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On Derived Unique Gentle Algebras

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Abstract An algebra A is called derived-unique provided that any algebra which is derived equivalent to A is necessarily Morita equivalent to A. We classify derived-unique gentle algebras with at most one cycle using the combinatorial invariant introduced by Avella-Alaminos and Geiss.

Keywords Derived equivalence, transformation, derived standard form

MR(2010) Subject Classification 16E35, 16G60, 16E05, 16G20

1 Introduction

Let k be an algebraically closed field. For a finite dimensional k-algebra A, we denote by mod A the category of finite dimensional right A-modules and by $D^b(A)$ the bounded derived category of mod A. By a derived equivalence between two algebras A and B, we mean a k-linear triangle equivalence between $D^b(A)$ and $D^b(B)$. Many homological properties preserve under the derived equivalences, such as the finiteness of global dimension, the rank of Grothendieck group and so on, see [9] for example. So it is natural to explore the condition to ensure the derived equivalence, see [12]. However, it seems difficult to judge whether two general algebras are derived equivalent.

The gentle algebras are known as an important class of algebras, whose derived categories have been extensively studied. This class of algebras is closed under derived equivalences [14]. The indecomposable objects in derived categories of gentle algebras and the morphisms between them have been explicitly described by Bekkert–Merklen [7] and Arnesen–Laking–Pauksztello [2] respectively. Moreover, Avella-Alaminos and Geiss [6] constructed a function which turns out to be a derived invariant. As a result, many explicit computations can be carried out, and gentle algebras can be seen as a useful test-subject for many more general conjectures related to derived categories.

Derived unique algebras, originally introduced by Kalck in [11], are those algebras for which the notions of derived equivalence and Morita equivalence coincide. In this paper, by using the combinatorial function introduced by Avella-Alaminos and Geiss for gentle algebras, we obtain a standard form of gentle algebras with at most one cycle under the derived equivalences and

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classify these derived unique algebras. To be more precise, the mainly theorem in present paper is as follows.

Theorem 1.1 *Let $A = kQ/I$ be a nonsimple gentle algebra with at most one cycle. Then A is derived unique if and only if A is hereditary of type $\mathbb{A}_2$, or the 2-Kronecker algebra, or the quiver Q of A is an oriented cycle with n arrows and I is generated by m paths of length two, where $m = n - 1$ or n.*

The paper is organized as follows: in Section 2, we shall introduce some basic notions and definitions about gentle algebras including the combinatorial invariant introduced by Avella-Alaminos and Geiss. In the third section, we provide a standard form of gentle algebras with at most one cycle by observing changes of the invariant under certain transformations with respect to the quiver and relation. Finally, we prove the main theorem in the last section.

2 The Combinatorial Derived Invariant for Gentle Algebras

Throughout this paper, we denote by kQ/I the quotient algebra of the path algebra kQ by an admissible ideal with $Q = (Q_0, Q_1, s, t)$ a finite quiver. Moreover, we always write the path in Q from left to right, see [4]. In this section, we mainly recall some notions about gentle algebras and the definition of combinatorial derived invariant introduced in [6].

Definition 2.1 *Let $A = kQ/I$ be a connected finite-dimensional k-algebra with I an admissible ideal of kQ. A is called a special biserial algebra if the following conditions hold*:

(1) $\forall v \in Q_0$, $\#\{\alpha \in Q_1 | s(\alpha) = v\} \leq 2$ *and* $\#\{\beta \in Q_1 | e(\beta) = v\} \leq 2$;

(2) $\forall \gamma \in Q_1$, $\#\{\alpha \in Q_1 | s(\gamma) = e(\alpha)$ *and* $\alpha\gamma \notin I\} \leq 1$ *and* $\#\{\beta \in Q_1 | e(\gamma) = s(\beta)$ *and* $\gamma\beta \notin I\} \leq 1$;

(3) $\forall \beta \in Q_1$, *there exists a bound n, n' such that $\beta_1\beta_2 \cdots \beta_n$ with $\beta_n = \beta$ contains a subpath in I and any path $\beta_1\beta_2 \cdots \beta_{n'}$ with $\beta_1 = \beta$ contains a subpath in I.*

Moreover, A is called a gentle algebra if A satisfies the additional condition:

(4) *All relations in I are monomials of length* 2.

(5) $\forall \gamma \in Q_1$, $\#\{\alpha \in Q_1 | e(\alpha) = s(\gamma)$ *and* $\alpha\gamma \in I\} \leq 1$ *and* $\#\{\beta \in Q_1 | s(\beta) = e(\gamma)$ *and* $\gamma\beta \in I\} \leq 1$.

For any gentle algebra $A = kQ/I$, Avella-Alaminos and Geiss defined a function ϕ which turn out to be a perfect derived invariant for those gentle algebras with at most one cycle, see [6]. The notations of permitted threads and forbidden threads are useful in the definition of the function.

Definition 2.2 *Let $A = kQ/I$ be a gentle algebra.*

(1) *A permitted path of A is a path $C = \alpha_1\alpha_2 \cdots \alpha_n$ with no zero relations. A nontrivial permitted thread of A is a maximal permitted path, i.e. for all $\beta \in Q_1$, neither βC nor $C\beta$ is a permitted path. A trivial permitted thread is a trivial path ε_v over the vertex v of Q where the v is such that $\#\{\alpha \in Q_1 | s(\alpha) = v\} \leq 1$, $\#\{\alpha \in Q_1 | e(\alpha) = v\} \leq 1$, and if $\beta, \gamma \in Q_1$ are such that $e(\beta) = v = s(\gamma)$ then $\beta\gamma \notin I$.*

(2) *A forbidden path of A is a path $C = \alpha_1\alpha_2 \cdots \alpha_n$ formed by pairwise different arrows with $\alpha_i\alpha_{i+1} \in I$ for all $i \in \{1, 2, \ldots, n-1\}$. A non-trivial forbidden thread of A is a maximal forbidden path, i.e. for all $\beta \in Q_1$, neither βC nor $C\beta$ is a forbidden path. A trivial forbidden*

thread is a trivial path ε_v over the vertex v of Q where the v is such that $\#\{\alpha \in Q_1 | s(\alpha) = v\} \leq 1$, $\#\{\alpha \in Q_1 | e(\alpha) = v\} \leq 1$, and if $\beta, \gamma \in Q_1$ are such that $e(\beta) = v = s(\gamma)$ then $\beta\gamma \in I$.

Now, we can define the function ϕ for gentle algebras.

Firstly, for a special biserial algebra, we can define two functions $\sigma, \epsilon : Q_1 \to \{1, -1\}$ as follows:

(1) If $\beta_1 \neq \beta_2$ are arrows with $s(\beta_1) = s(\beta_2)$, then $\sigma(\beta_1) = -\sigma(\beta_2)$.

(2) If $\gamma_1 \neq \gamma_2$ are arrows with $e(\gamma_1) = e(\gamma_2)$, then $\epsilon(\gamma_1) = -\epsilon(\gamma_2)$.

(3) If β, γ are arrows with $s(\gamma) = e(\beta)$ and $\beta\gamma \notin I$, then $\sigma(\gamma) = -\epsilon(\beta)$.

The function ϕ can be defined by the following algorithm.

Definition 2.3 (algorithm)

Step 1 (1) *Begin with a permitted thread of A, say H_0.*

(2) *If H_i is defined, consider Π_i, the forbidden thread which ends in $e(H_i)$ and such that $\epsilon(H_i) = -\epsilon(\Pi_i)$*

(3) *Let H_{i+1} be the permitted thread which starts in $s(\Pi_i)$ and such that $\sigma(H_{i+1}) = -\sigma(\Pi_i)$.*

This process stops when $H_n = H_0$ for some $n \in \mathbb{N}$. Let $m = \sum_{1 \leq i \leq n} l(\Pi_{i-1})$, where $l(C)$ is the length of path C, then we obtain a pair (n, m).

Step 2 *Repeat the Step 1 until all permitted threads of A appears.*

Step 3 *If there are oriented cycle in which each pair of consecutive arrows form a relation, we add a pair $(0, m)$ for each those cycles, where m is the length of the oriented cycle.*

Step 4 *Define $\phi_A : \mathbb{N}^2 \to \mathbb{N}$ where $\phi_A(n, m)$ is the number of times that the pair (n, m) arises in the algorithm.*

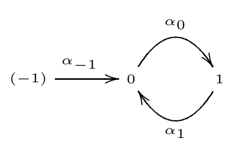
Remark 2.4 By the Definition 2.3, for a gentle algebra, we can obtain a set of pairs

$$\{(n_1, m_1), \ldots, (n_s, m_s)\},$$

which is determined by the algorithm in Definition 2.3. Therefore, we denote $[(n_1, m_1), \ldots, (n_s, m_s)]$ by the function ϕ_A, see [6].

We illustrate the algorithm by the following examples.

Example 2.5 (1) Let $A = kQ_A/I_A$ be a k-algebra and Q_A be of the form



and $I = I_{A_1} = \langle \alpha_1\alpha_0 \rangle$. Obviously, the algebra $A_1 = kQ/I_{A_1}$ has three permitted threads $\alpha_{-1}\alpha_0\alpha_1$, ε_{-1} and ε_1. If beginning with the permitted thread $\alpha_{-1}\alpha_0\alpha_1$ of A_1 and say H_0, then we have

$$\mathrm{H}_0 = \alpha_{-1}\alpha_0\alpha_1, \qquad \Pi_0^{-1} = \alpha_{-1}^{-1},$$
$$\mathrm{H}_1 = \varepsilon_{-1}, \qquad \Pi_1^{-1} = \varepsilon_{-1},$$
$$\mathrm{H}_2 = \alpha_{-1}\alpha_0\alpha_1 = \mathrm{H}_0.$$

Thus, we obtain a pair $(2,1)$. Note that the trivial permitted threads ε_1 does not appear, so beginning with ε_1, we have

$$\mathrm{H}_0 = \varepsilon_1, \qquad \Pi_0^{-1} = \alpha_0^{-1}\alpha_1^{-1},$$
$$\mathrm{H}_1 = \varepsilon_1 = \mathrm{H}_0,$$

and obtain another pair $(1,2)$. By Definitions 2.3 and 2.4,

$$\phi_{A_1} = [(1,2),(2,1)]$$

and

$$\phi_{A_1}(1,2) = 1,\ \phi_{A_1}(2,1) = 1.$$

If $I = I_{A_2} = \langle \alpha_1\alpha_0, \alpha_0\alpha_1 \rangle$, the algebra $A_2 = kQ/I_{A_2}$ has three permitted threads $\alpha_{-1}\alpha_0$, α_1 and ε_{-1}. Similarly, if we assume that H_0 is the permitted thread $\alpha_{-1}\alpha_0$, then

$$\mathrm{H}_0 = \alpha_{-1}\alpha_0, \qquad \Pi_0^{-1} = \varepsilon_1,$$
$$\mathrm{H}_1 = \alpha_1, \qquad \Pi_1^{-1} = \alpha_{-1}^{-1},$$
$$\mathrm{H}_2 = \varepsilon_{-1}, \qquad \Pi_2^{-1} = \varepsilon_{-1},$$
$$\mathrm{H}_3 = \alpha_{-1}\alpha_0 = \mathrm{H}_0$$

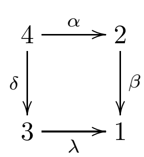
and we get the pair $(3,1)$. Moreover, A_2 satisfies the conditions of Step 3 in Definition 2.3,

$$\phi_{A_2} = [(3,1),(0,2)],$$

and

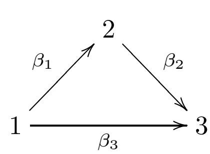
$$\phi_{A_2}(3,1) = 1, \phi_{A_2}(0,2) = 1.$$

(2) Let $B = KQ_B/I_B$, where Q_B has the form of



and $I_B = \langle \alpha\beta, \delta\lambda \rangle$. Then we have $\phi_B = [(2,2),(2,2)]$ and $\phi_B(2,2) = 2$.

(3) Let $C = kQ_C/I_C$ be a k-algebra with Q_C of the form

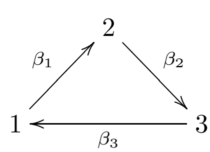


and $I_C = \langle \beta_1\beta_2 \rangle$. Then we have

$$\phi_C = [(2,1),(1,2)].$$

By the example (1) as above, we know that $\phi_{A_1} = \phi_C$.

(4) Let $D = kQ_D/I_D$, Q_D be the quiver given by



and $I_{D_1} = \langle \beta_1\beta_2 \rangle$, $I_{D_2} = \langle \beta_1\beta_2, \beta_2\beta_3 \rangle$ or $I_{D_3} = \langle \beta_1\beta_2, \beta_2\beta_3, \beta_3\beta_1 \rangle$ respectively. With a similar argument as above, we have

$$\phi_{kQ_D/I_{D_1}} = [(1,0),(2,3)],$$
$$\phi_{kQ_D/I_{D_2}} = [(2,0),(1,3)],$$

and

$$\phi_{kQ_D/I_{D_3}} = [(3,0),(0,3)].$$

Due to the work of Avella-Alaminos and Geiss [6], we know that the combinatorial function ϕ is a perfect invariant to differentiate derived equivalences between gentle algebras with at most one cycle.

Proposition 2.6 *Let $A = kQ/I$ and $B = kQ'/I'$ be two gentle algebras. If A and B are derived equivalent, then $\phi_A = \phi_B$. Moreover, the converse is also ture if A and B are gentle algebras with at most one cycle.*

3 Derived Unique Algebras

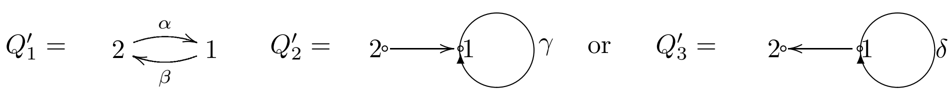
In this section, we shall introduce the definition of derived unique algebras. Gentle algebras, as an important class of algebras, provide some examples.

The definition of derived unique algebras was originally due to [11].

Definition 3.1 *A k-algebra A is called derived unique if every k-algebra B derived equivalent to A is already Morita equivalent to A.*

Remark 3.2 Commutative algebras [13], local algebras [16], path algebras of n-Kronecker quivers, preprojective algebras of Dynkin type [1] and of extended Dynkin type [10] are known classes of derived unique algebras, which was also pointed out in [11].

Indeed, the 2-Kronecker algebras $A = kQ$ with $Q = 1 \rightrightarrows 2$, is a gentle algebra, whose derived uniqueness can also be obtained by the function ϕ introduced in previous section. Assume $A' = kQ'/I'$ is a gentle algebra such that $D^b(A') \simeq D^b(A)$. Since the Grothendieck group of gentle algebras a derived invariant, Q' has two vertices. By the work of Avella-Alaminos and Geiss [6], the number of arrows is also derived invariant, we know that Q' has two arrows. So if $Q' \neq Q$, then Q' is one of the following form



Then we can calculate the values of ϕ for all possible A', i.e., $A'_{11} = kQ'_1/\langle \alpha\beta \rangle$, $A'_{12} = kQ'_1/\langle \alpha\beta, \beta\alpha \rangle$, $A'_2 = kQ'_2/\langle \gamma^2 \rangle$ and $A'_3 = kQ'_3/\langle \delta^2 \rangle$. To be precise, the value $\phi_A = [(1,1),(1,1)]$, and the values of these possible A' are $\phi_{A'_{11}} = [(1,0),(1,2)]$, $\phi_{A'_{12}} = [(2,0),(0,2)]$, $\phi_{A'_2} = \phi_{A'_3} = [(2,1),(0,1)]$ respectively. By Proposition 2.6, 2-Kronecker algebra is derived unique.

Moreover, from the argument as above, we know that the algebras A'_2 and A'_3 are derived equivalent by Proposition 2.6, and thus neither A'_2 or A'_3 is derived unique. In general, we have the following example.

Example 3.3 Let $A = kQ/I$ be an algebra with $|Q_0| > 2$, where the quiver Q has a sink point or a source point, then A is not derived unique since the APR-tilting module or APR-cotilting module induces a derived equivalence, which is not a Morita equivalence.

4 The Standard Forms of Gentle Algebras with at Most One Cycle

In this section, we mainly deduce a standard form of gentle algebras with at most one cycle just using the combinatorial function ϕ. The strategy to achieve this is to introduce three *elementary transformations* on the quiver and relation for gentle algebras with precisely one cycle, and then observe the changes of value of combinatorial function ϕ under these transformations. Therefore, we need to define a notion $\Delta\phi_{A,A'}$ to describe this change from a gentle algebra A to another A' as the following two steps.

Step 1 For a gentle algebra $A = kQ_A/I$ with one cycle, one can calculate the combinatorial function ϕ_A. Moreover, ϕ_A precisely consists of two pairs. The first step is just to fix a order for the two pairs. Note that for any permitted thread H on the cycle of the A, it must be a clockwise thread, or a trivial one, or an anticlockwise one. Our order is to put the pair (a, b) obtained by Definition 2.3 with $\mathrm{H}_0 = H$ in the first place if there exists such a clockwise permitted thread H on the cycle; otherwise, there must be an anticlockwise permitted thread H', then we put the pair (c, d) obtained by Definition 2.3 with $\mathrm{H}_0 = H'$ in the second place.

Step 2 For two gentle algebras A and A' with one cycle, the value of ϕ are

$$\phi_A = [(a, b), (c, d)]$$

and

$$\phi_{A'} = [(a', b'), (c', d')],$$

as in the first step. Now we define

$$\phi_{A'} + \phi_A := [(a' + a, b' + b), (c' + c, d' + d)]$$

and

$$\phi_{A'} - \phi_A := [(a' - a, b' - b), (c' - c, d' - d)],$$

where $\Delta\phi_{A,A'} := \phi_{A'} - \phi_A$ is said to be the change of ϕ from the algebra A to A'. In addition, for gentle algebras $A_1, \cdots, A_n$ with one cycle,

$$\Delta\phi_{A_1,A_n} := \Delta\phi_{A_1,A_2} + \Delta\phi_{A_2,A_3} + \cdots + \Delta\phi_{A_{n-1},A_n} = \sum_{k=1}^{n-1} \Delta\phi_{A_k,A_{k-1}}.$$

In particular, $\Delta\phi_{A,A} = \Delta\phi_{A,A'} + \Delta\phi_{A',A} = [(0, 0), (0, 0)]$, which establishes the relation between the change $\Delta\phi_{A,A'}$ of a transformation from A to A' and the change $\Delta\phi_{A',A}$ of its inverse transformation from A' to A. As a direct consequence of Proposition 2.6, $\Delta\phi_{A,A'} = [(0, 0), (0, 0)]$ implies that the algebras A and A' are derived equivalent.

Now, we define three types of elementary transformations of gentle algebra with one cycle A as follows:

Definition 4.1 *Let $A = kQ/I$ be a gentle algebra with precisely one cycle, where $I = \langle S \rangle$ is generated by S. The elementary transformations of A is defined as follows.*

Transformations of type I *The transformations of type* I *are adding a relation of length two to S or removing a relation from S, such that the algebra A' we get is gentle.*

Transformations of type II *The transformations of type* II *are those diverting the direction of an arrow $\alpha \in Q_1$ which satisfies $\alpha\beta \notin I, \beta\alpha \notin I$ for any $\beta \in Q_1$, such that the new algebra A' is gentle.*

Transformations of type III *The transformations of type* III *are those splitting a point i where there is no relation into two points, say i' and i'', then adding an arrow to connect i' and i'', such that the algebra A' we obtain is gentle; or removing an arrow $\alpha \in Q_1$ which satisfies $\alpha\beta \notin I, \beta\alpha \notin I$ for any $\beta \in Q_1$, then identifying the starting point and endpoint of the arrow such that the algebra A' we get is gentle.*

The following lemma is to observe the changes $\Delta\phi_{A,A'}$ from the algebra A to another A' by applying the elementary transformations. Note that the adding-transformations of types I and II can be seen as the inverse transformations of the according removing-transformations. Then it suffices to describe the changes of the removing-transformations. Moreover, since the elementary transformations of type III on one branch in the quiver are much more complicated, we will deal with this case in Lemma 4.5.

Lemma 4.2 *Let $A = kQ/I$ be a gentle algebra with precisely one cycle, $A' = kQ'/I'$ be another gentle algebra with one cycle obtained by elementary transformations on A.*

(1) **Transformations of type I**

(i) (Remove a clockwise relation on the cycle) *If A' is obtained by removing a clockwise relation on the cycle, then $\Delta\phi_{A,A'} = [(-1, 0), (+1, 0)]$.*

(ii) (Remove an anticlockwise relation on the cycle) *If A' is obtained by removing an anticlockwise relation on the cycle, then $\Delta\phi_{A,A'} = [(+1, 0), (-1, 0)]$.*

(iii) (Remove a relation on a branch) *If A' is obtained by removing a relation on a branch in Q, then $\Delta\phi_{A,A'} = [(0, 0), (0, 0)]$. In particular, $D^b(A) \simeq D^b(A')$.*

(2) **Transformations of type II**

(i) (Divert an arrow on the cycle) *If A' is obtained by diverting a clockwise arrow or an anticlockwise arrow on the cycle, then $\Delta\phi_{A,A'} = [(+1, +1), (-1, -1)]$ or $\Delta\phi_{A,A'} = [(-1, -1), (+1, +1)]$ respectively.*

(ii) (Divert an arrow on a branch) *If A' is obtained by diverting an arrow on a branch, then $\Delta\phi_{A,A'} = [(0, 0), (0, 0)]$.*

(3) **Transformations of type III**

(i) (Remove a clockwise arrow on the cycle) *If A' is obtained by removing a clockwise arrow on the cycle, then $\Delta\phi_{A,A'} = [(0, 0), (-1, -1)]$.*

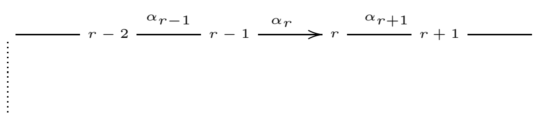
(ii) (Remove an anticlockwise arrow on the cycle) *If A' is obtained by removing an anticlockwise arrow on the cycle, then $\Delta\phi_{A,A'} = [(-1, -1), (0, 0)]$.*

Proof The proof is to calculate ϕ_A, $\phi_{A'}$, and then the change $\Delta\phi_{A,A'}$ in all cases. We only prove three typical cases: the statement (iii) of (1), the statement (ii) of (2) and the statements (i) of (3).

First of all, we prove the statement (iii) of (1), by the definition of elementary transformation of type I, a relation $\alpha\beta \in S$ with the vertex $t(\alpha) = s(\beta) = j$ can be removed if and only if $\#\{\gamma \in Q_1 | s(\gamma) = j\} = 1$ and $\#\{\gamma \in Q_1 | e(\gamma) = j\} = 1$. In this case, the permitted thread H which satisfies $e(H) = j$ and the permitted thread H' which satisfies $s(H') = j$ are connected to be a new permitted thread HH', and we get a trivial permitted thread ε_j. Obviously, both the number of permitted threads and the total length of forbidden threads are invariant, then the value of ϕ stays unchanged.

For the proof of statement (ii) of (2), we observe first that all the permitted threads and forbidden threads in a fixed branch should appear in exactly one pair in the algorithm of ϕ_A. If we divert an arrow $\alpha : i \to j$ on a branch of A, the permitted thread H of A which contains α is divided to three parts in A', the permitted thread H_1' such that $e(H_1') = i$, the forbidden thread Π'^{-1} such that $s(\Pi'^{-1}) = i$ and $e(\Pi') = j$ and the permitted thread H_2' such that $s(H_2') = j$ (it is possible that H_1' or H_2' is a trivial path). Moreover, the new arrow $\alpha' : j \to i$ should connect two permitted threads of A which ends at j and starts at i respectively. Therefore, the number of permitted threads is invariant. Note that the total length of forbidden threads is invariant in this pair since the number of the arrows in this branch stays unchanged.

In what follows, we will show the statement (i) of (3). We assume that $A = kQ/I$ with the quiver Q is of the following form



and the algebra $A' = kQ'/I$ is obtained by removing the clockwise arrow α_r (in this case, the arrow $\alpha_{r-1} : r-2 \to r-1$ is changed to $\alpha_{r-1} : r-2 \to r$). The following argument is divided into four cases: (a) both α_{r-1} and α_{r+1} are clockwise arrows; (b) α_{r-1} is a clockwise arrow but α_{r+1} is an anticlockwise arrow; (c) both α_{r-1} and α_{r+1} are anticlockwise arrows; (d) α_{r-1} is an anticlockwise arrow but α_{r+1} is a clockwise arrow.

For the case (a), we assume that the clockwise permitted thread H_0' of A' is $\cdots \alpha_{r-1}\alpha_{r+1}\cdots$, and the clockwise permitted thread $\mathrm{H}_0 = \cdots \alpha_{r-1}\alpha_r\alpha_{r+1}\cdots$ in $A = kQ/I$, then $\mathrm{H}_i' = \mathrm{H}_i$ for every $i = 1, 2, \ldots$, and $\Pi_j' = \Pi_j$ for every $j = 0, 1, 2, \ldots$. Thus, the first pair of $\Delta\phi_{A,A'}$ is $(0, 0)$. For the second pair, we assume Π_0' is the anticlockwise forbidden thread $\alpha_{r+1}\cdots$ in A', and the anticlockwise forbidden thread $\Pi_0 = \alpha_{r+1}\cdots$ in A. Then $(\Pi_0')^{-1}$ of A' is $\cdots\alpha_{r+1}^{-1}$, while $\Pi_0^{-1} = \cdots\alpha_{r+1}^{-1}$ in A. To be precise, we have

$$
\begin{array}{ll}
 & (\Pi_0')^{-1} = \cdots\alpha_{r+1}^{-1} \\
\mathrm{H}_1' = \varepsilon_r & (\Pi_1')^{-1} = \alpha_{r-1}^{-1}\cdots \\
\mathrm{H}_2' = \cdots & \quad\cdots
\end{array}
$$

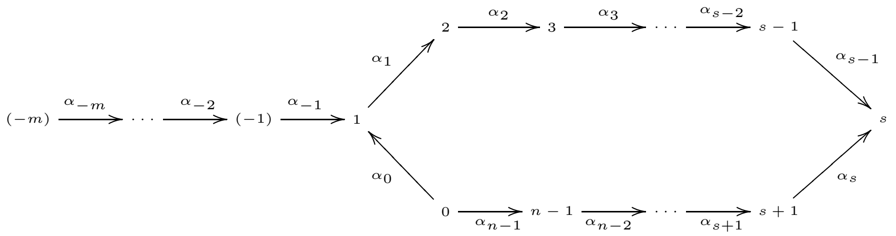
and for A,

$$
\begin{array}{ll}
 & \Pi_0^{-1} = \cdots\alpha_{r+1}^{-1} \\
\mathrm{H}_1 = \varepsilon_r & \Pi_1^{-1} = \alpha_r^{-1} \\
\mathrm{H}_2 = \varepsilon_{r-1} & \Pi_2^{-1} = \alpha_{r-1}^{-1}\cdots \\
\mathrm{H}_3 = \cdots & \quad\cdots
\end{array}
$$

The above comparison implies that $\mathrm{H}_i' = \mathrm{H}_{i+1}$ for $i \geq 2$, and $\Pi_j' = \Pi_{j+1}$ for any $j \geq 1$. So the second pair of $\Delta\phi_{A,A'}$ is $(-1, -1)$. Therefore, $\Delta\phi_{A,A'} = [(0, 0), (-1, -1)]$. The proof of case (b), (c) and (d) is similar. □

Let $m \geq 0, n \geq 0$ be two integers and $\Omega(m, n, s, r) = kQ/I$ be the gentle algebra where Q

is of the form



and $I = \langle \alpha_0\alpha_1, \alpha_1\alpha_2, \ldots, \alpha_{r-1}\alpha_r\rangle$ ($r \le s \le n$), where we identify the index n with 0 as in the cyclic group $\mathbb{Z}/n\mathbb{Z}$, see [8] for details. Note that the case $r = s$ does only make sense when $s = n$ and the relation $I = \langle \alpha_0\alpha_1, \ldots, \alpha_{n-2}\alpha_{n-1}, \alpha_{n-1}\alpha_0\rangle$. Moreover, $\phi_{\Omega(m,n,s,r)} = [(m+n-s+r, m+n-s), (s-r, s)]$.

The following theorem states that any gentle algebra A with at most one cycle is derived equivalent to $\Omega(m,n,s,r)$ for some $m \ge 0, n \ge 0$ and $r \le s \le n$. For this reason, we call $\Omega(m,n,s,r)$ is a *derived standard form* of A. Indeed, the derived classification of gentle algebras at most one cycle is already known by [3, 5, 15], see also [6, Section 7], but our description unifies their classification and we prove it totally by the combinatorial invariant ϕ.

Theorem 4.3 *Let $A = kQ/I$ be a gentle algebra with at most one cycle. Then there exist $m \ge 0, n \ge 0$ and $r \le s \le n$, such that $D^b(A) \simeq D^b(\Omega(m,n,s,r))$. To be more precise,*

(1) *if Q is a tree, then $D^b(A) \simeq D^b(\Omega(m,0,0,0))$ for some $m > 0$;*

(2) *if Q has precisely one cycle, then A is derived equivalent to either $\Omega(0,n,s,r)$ or $\Omega(m,n,n,n)$.*

Before the proof of the theorem, we need the following two lemmas.

Lemma 4.4 *Let $A = kQ/I$ be a gentle algebra with precisely one cycle. Then A is derived equivalent to $\Omega(m,n,n,n)$ or an algebra of type $\tilde{\mathbb{A}}_l$ for some integer l.*

Proof If Q is already a cycle, then we are done. Now we assume that Q is a cycle with some branches on it and $\phi_A = [(a,b),(c,d)]$. Then there is an arrow α on a branch such that: $s(\alpha)$ is a source point of A, or $e(\alpha)$ is a sink point of A. We only prove the former case, the latter one can be deduced with a similar argument. If there exists a relation $p = \alpha\beta$, then we define $A' = kQ'/I'$ where $Q'_0 = Q_0\backslash\{s(\alpha)\}$, $Q'_1 = Q_1\backslash\{\alpha\}$ and $I' = I\backslash\{\alpha\beta\}$; if there is no such relation containing α, then we define $A' = kQ'/I$. In fact, the algebra A' can be viewed as the gentle algebra obtained by removing α. We claim that $\phi_{A'}$ is either $[(a-1,b-1),(c,d)]$ or $[(a,b),(c-1,d-1)]$. To be more precise, let H be the permitted thread in Definition 2.3 containing the arrow α, if moreover:

(1) H is a permitted thread in the calculation of the first pair (a,b), then $\phi_{A'} = [(a-1,b-1),(c,d)]$;

(2) H is a permitted thread in the calculation of the second pair (c,d), then $\phi_{A'} = [(a,b),(c-1,d-1)]$.

We only consider the case (1), and the proof of (2) is similar. When we remove the arrow α of A, the trivial permitted thread $\varepsilon_{s(\alpha)}$ of A is deleted, so the number of permitted thread is decreased by 1 in the calculation of the first pair. Obviously, the length of the forbidden thread containing the arrow α is also decreased by 1. Therefore, both a and b are decreased by 1 for

A' in this case.

By the argument as above, we can remove all branches of A to get a new gentle algebra A'', whose quiver is a cycle, such that $\phi_{A''} = [(a-s, b-s), (c-t, d-t)]$ for some $s, t \in \mathbb{N}$. Here, $s+t$ is the number of arrows on all branches.

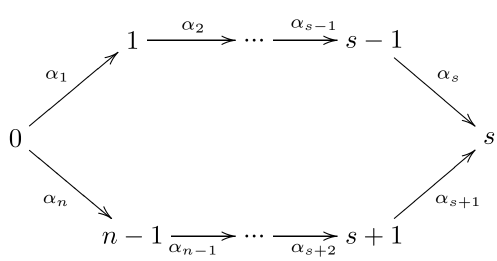
Next we divide the proof of the lemma into two cases as follows.

If the cycle of A'' is an oriented cycle and the composition of every two successive arrows on the cycle is zero. Without loss of generality, we assume all arrows on the cycle of A'' are clockwise, i.e., $A'' = \Omega(0, r, r, r)$. Then $\phi_{A''} = [(a-s, b-s), (c-t, d-t)] = [(r, 0), (0, r)]$, which implies $\phi_A = [(a, b), (c, d)] = [(r+s, s), (t, r+t)]$. Note that in this case, all the arrows on the branches appear in the first pair in the algorithm of ϕ_A since the cycle is a 2-truncated cycle, which implies $t = 0$. Set $s = m$ and $r = n$. Then we have $\phi_A = [(m+n, m), (0, n)] = \phi_{\Omega(m,n,n,n)}$. By the result from [6], $D^b(A) \simeq \Omega(m, n, n, n)$.

If the number of vertices on the cycle of A'' is greater than the number of relations on the cycle, then we can find a point i with no relations. Note that $\phi_{A''} = [(a-s, b-s), (c-t, d-t)]$ for some $s, t \in \mathbb{N}$. Then, by the transformations of type III, we can split this point i into two points i' and i'', then adding s anticlockwise arrows and t clockwise arrows on the cycle. By (i) and (ii) of (3) in Lemma 4.2, we get the algebra B of type $\tilde{\mathbb{A}}_l$ for some l, such that $D^b(B) \simeq D^b(A)$ since $\phi_B = [(a, b), (c, d)] = \phi_A$. □

The next lemma can be directly deduced from Lemma 4.2.

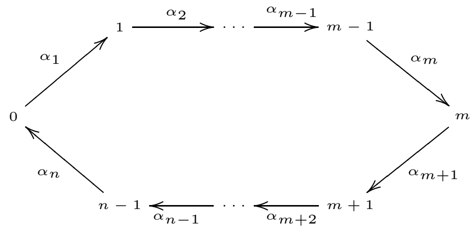
Lemma 4.5 *Let A be a gentle algebra of type $\tilde{\mathbb{A}}_{n-1}$. Then there exists a gentle algebra $B = kQ_B/I$ with Q_B is of the following form*



and $I = \langle \alpha_1\alpha_2, ..., \alpha_r\alpha_{r+1} \rangle$, where $0 < s \leq n$, $0 \leq r \leq s$ and $r = 0$ means that B is hereditary, such that $D^b(A) \simeq D^b(B)$. In this case, $\phi_A = [(n-s+r, n-s), (s-r, s)]$.

Proof Suppose that there are m clockwise arrows and m' anticlockwise arrows in the quiver of A $(m+m' = n)$. We assume that the number of clockwise relations and anticlockwise relations of A are $r+r'$ and r' without loss of generality, and we can obtain the gentle algebra $A_1 = kQ/I_1$ by removing r' anticlockwise relations and r' clockwise relations of A. Let $\phi_A = [(a, b), (c, d)]$. By Lemma 4.2, we have $\Delta\phi_{A,A_1} = [(0, 0), (0, 0)]$.

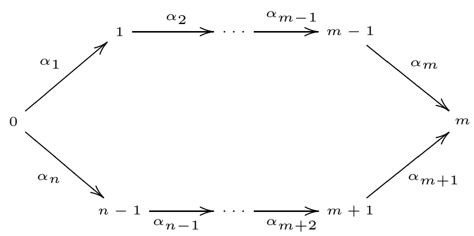
Now we divert the direction of all anticlockwise arrows by m' transformations of type III, we obtain the k-algebra $A_2 = kQ_2/I_2$ where Q_2 is a oriented cycle with n points, i.e., it has the form of



and I_2 is an ideal generated by a set of clockwise relations of A. Moreover, by Lemma 4.2, we have $\Delta\phi_{A_1,A_2} = [(-m',-m'),(+m',+m')]$.

Note that the changes of ϕ of removing one clockwise relation or adding one clockwise relation on the cycle is independent of the their position. So if we move the position of a relation, then the value of ϕ is unchanged. Therefore, we can suppose that the $I_2 = \langle \alpha_1\alpha_2, \alpha_2\alpha_3, \ldots, \alpha_r\alpha_{r+1}\rangle$.

Next, we divert the direction of arrows $\alpha_{m+1}, \alpha_{m+2}, \ldots, \alpha_{n-1}$ and α_n be changed to anticlockwise, and then we get the algebra $A_3 = kQ_3/I_3$ where the quiver Q_3 has the following form



where $I_3 = \langle \alpha_1\alpha_2, \alpha_2\alpha_3, \ldots, \alpha_r\alpha_{r+1}\rangle$. By Lemma 4.2, we know the change of ϕ is $\Delta\phi_{A_2,A_3} = [(+m',+m'),(-m',-m')]$. Now we obtain a gentle algebra A_3 of required form by a series of elementary transformations. Moreover, $D^b(A) \simeq D^b(A_3)$ since

$$\Delta\phi_{A,A_3} = \Delta\phi_{A,A_1} + \Delta\phi_{A_1,A_2} + \Delta\phi_{A_2,A_3} = [(0,0),(0,0)]. \qquad \square$$

Now we are ready to prove Theorem 4.3.

Proof (1) If $A = kQ/I$, where Q is a tree with m arrows, we claim that the ϕ_A is $[(m+2,m)]$. The method of this proof is similar to that of Lemma 4.4. Observe that there is an arrow α such that $s(\alpha)$ is a source or $e(\alpha)$ is a sink.

We only discuss the case that $s(\alpha)$ is a source. If there is a relation $\alpha\beta$ of length two, we define $A' = kQ'/I'$ where $Q'_0 = Q_0\backslash\{s(\alpha)\}$, $Q'_1 = Q_1\backslash\{\alpha\}$ and $I' = I\backslash\{\gamma\}$. Otherwise, we define $A' = kQ'/I$. When we remove α, the trivial permitted thread $\varepsilon_{s(\alpha)}$ of A is deleted, and both the number of the permitted threads and the length of the forbidden thread containing α are decreased by 1, so $\phi_{A'} = [(a-1,b-1)]$. We can remove $m-1$ arrow α step by step, until we get the algebra $B = kQ$ where $Q = 2 \to 1$. Then $\phi_B = [(3,1)] = [(a-m+1, b-m+1)]$. Thus $\phi_A = [(m+2,m)]$. Therefore, $D^b(A) \simeq D^b(\Omega(m,0,0,0))$ since $\phi_{\Omega(m,0,0,0)} = [(m+2,m)] = \phi_A$.

(2) For any gentle algebra A with one cycle, by Lemma 4.4, is derived equivalent to $\Omega(m,n,n,n)$ or an algebra of type $\tilde{\mathbb{A}}_l$ for some integer l. Then we have the statement as required by Lemma 4.5. $\square$

5 The Proof of Our Main Theorem

Throughout this section, we denote by $Q(n,s)$ the quiver and by $I(n,s,r)$ the ideal of the gentle algebra $\Omega(0,n,s,r)$ respectively for our convenience.

Theorem 5.1 *Let $A = kQ/I$ be a nonsimple gentle algebra with at most one cycle. Then A is derived unique if and only if it is one of following cases*:

(1) *A is hereditary algebra of type $\mathbb{A}_2$;*

(2) *$A = \Omega(0,2,1,0)$, i.e., A is the 2-Kronecker algebra;*

(3) *$A = \Omega(0,n,n,r)$, $r = n-1$, or n, that is, the quiver Q of A is an oriented cycle with n arrows, I is generated by m path of length two and $m = n-1$ or n.*

Proof We assume that $n = |Q_0|$. Since the Grothendieck group is a derived invariant, the number of vertices preserves under derived equivalences and we classify the derived unique algebras according to $n = |Q_0|$. We observe first that the nonsimple gentle algebra with $n = 1$ must be $\Omega(0,1,1,1)$, which is derived unique obviously.

(1) For the case of $n = 2$, we should note that the hereditary of type $\mathbb{A}_2$ is derived unique obviously. Then the quiver Q of A must be $Q(2,1) = 1 \rightrightarrows 2$ or $Q(2,2) = 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$. If $Q = Q(2,2)$, then I is either $I(2,2,1) = \langle \alpha\beta \rangle$ or $I(2,2,2) = \langle \alpha\beta, \beta\alpha \rangle$ since A is finite-dimensional. Note that both $\Omega(0,2,2,1) = Q(2,2)/I(2,2,1)$ and $\Omega(0,2,2,2) = Q(2,2)/I(2,2,2)$ are derived unique by Proposition 2.6. If $Q = Q(2,1)$, then $A = kQ = kQ(2,1)$ is 2-Kronecker algebra, which is also derived unique.

(2) Now we consider the case of $n \geq 3$. If A is derived unique, then Q must be of type $\tilde{\mathbb{A}}_{n-1}$ by Example 3.3. Moreover, we can assume that $A = \Omega(0,n,s,r)$ by Theorem 4.3. For $A = \Omega(0,n,s,r)$ where $1 \leq s < n$, $n \geq 3$, that is, the quiver Q is a cycle but not oriented, we claim that A is not derived unique. Indeed, if $r \geq 1$, by the transformation of type III, we can remove an anticlockwise arrow on the cycle and add another arrow $\alpha : (-1) \to 1$ to obtain another gentle algebra $\Omega(1,n-1,s,r)$. Obviously, $D^b(A) \simeq D^b(\Omega(1,n-1,s,r))$ by the proof of Lemma 4.4. If $r = 0$, $A = \Omega(0,n,s,0)$ is a hereditary algebra with at least 3 vertices. We can consider the algebra A' which is obtained by adding an arrow $\beta : (-1) \to 0$ and a relation $\beta\alpha_0$, Lemma 4.4 implies that $\Delta\phi_{A,A'} = [(+1,+1),(0,0)]$. Then by removing the clockwise arrow α_2, we obtain a gentle algebra A'' such that $\Delta\phi_{A',A''} = [(-1,-1),(0,0)]$ by Lemma 4.2. Therefore, $\Delta\phi_{A,A''} = \Delta\phi_{A,A'} + \Delta\phi_{A',A''} = [(0,0),(0,0)]$, which implies that $D^b(A) \simeq D^b(A'')$. So it suffices to prove that $A = \Omega(0,n,n,r)$ with $1 \leq r \leq n$ is derived unique if and only if $r = n-1$ or n.

For the sufficiency, we only prove the case $r = n-1$ and the proof of the derived uniqueness of $\Omega(0,n,n,n)$ is similar. Let X be a gentle algebra such that $D^b(X) \simeq D^b(\Omega(0,n,n,1))$. We can suppose that $X = \Omega(m,n-m,s,r)$ by Theorem 4.3, then $\phi_X = [(n-s+r,n-s),(s-r,s)]$, then

$$\phi_X = [(n-s+r,n-s),(s-r,s)] = [(n-1,0),(1,n)] = \phi_A.$$

Thus $(n-s+r,n-s) = (n-1,0)$; $(s-r,s) = (1,n)$ by the order we fix, which implies that $s = n$ and $r = n-1$. Then $X = \Omega(m,n-m,n,1)$. Moreover, by $n = s \leq n-m$ and $m \geq 0$, we have $m = 0$ and $X = \Omega(0,n,n,n-1) = A$. Therefore, $\Omega(0,n,n,n-1)$ is derived unique.

Conversely, we claim that A is not derived unique if $1 < r < n-1$ (which forces that $n > 3$), since $\Omega(0,n,n,r) = kQ(n,n)/\langle \alpha_0\alpha_1, \dots, \alpha_{r-2}\alpha_{r-1}, \alpha_{r-1}\alpha_r \rangle$ is derived equivalent to the algebra $A' = kQ(n,n)/\langle \alpha_0\alpha_1, \dots, \alpha_{r-2}\alpha_{r-1}, \alpha_r\alpha_{r+1} \rangle$. Indeed, the algebra A' can be obtained from A by moving the last relation $\alpha_{r-1}\alpha_r$ one step anticlockwise, which can be viewed as taking removing and adding transformations of type I successively. By statement (1) of Lemma 4.2, we have $\Delta\phi_{A,A'} = 0$. Thus A is not derived unique. □

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