Q_{p,q}$ *of the form*



Proof First of all, the quiver Q is not an oriented cycle since A is finite-dimensional. To prove the lemma, it suffices to show that $D^b(kQ) \simeq D^b(kQ')$ if we exchange the position of two arrows α and β in Q to the quiver Q'. Then, we need to prove the values of ϕ of these two algebras coincide. Suppose that α is an anticlockwise arrow and β is a clockwise one. Then, S_A is the form of the first picture in Fig. 7. If we exchange the position of α and β, then the edge corresponding to $t(\alpha)$ flips on the surface as in the second figure of Fig. 7, i.e., two endpoints move to next marked points along fixed direction on their own boundary components, and the value of ϕ stays unchanged by Theorem 3.3. □

Theorem 4.6 *Let $A = kQ_A/I_A$ be a gentle one-cycle algebra. Then, A is derived equivalent to such a gentle algebra $B = kQ_B/I_B$ that the marked ribbon surface S_B is of the form S_{I} or S_{II}, where*



$S_{\mathrm{I}} = S(0, s, s', r)$
$s \geq 1, s' \geq 0; 0 < r < s$ *if* $s' = 0$

$S_{\mathrm{II}} = S(m, s, 0, s)$
$m \geq 0, s \geq 2$

Proof We divide our theorem into two statements as follows.

(1) If the cycle of A is not oriented, or the cycle of A is oriented and the number of relations on the cycle is less than that of arrows, then S_B is of the form S_{I}.
(2) If the cycle of A is oriented with full relations, i.e., the number of relations on the cycle equals that of arrows, then S_B is of the form S_{II}.

Suppose that both the number of vertices and arrows of A are n in this proof.

(1) For this case, there always exists a vertex on the cycle without relation.

   (i) If Q_A has no branch, then by the proof of Lemma 4.4 and Lemma 4.5, we can remove the same number of clockwise relations and anticlockwise relation such that there only exist relations in one direction, and exchange the position of arrows and relations and then obtain a gentle one-cycle algebra B such that $D^b(A) \simeq D^b(B)$ and the marked ribbon surface S_B is of the form $S_{\mathrm{I}} = S(0, s, s', r)$, where s and s' are the number of clockwise arrows and anticlockwise arrows on the cycle of Q_A, respectively, and r is the number of relations.

   (ii) If Q_A has at least one branch, then we write $A^0 = kQ^0/I^0$ the original algebra $A = kQ_A/I_A$. By removing all branches, we obtain a gentle algebra $A^1 = kQ^1/I^1$ of type $\tilde{\mathbb{A}}_n$. By Lemma 4.3, we have $\Delta\phi_{A^0,A^1} = [(-u, -u), (-v, -v)]$ for some $u, v \in \mathbb{N}$ and Q^1 is a cycle of length $t = n - (u + v)$. Let $A^2 = kQ^2/I^2$ be the algebra obtained by removing all relations of A^1, that is, $Q^2 = Q^1$, $I^2 = 0$, then we have $\Delta\phi_{A^1,A^2} = [(-r_1 + r_2, 0), (r_1 - r_2, 0)]$ by Lemma 4.4 with r_1 and r_2 the number of clockwise relations and anticlockwise relations, respectively. By Lemma 4.5, we exchange the position of arrows and obtain an algebra $A^3 = kQ^3 = kQ_{p,q}$ satisfying $p + q = n - (u + v) = t$ and $\Delta\phi_{A^2,A^3} = 0$. Note that the marked ribbon surface of A^3 is $S(0, p, q, 0)$ and $\phi_{A^3} = [(q, q), (p, p)]$. Now we split the vertex $p + 1$ of $Q^3 = \tilde{\mathbb{A}}_{p,q}$ into two vertices $q' + 1$ and $p' + 1$ and then add a path of the form

$$q'+1 \xrightarrow{a} q'+2 \cdots \longrightarrow \circ \longleftarrow \cdots p'+2 \xleftarrow{b} p'+1$$

to connect $q'+1$ and $p'+1$, such that the number of clockwise arrows is v and the number of anticlockwise arrows is u. We have a path algebra $A^4 = kQ^4 = A_{p+v,q+u}$ which is of the type $\tilde{\mathbb{A}}_n$ with $\phi_{A^4} = \phi_{A_{p+v,q+u}} = [(q+u, q+u), (p+v, p+v)]$ by Theorem 3.3. Thus, $\Delta\phi_{A^3,A^4} = \phi_{A^4} - \phi_{A^3} = [(u,u),(v,v)]$. Without loss of generality, we assume that $r_1 > r_2$. We construct $B = A_5 = kQ^5/I^5$ with $Q^5 = Q^4$ and $I^5 = \langle \alpha_1\alpha_2, \cdots, \alpha_{r_1-r_2}\alpha_{r_1-r_2+1}\rangle$ and $\Delta\phi_{A^4,B} = [(r_1-r_2, 0), (-(r_1-r_2), 0)]$ by Lemma 4.4. Hence,

$$\begin{aligned}\Delta\phi_{A,B} = \sum_{i=1}^{5} \Delta\phi_{A_{i-1},A_i} &= [(-u,-u),(-v,-v)] + [(-r_1+r_2, 0), (r_1-r_2, 0)] \\ &+ 0 + [(u,u),(v,v)] + [(r_1-r_2,0),(-(r_1-r_2),0)] = 0 \qquad (4.1)\end{aligned}$$

Therefore, we finally get a gentle one-cycle algebra B such that $D^b(A) \simeq D^b(B)$ and the marked ribbon surface S_B is of the form S_{I}.

(2) If the cycle of A is oriented with full relations, then we suppose that all arrows on the cycle are clockwise. If, moreover, A has no branch, $S_A = S(0,n,0,n)$ is of the form S_{II} such that $m=0$. If else, by removing all branches, we obtain an algebra $A^1 = kQ^1/I^1$ of $\tilde{\mathbb{A}}_t$-type with Q^1 an oriented cycle with full relations, and $\Delta\phi_{A,A^1} = [(-u,-u),(0,0)]$. Let $B = kQ_B/I_B$ be a gentle one-cycle algebra with one branch of the following form on the cycle Q_1 with full relations

$$1 \longrightarrow \cdots \longrightarrow u \longrightarrow u+1,$$

then $\Delta\phi_{A^1,B} = [(u,u),(0,0)]$ by Lemma 4.3, Since $\Delta\phi_{A,B} = \Delta\phi_{A,A^1} + \Delta\phi_{A^1,B} = 0$, we have $D^b(A) \simeq D^b(B)$, and in this case, S_B is of the form S_{II} with $m > 0$. □

The above theorem provides a standard form of marked surfaces of gentle one-cycle algebras up to derived equivalence. To be more precise, if A is a gentle one-cycle algebra derived equivalent to such a gentle algebra that the marked ribbon surface S is of the form S_{I} or S_{II}, then S is called to be *the standard form of the marked surface* of A.

5 Derived-Unique Gentle One-Cycle Algebras

A k-algebra A is called *derived-unique*, if any algebra B which is derived equivalent to A is Morita equivalent to A, see [19]. In this section, we characterize the derived-unique gentle one-cycle algebras in terms of the marked ribbon surfaces.

Theorem 5.1 *Let $A = kQ/I$ be a gentle one-cycle algebra with marked ribbon surface S_A. Then, A is derived-unique if and only if S_A is one of the following cases:*

(1) $S(0,1,1,0)$, *i.e., A is 2-Kronecker algebra;*

(2) *$S(0, n, 0, r)$, $r = n-1, n$, i.e., Q is an oriented cycle and the number of relations is n or $n-1$.*

Proof It suffices to establish derived-unique gentle once-cycle algebras in the standard form provided in Theorem 4.6.

(1) If $S_A = S(0, s, s', r)$ is of the form S_{I} with $s \geq 1, s' \geq 1, r < s$, then A is type of $\widetilde{\mathbb{A}}_n$, and we claim that A is not derived-unique except the case that $S_A = (0, 1, 1, 0)$.

(i) The case that $s' \geq 2$. Let B be a gentle algebra with marked ribbon surface S_B as the first one in Fig. 8. By Corollary 3.3, A and B are derived equivalent but not Morita equivalent, then A is not derived-unique.

(ii) If $s' = 1$, $r \leq s-2$ and $s \geq 2$, then as in the previous case, we can move both endpoints of the arc indexed by s one step on two different boundary component in the standard surface S_I in Theorem 4.6 to obtain a surface such that the corresponding gentle algebra is derived equivalent but not Morita equivalent to A. Thus, A is not derived-unique in this case.

(iii) We finally come to the case that $s' = 1$, $r = s-1$ and $s \geq 2$. In this case, $\phi_A = [(s-1, 0), (1, s)]$. Now we construct a gentle algebra

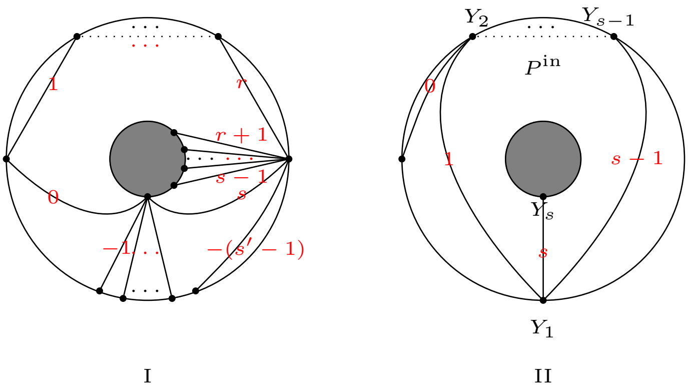


Fig. 8 Two marked surfaces providing derived equivalences for the form of S_I

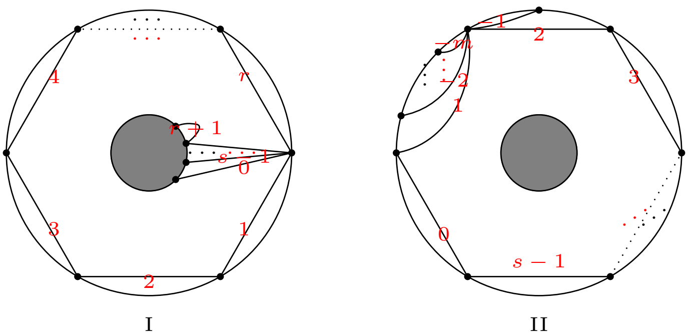


Fig. 9 Two marked surfaces providing derived equivalences for gentle algebras with a oriented cycle

B whose marked ribbon surface S_B is the second one in Fig. 8. Then, $\sharp\mathfrak{C}P^{\text{in}} = s+1$ since $\mathfrak{C}P^{\text{in}} = \{Y_1Y_{s-1}, Y_{s-1}Y_{s-2}, \ldots, Y_2Y_1, Y_1Y_s, Y_sY_1\}$. By Corollary 3.3, the second integer pair of ϕ_B is $(1, \sharp\mathfrak{C}P^{\text{in}} - 1) = (1, s)$; then, $\phi_B = [(s-1, 0), (1, s)] = \phi_A$. Therefore, A is derived equivalent but not Morita equivalent to B and thus not derived-unique.

(iv) If A is 2-Kronecker algebra with the associated marked surface $S(0, 1, 1, 0)$, then it is derived unique. In fact, if $B = kQ_B/I$ is a gentle one-cycle algebra derived equivalent to A, then Q_B has two vertices and two arrows. By a case-by-case discussion using the value of ϕ, B must be 2-Kronecker algebra.

(2) If $S_A = S(0, n, 0, r)$ is of the form S_{I} with $n \geq 1, r < n$, then we claim that A is derived unique if and only if $r = n-1$. Let $B = kQ_B/I_B$ be a gentle algebra derived equivalent to A, then Q_B has one cycle with n arrows and n vertices.

(i) If $r = n-1$, i.e., $S_A = S(0, n, 0, n-1)$, then we have $\phi_A = [(n-1, 0), (1, n)] = \phi_B$, and B is of the type $\tilde{\mathbb{A}}_n$. Indeed, if B has at least one branch, then Q_B has at least one arrow α such that $s(\alpha)$ is a source, or $t(\alpha)$ is a source. Thus, there is an arc Y_1Y_2 corresponding to $s(\alpha)$ or $t(\alpha)$ satisfying its endpoints both are on the same boundary-component of S_B. Since $\phi_B = [(n-1, 0), (1, n)]$, there is a unique marked point on the inner boundary-component γ^{in}. By Remark 2.12, there exists a unique polygon P of the type (I). Note that Y_1Y_2 is not an edge of the inner polygon P. Therefore, the second integer pair (b, c) of ϕ_B satisfies that $b = 1, c = \sharp\mathfrak{C}P - b < n+1-1 = n$, which is impossible. Now we know that Q_B is a cycle and $\phi_B = [(n-1, 0), (1, n)]$. So S_B has only one marked point on the inner boundary component and then only one polygon P of the type (I). Since $c = \sharp\mathfrak{C}P - b = n$, the polygon P satisfies that $\sharp\mathfrak{C}P = n+1$, and then, S_B must be of the form $S(0, n, 0, n-1)$. Therefore, A is Morita equivalent to B and thus is derived-unique.

(ii) Now we prove that A is not derived-unique if $r \leq n-2$. Let B be a gentle algebra with S_B of the form as the first one in Fig. 9. Then, $\phi_A = \phi_B$; hence, $D^b(A) \simeq D^b(B)$ by Theorem 3.2 and A is not derived-unique.

(3) The case that $S_A = S(m, s, 0, s)$ is of the form S_{II} with $s + m = n$.

(i) If $m > 0$, i.e., the quiver of A has one branch, then A is not derived-unique, since the gentle algebra B with S_B of the form as the second one in Fig. 9 shares the same value of ϕ with A by Corollary 3.3. Then, A is not derived-unique.

(ii) If $r = n$, i.e., $S_A = S(0, n, 0, n)$, then we obtain $\phi_A = [(n, 0), (0, n)]$ by Corollary 3.3. $D^b(A) \simeq D^b(B)$ yields that B has no branch with a similar argument as above for the case that $r = n-1$. Since $\phi_B = [(n, 0), (0, n)]$, there is no marked point in the inner boundary component and then S_B must be $S(0, n, 0, n)$ by the property that the full formal arc system cuts the marked surface into polygons with a boundary edge. Therefore, A is derived-unique.

□

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