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On algebras of strongly derived unbounded type

Chao Zhang [a,b,*,1]

[a] *Department of Mathematics, Guizhou University, Guiyang 550025, PR China*
[b] *Faculty of Mathematics, Bielefeld University, 33501 Bielefeld, Germany*

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ABSTRACT

Let A be a finite-dimensional algebra over an algebraically closed field. We prove that A is of strongly derived unbounded type (see Definition 1.1) if and only if there exists an integer m such that $C_m(\text{proj}\, A)$, the category of all minimal projective A-module complexes with degree concentrated in $[0, m]$, is of strongly unbounded type, which is also equivalent to the statement that the repetitive algebra $\hat{A}$ is of strongly unbounded representation type. As a corollary, we can establish the Finite–Strongly unbounded dichotomy on the representation type of $C_m(\text{proj}\, A)$, and also the Discrete–Strongly unbounded dichotomy on the representation type of homotopy category $K^b(\text{proj}\, A)$ and the repetitive algebra $\hat{A}$.

0. Introduction

Throughout this article, k is an algebraically closed field and all the algebras are associative finite-dimensional connected basic k-algebras with identity. In representation theory of algebras, one of the main topics is to study their representation type. As early as 1940s, Brauer and Thrall began the investigation of representation type of finite-dimensional algebras [9,25]. Jans formulated the first and second Brauer–Thrall conjectures in his paper [20]. Roughly speaking, the first Brauer–Thrall conjecture says that an algebra is of bounded representation type if and only if it is of finite representation type, whereas the second Brauer–Thrall conjecture states that the algebras of unbounded representation type are of strongly unbounded representation type. Here, we say an algebra is *of strongly unbounded representation type* if there are infinitely many $d \in \mathbb{N}$ such that for each d, there exist infinitely many isomorphism classes of indecomposable

* Correspondence to: Department of Mathematics, Guizhou University, Guiyang 550025, PR China
E-mail address: zhangc@amss.ac.cn.

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modules of dimension d. The study of the Brauer–Thrall conjectures stimulated the development of representation theory of algebras to a large extent [3,4,21,23,24].

In recent years, the bounded derived categories of algebras have been studied widely since Happel's work [17]. By a celebrated theorem from [17], there is a full embedding from the bounded derived category of a finite-dimensional algebra to the stable module category of its repetitive algebra, which is an equivalence if and only if its global dimension is finite. The theorem connects the bounded derived category and the module category, and hence provides a method to explore the property of bounded derived category of algebras by studying their repetitive algebra, for example the derived representation type of algebras [11,15]. Moreover, Vossieck established his definitive work on the classification and distribution of indecomposable objects in the bounded derived category of an algebra in terms of its repetitive algebra. He introduced and classified derived discrete algebras, and proved that an algebra is derived discrete if and only if its repetitive algebra is of discrete representation type [26, Theorem]. Motivated by Vossieck's work, Han and the author introduced the cohomological range of a bounded complex, which leads to the concept of strongly derived unbounded algebras naturally [16, Definition 5]. We say an algebra is *of strongly derived unbounded type* if there are infinitely many $r \in \mathbb{N}$ such that for each r, there exist infinitely many isomorphism classes of indecomposable objects of cohomological range r in its bounded derived category. Han and the author also proved the dichotomy theorem on the representation type of bounded derived category, i.e., a finite-dimensional algebra is either derived discrete or of strongly derived unbounded type, but not both [16, Theorem 2]. The main purpose of this paper is to characterize the strongly derived unbounded algebras using the representation type of their repetitive algebras, which in turn provides a proof of the Discrete–Strongly unbounded dichotomy of the repetitive algebras combined with Han and the author's theorem.

During the research on the bounded derived category $D^b(A)$ of an algebra A, another category turns out to be very crucial, that is $C_m(\operatorname{proj} A)$, the category of all minimal complexes of finite-dimensional projective modules with degree concentrated in $[0, m]$ for any fixed integer $m \geq 0$. Bautista generalized the definition of derived discreteness for the Artin algebras over commutative Artin rings and characterized the derived discrete algebras in terms of generic objects in the category $C_m(\operatorname{proj} A)$ (Ref. [5]). In the context of the representation type, Bautista introduced the finite, tame and wild representation type for $C_m(\operatorname{proj} A)$, and then established the Tame–Wild dichotomy theorem of $C_m(\operatorname{proj} A)$. Moreover, the description that, A is derived discrete if and only if $C_m(\operatorname{proj} A)$ is of finite representation type for all m, is obtained [6]. In present paper, we define the strongly unboundedness of the category $C_m(\operatorname{proj} A)$ for any fixed integer m in a natural way, and describe the algebras of strongly derived unbounded type as those such that the associated category $C_m(\operatorname{proj} A)$ is of strongly unbounded type for some m. The characterization provides us a bridge to connect the strongly unboundedness of bounded derived category $D^b(A)$ and the repetitive algebra $\hat{A}$. Indeed, we prove the following main theorem.

Theorem. *Let A be a finite-dimensional algebra. Then the following statements are equivalent:*

(1) *A is strongly derived unbounded;*
(2) *There exists an integer $m \geq 1$, such that the category $C_m(\operatorname{proj} A)$ is of strongly unbounded type;*
(3) *$K^b(\operatorname{proj} A)$ is of strongly unbounded type;*
(4) *The repetitive algebra $\hat{A}$ is of strongly unbounded representation type.*

By the dichotomy theorem for bounded derived category mentioned above, a finite-dimensional algebra A is derived discrete or of strongly derived unbounded type [16, Theorem 2]. Combined with the equivalent characterizations of derived discrete algebras with representation type of $C_m(\operatorname{proj} A)$ [6], the homotopy category $K^b(\operatorname{proj} A)$ and repetitive algebra $\hat{A}$ [26], we obtain the dichotomy on the representation type of $C_m(\operatorname{proj} A)$, $K^b(\operatorname{proj} A)$ and $\hat{A}$ as follows.

Corollary. *Let A be a finite-dimensional algebra. Then we have*

(1) $C_m(\text{proj } A)$ *is either of finite representation type for any m, or of strongly unbounded type for all but finitely many m;*
(2) $K^b(\text{proj } A)$ *is either discrete or of strongly unbounded type;*
(3) *The repetitive algebra $\hat{A}$ is either of discrete representation type or of strongly unbounded representation type.*

The present paper is organized as follows. In the first section, we define the strongly unboundedness of $C_m(\text{proj } A)$ and prove some basic lemmas. In Section 2, we observe the strongly unboundedness of $C_m(\text{proj } A)$ under the derived equivalences and cleaving functors. Moreover, we study the strong unboundedness of $C_m(\text{proj } A)$ for representation-infinite algebras, simply connected algebras and finally prove the main theorem.

1. The strongly unboundedness of $C_m(\text{proj } A)$

1.1. Notations and definitions

Let A be a finite-dimensional algebra, and mod A be the category of all finite-dimensional right A-modules and proj A be its full subcategory consisting of all projective right A-modules. Assume $C(A)$ is the category of all complexes of finite-dimensional right A-modules. Denote by $C^b(A)$ and $C^{-,b}(A)$ its full subcategories consisting of all bounded complexes and right bounded complexes with bounded cohomology respectively, by $C^b(\text{proj } A)$ and $C^{-,b}(\text{proj } A)$ the full subcategories of $C^b(A)$ and $C^{-,b}(A)$ respectively consisting of all complexes of projective modules. Moreover, $K(A)$, $K^b(\text{proj } A)$ and $K^{-,b}(\text{proj } A)$ are the homotopy categories of $C(A)$, $C^b(\text{proj } A)$ and $C^{-,b}(\text{proj } A)$ respectively, and $D^b(A)$ is the bounded derived category of mod A with [1] the shift functor.

From [16], for any complex $X^\bullet \in D^b(A)$, the *cohomological length* is

$$\text{hl}(X^\bullet) := \max\{\dim H^i(X^\bullet) \mid i \in \mathbb{Z}\},$$

the *cohomological width* of $X^\bullet$ is

$$\text{hw}(X^\bullet) := \max\{j - i + 1 \mid H^i(X^\bullet) \neq 0 \neq H^j(X^\bullet)\},$$

and the *cohomological range* of $X^\bullet$ is defined as

$$\text{hr}(X^\bullet) := \text{hl}(X^\bullet) \cdot \text{hw}(X^\bullet).$$

Note that these numerical invariants preserve under shifts and isomorphisms. Moreover, the dimension of an A-module M is equal to the cohomological range of the stalk complex with M in degree 0.

Definition 1.1. (See [16, Definition 5].) An algebra A is said to be *of strongly derived unbounded type* or *strongly derived unbounded* if there is a strictly increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ such that for each r_i, up to shifts and isomorphisms, there are infinitely many indecomposable objects in $D^b(A)$ of cohomological range r_i.

Recall that a complex $X^\bullet = (X^i, d^i) \in C^b(A)$ is said to be *minimal* if $\text{Im}\, d^i \subseteq \text{rad}\, X^{i+1}$ for all $i \in \mathbb{Z}$, and the *width* of $X^\bullet$ is

$$w(X^\bullet) := \max\{j - i + 1 \mid X^j \neq 0 \neq X^i\}.$$

For any integer $m \geq 1$, $C_m(\operatorname{proj} A)$ is the subcategory of $C^b(\operatorname{proj} A)$ consisting of all minimal complexes $P^\bullet = (P^i, d^i)$ such that $P^i = 0$ for any $i \notin \{0, 1, \cdots, m\}$. Following [5,6], for $P^\bullet \in C_m(\operatorname{proj} A)$, the *dimension* of $P^\bullet$ is

$$\dim(P^\bullet) = \sum_{i=0}^{m} \dim P^i.$$

Now we shall define the strongly unboundedness of $C_m(\operatorname{proj} A)$.

Definition 1.2. Let A be a finite-dimensional algebra and $m \geq 1$ be an integer. The category $C_m(\operatorname{proj} A)$ is said to be *strongly unbounded* or *of strongly unbounded type* if there is a strictly increasing sequence $\{d_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ such that for each d_i, up to isomorphisms, there are infinitely many indecomposable objects in $C_m(\operatorname{proj} A)$ of dimension d_i.

Remark 1.3. For any algebra A and fixed $m \geq 1$, there is a full embedding from the category $C_m(\operatorname{proj} A)$ to $C_{m+1}(\operatorname{proj} A)$, thus the strongly unboundedness of $C_m(\operatorname{proj} A)$ implies the strongly unboundedness of $C_{m+1}(\operatorname{proj} A)$. In particular, the statement that $C_m(\operatorname{proj} A)$ is strongly unbounded for some m is equivalent to that $C_m(\operatorname{proj} A)$ is of strongly unbounded type for all but finitely many m.

To study the strongly derived unboundedness of $C_m(\operatorname{proj} A)$, we need two lemmas as follows.

Lemma 1.4. *(See [5, Lemma 2.2].) Let A be a finite-dimensional algebra with $\dim A = d$, and $P^\bullet \in C_m(\operatorname{proj} A)$ such that $\operatorname{hl}(P^\bullet) = c$. Then for any $i \in [0, m]$,*

$$\dim P^i \leq c(d + d^2 + \cdots + d^{m-i+1}).$$

Proof. Since $P^\bullet = (P^i, d^i)$ is a minimal complex, i.e., $\operatorname{Im} d^i \subseteq \operatorname{rad} P^{i+1}$, then for any $i \in [0, m]$,

$$\begin{aligned}
\dim P^i &\leq \dim A \cdot \dim(P^i / \operatorname{rad} P^i) \\
&\leq \dim A \cdot \dim(P^i / \operatorname{Im} d^{i-1}) \\
&= \dim A \cdot \big(\dim(P^i / \operatorname{Ker} d^i) + \dim(\operatorname{Ker} d^i / \operatorname{Im} d^{i-1})\big) \\
&= \dim A \cdot \big(\dim \operatorname{Im} d^i + \dim H^i(P^\bullet)\big) \\
&\leq \dim A \cdot \big(\dim P^{i+1} + \dim H^i(P^\bullet)\big) \\
&\leq d \cdot \big(\dim P^{i+1} + c\big).
\end{aligned}$$

Thus we can get the inequality as required recursively. □

Lemma 1.5. *Let A be a finite-dimensional algebra and $m \geq 0$ be an integer. Suppose $P^\bullet$, $Q^\bullet$ are two objects in $C_m(\operatorname{proj} A)$. Then*

1. *$P^\bullet$ is indecomposable in $C_m(\operatorname{proj} A)$ if and only if $P^\bullet$ is indecomposable as an object in $D^b(A)$.*
2. *$P^\bullet \cong Q^\bullet$ in $C_m(\operatorname{proj} A)$ if and only if $P^\bullet \cong Q^\bullet$ in $D^b(A)$.*

Proof. (1) Since $D^b(A) \simeq K^{-,b}(\operatorname{proj} A)$, which is Krull–Schmidt, the complex $P^\bullet$ is indecomposable in $D^b(A)$ if and only if it is an indecomposable complex in $K^b(\operatorname{proj} A)$, and if and only if its endomorphism algebra $\operatorname{End}_{K(A)}(P^\bullet)$ is a local algebra. Moreover, the minimality of complex $P^\bullet$ implies that all null homotopic cochain maps in $\operatorname{End}_{C(A)}(P^\bullet)$ are in $\operatorname{rad} \operatorname{End}_{C(A)}(P^\bullet)$. Thus

$$\mathrm{End}_{K(A)}(P^\bullet)/\operatorname{rad}\mathrm{End}_{K(A)}(P^\bullet) \cong \mathrm{End}_{C(A)}(P^\bullet)/\operatorname{rad}\mathrm{End}_{C(A)}(P^\bullet),$$

which implies $P^\bullet$ is indecomposable in $K^b(\operatorname{proj} A)$ if and only if $P^\bullet$ is indecomposable in $C_m(\operatorname{proj} A)$.

(2) If $P^\bullet \cong Q^\bullet$ in $C_m(\operatorname{proj} A)$, then they are isomorphic in $D^b(A)$. Conversely, suppose $P^\bullet \cong Q^\bullet$ in $D^b(A)$ and there is a quasi-isomorphism $f^\bullet : P^\bullet \to Q^\bullet$. Then we have a triangle in $K(A)$ of form

$$P^\bullet \xrightarrow{f^\bullet} Q^\bullet \to L^\bullet \to P^\bullet[1]$$

such that $L^\bullet$ is an acyclic complex. Applying $\mathrm{Hom}_{K(A)}(Q^\bullet, -)$ to the triangle, we have $\mathrm{Hom}_{K(A)}(Q^\bullet, P^\bullet) \cong \mathrm{Hom}_{K(A)}(Q^\bullet, Q^\bullet)$ induced by $f^\bullet$ since $\mathrm{Hom}_{K(A)}(Q^\bullet, L^\bullet[n]) = 0$ for any $n \in \mathbb{Z}$, which implies $f^\bullet$ is a split epimorphism in $K(A)$. Note that $P^\bullet$ and $Q^\bullet$ are quasi-isomorphic. Then $f^\bullet$ is a chain homotopy equivalence, i.e., there is a morphism $g^\bullet$ such that $1 - g^\bullet f^\bullet$ and $1 - f^\bullet g^\bullet$ are null homotopic. Since $P^\bullet$, $Q^\bullet$ are minimal, $1 - g^\bullet f^\bullet$ and $1 - f^\bullet g^\bullet$ are nilpotent. Thus $f^\bullet$ and $g^\bullet$ are split monomorphisms in $C_m(\operatorname{proj} A)$. Therefore, $P^\bullet \cong Q^\bullet$ in $C_m(\operatorname{proj} A)$. □

The following lemma implies the strongly unboundedness of $C_m(\operatorname{proj} A)$ can be also defined in terms of the cohomological range.

Lemma 1.6. *Let A be a finite-dimensional algebra and $m \geq 1$ be an integer. The category $C_m(\operatorname{proj} A)$ is strongly unbounded if and only if there is an increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ such that for each r_i, there are infinitely many indecomposable objects in $C_m(\operatorname{proj} A)$ of cohomological range r_i up to isomorphisms.*

Proof. We suppose that there exist an increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and pairwise non-isomorphic objects $\{P^\bullet_{ij} \mid i, j \in \mathbb{N}\}$ in $C_m(\operatorname{proj} A)$ such that $\mathrm{hr}(P^\bullet_{ij}) = r_i$. Note that for any object $P^\bullet \in C_m(\operatorname{proj} A)$, $\mathrm{hr}(P^\bullet) \leq (m+1) \cdot \dim(P^\bullet)$. Moreover by Lemma 1.4, $\dim(P^\bullet) \leq \mathrm{hr}(P^\bullet) \cdot (m+1) \cdot (d + d^2 + \cdots + d^{m+1})$. Set $N = (m+1) \cdot (d + d^2 + \cdots + d^{m+1})$, then for any $i, j \in \mathbb{N}$, we have

$$\frac{1}{m+1} \cdot \mathrm{hr}(P^\bullet_{ij}) \leq \dim(P^\bullet_{ij}) \leq N \cdot \mathrm{hr}(P^\bullet_{ij}).$$

In order to show that $C_m(\operatorname{proj} A)$ is of strongly unbounded type, we shall find inductively an increasing sequence $\{d_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and infinitely many indecomposable objects $\{Q^\bullet_{ij} \in C_m(\operatorname{proj} A) \mid i, j \in \mathbb{N}\}$ which are pairwise different up to isomorphisms such that $\dim(Q^\bullet_{ij}) = d_i$ for all $j \in \mathbb{N}$. For $i = 1$, $0 < \dim(P^\bullet_{1j}) \leq Nr_1$. Then there is $0 < d_1 \leq Nr_1$ and infinitely many indecomposable objects $\{Q^\bullet_{1j} \mid j \in \mathbb{N}\} \subseteq \{P^\bullet_{1j} \mid j \in \mathbb{N}\}$ of dimension d_1. Assume that we have found d_i. We choose some r_l with $r_l > (m+1) \cdot d_i$. Since

$$d_i < \frac{1}{m+1} \cdot r_l = \frac{1}{m+1} \cdot \mathrm{hr}(X^\bullet_{lj}) \leq \dim(P^\bullet_{lj}) \leq N \cdot \mathrm{hr}(X^\bullet_{lj}) = N \cdot r_l,$$

we can choose $d_i < d_{i+1} \leq N \cdot r_l$ and infinitely many indecomposable objects $\{Q^\bullet_{i+1,j} \mid j \in \mathbb{N}\} \subseteq \{P^\bullet_{lj} \mid j \in \mathbb{N}\}$ which are pairwise non-isomorphic such that $\dim(Q^\bullet_{i+1,j}) = d_{i+1}$ for all $j \in \mathbb{N}$.

Conversely, if $C_m(\operatorname{proj} A)$ is of strongly unbounded, then we can construct an increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and pairwise non-isomorphic objects $\{Q^\bullet_{ij} \mid i, j \in \mathbb{N}\}$ such that $\mathrm{hr}(Q^\bullet_{ij}) = r_i$ in a similar way by the inequality

$$\frac{1}{N} \cdot \dim(P^\bullet) \leq \mathrm{hr}(P^\bullet) \leq (m+1) \cdot \dim(P^\bullet),$$

for any $P^\bullet \in C_m(\operatorname{proj} A)$. □

2. The proof of Theorem

2.1. Simply connected algebras

Simply connected algebras play an important role in the representation theory of algebras since any representation-finite algebra can be transformed to a simply connected algebra using covering technique. We first recall the definition of simply connected algebras from [2]. Fix a connected quiver (Q, I, s, t) with I admissible. For any $\alpha \in Q_1$, we write its formal inverse α^{-1} with source $s(\alpha^{-1}) = t(\alpha)$ and target $t(\alpha^{-1}) = s(\alpha)$. A walk in Q is a path $w = w_1 w_2 \cdots w_n$ with $w_i \in Q_1$ or $w_i^{-1} \in Q_1$ such that $s(w_{i+1}) = t(w_i)$. A relation $r = \sum_{i=1}^{m} t_i u_i \in I$ $(m \geq 1)$ with u_i pairwise distinct and $t_i \in k \setminus \{0\}$ is called *minimal* if $r = \sum_{i \in S} t_i u_i \notin I$ for any non-empty proper subset $S \subset \{1, 2, \cdots, m\}$. The *homotopy relation* is the smallest equivalence relation $\sim_I$ on the set of walks such that

(1) $\alpha\alpha^{-1} \sim_I e_x$ and $\alpha^{-1}\alpha \sim_I e_y$ for any $x \xrightarrow{\alpha} y$;
(2) $u_1 \sim_I u_2$ for any minimal relation $t_1 u_1 + t_2 u_2 + \cdots + t_m u_m$;
(3) $u \sim_I v$ implies $uw \sim_I vw$ and $wu \sim_I wv$ for any w.

The *fundamental group* $\Pi_1(Q, I, x_0)$ of (Q, I) is defined to be the group consisting of homotopy classes of walks from x_0 to x_0 for any vertex $x_0 \in Q_0$ [12]. Note that the definition is independent of the choice of x_0, and we write $\Pi_1(Q, I)$ for short. A triangular algebra A is said to be *simply connected* if for any presentation $A \cong kQ/I$, the fundamental group $\Pi_1(Q, I)$ is trivial.

The following lemma implies that for a representation-infinite algebra A, the category $C_1(\mathrm{proj}\, A)$ is of strongly unbounded type.

Lemma 2.1. *If A is a representation-infinite algebra, then $C_1(\mathrm{proj}\, A)$ is of strongly unbounded type.*

Proof. If A is representation-infinite, then A is of strongly representation unbounded type by Nazarova–Roiter's theorem on the typical Brauer–Thrall conjecture II [21], i.e., there exist an infinite sequence $\{d_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and infinitely many indecomposable A-modules $\{M_{ij} \mid i, j \in \mathbb{N}\}$ which are pairwise different up to isomorphisms such that $\dim(M_{ij}) = d_i$ for all $j \in \mathbb{N}$. For any M_{ij}, we can take a minimal presentation $P^{-1} \xrightarrow{d} P^0 \to M_{ij} \to 0$. Let

$$P^{\bullet}_{ij} = \cdots \to 0 \to P^{-1} \xrightarrow{d} P^0 \to 0 \to \cdots$$

with P^{-1} in degree 0. Then $P^{\bullet}_{ij} \in C_1(\mathrm{proj}\, A)$ is indecomposable by [16, Proposition 2] with $\dim H^1(P^{\bullet}_{ij}) = d_i$. Moreover, $P^{\bullet}_{ij}$ are non-isomorphic for different $i, j \in \mathbb{N}$. Since $P^{\bullet}_{ij}$ is a minimal presentation of M_{ij}, $\dim P^{-1} \leq (\dim A)^2 \cdot d_i$ and we have

$$d_i \leq \mathrm{hr}(P^{\bullet}_{ij}) \leq 2 \cdot (\dim A)^2 \cdot d_i.$$

With the similar argument in the proof of Lemma 1.6, we can construct a sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and pairwise non-isomorphic objects $\{Q^{\bullet}_{ij} \mid i, j \in \mathbb{N}\}$ such that $\mathrm{hr}(Q^{\bullet}_{ij}) = r_i$. Thus $C_1(\mathrm{proj}\, A)$ is of strongly unbounded type by Lemma 1.6. □

The following lemma observes the strongly unboundedness of $C_m(\mathrm{proj}\, A)$ under the derived equivalences.

Proposition 2.2. *Let A be a finite-dimensional algebra with $C_m(\mathrm{proj}\, A)$ strongly unbounded for some m and* $\mathrm{gl.dim} A < \infty$. *If there is an algebra B derived equivalent to A, then $C_{m'}(\mathrm{proj}\, B)$ is of strongly unbounded type for some m'.*

Proof. Assume that $C_m(\text{proj}\, A)$ is strongly unbounded, then there exist an increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and pairwise non-isomorphic objects $\{P^\bullet_{ij} \mid i, j \in \mathbb{N}\}$ in $C_m(\text{proj}\, A)$ such that $\text{hr}(P^\bullet_{ij}) = r_i$ by Lemma 1.6. Moreover, since A and B are derived equivalent, there is a two-sided tilting complex in $D^b(A^{op} \otimes B)$

$$ {}_AT^\bullet_B = 0 \to T^{-l} \to T^{-l+1} \to \cdots \to T^{-1} \to T^0 \to 0, $$

such that $F = - \otimes^L_A T^\bullet_B : D^b(A) \to D^b(B)$ is an equivalence [22]. Note that gl.dim$A < \infty$ implies gl.dim$B < \infty$ [17, Chapter III, Lemma 1.5]. If gl.dim$B = n$, then there is a minimal projective B-B-bimodule resolution of B [18, Lemma 1.5]

$$ R^\bullet = 0 \longrightarrow R^{-n} \xrightarrow{d^{-n}} R^{-n+1} \xrightarrow{d^{-n+1}} \cdots \longrightarrow R^{-1} \xrightarrow{d^{-1}} R^0 \longrightarrow 0. $$

Then for any $i, j \in \mathbb{N}$, $F(P^\bullet_{ij}) = P^\bullet_{ij} \otimes^L_A T^\bullet_B \cong P^\bullet_{ij} \otimes^L_A T^\bullet \otimes^L_B R^\bullet_B$, which is a projective B-module complex of width less than $m + l + n$. Thus, without loss of generality, we can assume $F(P^\bullet_{ij}) \in C_{m+l+n}(\text{proj}\, B)$ with suitable shifts and isomorphisms for any $i, j \in \mathbb{N}$. By [16, Proposition 1(3)], we have two integers N, N', such that

$$ \frac{1}{N'} \cdot \text{hr}(P^\bullet_{ij}) \le \text{hr}(F(P^\bullet_{ij})) \le N \cdot \text{hr}(P^\bullet_{ij}). $$

With a similar discussion as the proof of Lemma 1.6, we shall find inductively an increasing sequence $\{r'_s \mid s \in \mathbb{N}\}$ and infinitely many indecomposable pairwise non-isomorphic objects $\{Q^\bullet_{st} \in C_{m+l+n}(\text{proj}\, B) \mid s, t \in \mathbb{N}\} \subseteq \{F(P^\bullet_{ij}) \mid i, j \in \mathbb{N}\}$ such that $\text{hr}(Q^\bullet_{st}) = r'_s$. Thus the lemma follows by Lemma 1.6. □

Corollary 2.3. *Let A be a simply connected algebra. If A is strongly derived unbounded, then there exists an integer m such that $C_m(\text{proj}\, A)$ is of strongly unbounded type.*

Proof. By the proof of [16, Lemma 2], any simply connected algebra is tilting equivalent to a hereditary algebra of Dynkin type or a representation-infinite algebra. If A is strongly derived unbounded, then A is tilting equivalent to a representation-infinite algebra. Since simply connected algebras are triangular algebras and then of finite global dimension, by the previous proposition and Lemma 2.1, $C_m(\text{proj}\, A)$ is of strongly unbounded type for some integer m. □

2.2. Cleaving functors and the strongly unboundedness of $C_m(\text{proj}\, A)$

In the context of cleaving functors, bound quiver algebras are viewed as bounded categories, see [14] for details. In the rest of this paper, we will replace bound quiver algebras by bounded categories.

A k-linear category A is a category together with k-vector space structure on the set $A(x, y)$ of all morphisms from $x \in A$ to $y \in A$ such that the composition of morphisms is bilinear. We say a k-linear category A is a *locally bounded category* if

(1) different objects in A are non-isomorphic;
(2) for any $a \in A$, the endomorphism algebra $A(a, a)$ is local;
(3) $\dim_k \sum_{x \in A} A(a, x) < \infty$ and $\dim_k \sum_{x \in A} A(x, a) < \infty$ for all $a \in A$.

A locally bounded category is a *bounded category* if it has only finitely many objects. Note that a bound quiver algebra $A = kQ/I$ with I admissible can be viewed as a bounded category by seeing the vertexes $i \in Q_0$ as objects and the combinations of paths in kQ/I as morphisms. Conversely, a bounded category A admits a presentation $A \cong kQ_A/I_A$ with Q_A finite and I_A admissible.

Let A be a locally bounded category. A *right A-module* M is just a covariant k-linear functor from A to the category of k-vector spaces. Denote by $\mathrm{Mod}\, A$ the category of all right A-modules M with $\dim M(a) < \infty$ for any $a \in A$. For any $M \in \mathrm{Mod}\, A$, the *dimension vector* of M is $\mathbf{dim} M := (\dim M(a))_{a \in A}$, and the *support* of M is $\mathrm{Supp}\, M := \{a \in A \mid M(a) \neq 0\}$. Denote by $\mathrm{mod}\, A$ the full subcategory of $\mathrm{Mod}\, A$ consisting of all A-modules M such that $\mathrm{Supp}\, M$ is finite. The *dimension* of $M \in \mathrm{mod}\, A$ is $\dim M := \sum_{a \in A} \dim_k M(a)$. The indecomposable projective A-modules are $P_a = A(a, -)$ and indecomposable injective A-modules are $I_a = DA(-, a)$ for all $a \in A$, where $D = \mathrm{Hom}_k(-, k)$. Moreover, all the concepts and notations defined for a bound quiver algebra make sense for a bounded category.

To a k-linear functor $F : B \to A$ between bounded categories, we associate a *restriction functor* $F_* : \mathrm{mod}\, A \to \mathrm{mod}\, B$, which is given by $F_*(M) = M \circ F$ and exact. The restriction functor F_* admits a left adjoint functor F^*, called the *extension functor*, which sends a projective B-module $B(b, -)$ to a projective A-module $A(Fb, -)$. If $\mathrm{gl.dim} B < \infty$ then F_* extends naturally to a derived functor $F_* : D^b(A) \to D^b(B)$, which has a left adjoint $\mathbf{L}F^* : D^b(B) \to D^b(A)$. Note that $\mathbf{L}F^*$ is the left derived functor associated with F^* and maps $K^b(\mathrm{proj}\, B)$ into $K^b(\mathrm{proj}\, A)$. We refer to [27] for the definition of derived functors.

A k-linear functor $F : B \to A$ between bounded categories with $\mathrm{gl.dim} B < \infty$ is called a *cleaving functor* [7,26] if it satisfies the following equivalent conditions:

(1) The linear map $B(b, b') \to A(Fb, Fb')$ associated with F admits a natural retraction for all $b, b' \in B$;
(2) The adjunction morphism $\phi_M : M \to (F_* \circ F^*)(M)$ admits a natural retraction for all $M \in \mathrm{mod}\, B$;
(3) The adjunction morphism $\Phi_{X^\bullet} : X^\bullet \to (F_* \circ \mathbf{L}F^*)(X^\bullet)$ admits a natural retraction for all $X^\bullet \in D^b(B)$.

Proposition 2.4. *Let B be a bounded category of finite global dimension and $C_m(\mathrm{proj}\, B)$ be of strongly unbounded type for some m. If there is a cleaving functor $F : B \to A$, then $C_m(\mathrm{proj}\, A)$ is of strongly unbounded type.*

Proof. Suppose there exist an increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and pairwise non-isomorphic objects $\{P_{ij}^\bullet \mid i, j \in \mathbb{N}\}$ in $C_m(\mathrm{proj}\, B)$ such that $\mathrm{hr}(P_{ij}^\bullet) = r_i$. Since F is a cleaving functor, $\mathbf{L}F^*(P_{ij}^\bullet) = F^*(P_{ij}^\bullet)$ for any $i, j \in \mathbb{N}$, which is projective A-module complex of width less than m by the definition of F^*. Then, with suitable isomorphisms, we can assume $\mathbf{L}F^*(P_{ij}^\bullet)$ lies in $C_m(\mathrm{proj}\, A)$. Moreover, for any $i, j \in \mathbb{N}$, $P_{ij}^\bullet$ is a direct summand of $(F_* \circ \mathbf{L}F^*)(P_{ij}^\bullet)$. Thus for any $P_{ij}^\bullet$, we can choose an indecomposable direct summand $Q_{ij}^\bullet$ of $\mathbf{L}F^*(P_{ij}^\bullet)$, such that $P_{ij}^\bullet$ is a direct summand of $F_*(Q_{ij}^\bullet)$. Note that for any $i \in \mathbb{N}$, the set $\{Q_{ij}^\bullet \mid j \in \mathbb{N}\}$ contains infinitely many elements which are pairwise non-isomorphic since the set $\{P_{ij}^\bullet \mid j \in \mathbb{N}\}$ contains infinitely many pairwise non-isomorphic elements. Moreover, by the proof of [16, Proposition 5(1)], there exist two integers N, N', such that for any $i, j \in \mathbb{N}$, we have the inequality $\frac{1}{N'} \cdot \mathrm{hr}(P_{ij}^\bullet) \leq \mathrm{hr}(Q_{ij}^\bullet) \leq N \cdot \mathrm{hr}(P_{ij}^\bullet)$. Here we give the proof for the convenience of readers.

On one hand, for any $a \in A$, we have

$$\begin{aligned} H^m(\mathbf{L}F^*(P_{ij}^\bullet))(a) &\cong \mathrm{Hom}_{D^b(A)}(\mathbf{L}F^*(P_{ij}^\bullet), I_a[m]) \\ &\cong \mathrm{Hom}_{D^b(B)}(P_{ij}^\bullet, F_*(I_a)[m]) \\ &\cong H^m(\mathbf{R}\mathrm{Hom}_B(P_{ij}^\bullet, F_*(I_a))). \end{aligned}$$

Since $\mathrm{gl.dim} B < \infty$, $F_*(I_a)$ admits a minimal injective resolution

$$0 \to F_*(I_a) \to E_a^0 \to E_a^1 \to \cdots \to E_a^{r_a} \to 0,$$

and there is a bounded converging spectral sequence

$$\mathrm{Ext}_B^p(H^{-q}(P_{ij}^\bullet), F_*(I_a)) \Rightarrow H^{p+q}(\mathbf{R}\mathrm{Hom}_B(P_{ij}^\bullet, F_*(I_a))),$$

thus $\mathrm{hw}(Q^\bullet_{ij}) \leq \mathrm{hw}(\mathbf{L}F^*(P^\bullet_{ij})) \leq \mathrm{hw}(P^\bullet_{ij}) + \mathrm{gl.dim}B$, and

$$\begin{aligned}
\dim H^m(Q^\bullet_{ij}) &= \sum_{a\in A} \dim H^m(Q^\bullet_{ij})(a) \\
&\leq \sum_{a\in A} \dim H^m(\mathbf{L}F^*(P^\bullet_{ij}))(a) \\
&= \sum_{a\in A} \dim H^m(\mathbf{R}\mathrm{Hom}_B(P^\bullet_{ij}, F_*(I_a))) \\
&\leq \sum_{a\in A} \sum_{p+q=m} \dim \mathrm{Ext}^p_B(H^{-q}(P^\bullet_{ij}), F_*(I_a)) \\
&\leq \sum_{a\in A} \sum_{p=0}^{r_a} \dim H^{p-m}(P^\bullet_{ij}) \cdot \dim E^p_a \\
&\leq \sum_{a\in A} \mathrm{hl}(P^\bullet_{ij}) \cdot (r_a+1) \cdot \max_{0\leq p\leq r_a} \{\dim E^p_a\} \\
&\leq n_0(A) \cdot \mathrm{hl}(P^\bullet_{ij}) \cdot (\mathrm{gl.dim}B + 1) \cdot \max_{a\in A,\, 0\leq p\leq r_a} \{\dim E^p_a\},
\end{aligned}$$

where $n_0(A)$ denotes the number of objects in A.

Set $N_0 = n_0(A) \cdot (\mathrm{gl.dim}B + 1) \cdot \max_{a\in A,\, 0\leq p\leq r_a} \{\dim E^p_a\}$. Then

$$\begin{aligned}
\mathrm{hr}(Q^\bullet_{ij}) &= \mathrm{hw}(Q^\bullet_{ij}) \cdot \mathrm{hl}(Q^\bullet_{ij}) \\
&\leq (\mathrm{hw}(P^\bullet_{ij}) + \mathrm{gl.dim}B) \cdot N_0 \cdot \mathrm{hl}(P^\bullet_{ij}) \\
&\leq N_0 \cdot (\mathrm{gl.dim}\, B + 1) \cdot \mathrm{hr}(P^\bullet_{ij}).
\end{aligned}$$

On the other hand, assume that the indecomposable projective B-module $P_b = B(b, -)$ for all $b \in B$ and indecomposable projective A-module $Q_a = A(a, -)$ for $a \in A$. Then

$$\begin{aligned}
\dim H^m(P^\bullet_{ij}) &\leq \dim H^m(F_*(Q^\bullet_{ij})) \\
&= \sum_{b\in B} \dim \mathrm{Hom}_{D^b(B)}(P_b, F_*(Q^\bullet_{ij})[m]) \\
&= \sum_{b\in B} \dim \mathrm{Hom}_{D^b(A)}(\mathbf{L}F^*(P_b), Q^\bullet_{ij}[m]) \\
&= \sum_{b\in B} \dim \mathrm{Hom}_{D^b(A)}(F^*(P_b), Q^\bullet_{ij}[m]) \\
&= \sum_{b\in B} \dim \mathrm{Hom}_{D^b(A)}(Q_{F(b)}, Q^\bullet_{ij}[m]) \\
&\leq n_0(B) \cdot \sum_{a\in A} \dim \mathrm{Hom}_{D^b(A)}(Q_a, Q^\bullet_{ij}[m]) \\
&\leq n_0(B) \cdot \dim H^m(Q^\bullet_{ij}),
\end{aligned}$$

where $n_0(B)$ denotes the number of objects in B. Thus $\mathrm{hl}(P^\bullet_{ij}) \leq n_0(B) \cdot \mathrm{hl}(Q^\bullet_{ij})$, $\mathrm{hw}(P^\bullet_{ij}) \leq \mathrm{hw}(Q^\bullet_{ij})$, and $\mathrm{hr}(Q^\bullet_{ij}) \geq \frac{1}{n_0(B)} \cdot \mathrm{hr}(P^\bullet_{ij})$.

Consequently, with a similar discussion as in the proof of Lemma 1.6, $C_m(\mathrm{proj}\, A)$ is of strongly unbounded type. □

2.3. The proof of the main theorem

Let A be a bounded category. Recall that the repetitive category $\hat{A}$ of A has the pairs (a,i) as objects, where $a \in A$ and $i \in \mathbb{Z}$, while the morphisms from (a,i) to (b,i) and $(b,i+1)$ are determined by $A(a,b)$ and $A(b,a)$ respectively, and zero else [19]. Note that $\hat{A}$ is self-injective locally bounded category. Moreover, there is a full embedding $F : D^b(A) \to \underline{\text{mod}}\,\hat{A}$ of triangulated categories [17].

Recall from [26], A is said to be *derived discrete* if for any $d \in \mathbb{N}$, there are only finitely many indecomposables in $D^b(A)$ with cohomological range d. Moreover, $K^b(\text{proj}\,A)$ is *discrete* if for any $d \in \mathbb{N}$, there are only finitely many indecomposables in $K^b(\text{proj}\,A)$ of cohomological range d.

Definition 2.5. A locally bounded category B is said to be *of discrete representation type* if for any $\mathbf{d} \in \mathbb{N}^{|B|}$, there are only finitely many indecomposable objects $M \in \text{mod}\,B$ with $\mathbf{dim}M = \mathbf{d}$. Moreover, we say B is *of strongly unbounded representation type* if there are infinitely many $\mathbf{d} \in \mathbb{N}^{|B|}$ such that for each $\mathbf{d}$, there are infinitely many indecomposables in $\text{mod}\,B$ with dimension vector $\mathbf{d}$.

The following lemma is the classification theorem of derived discrete algebras due to Vossieck [26, Theorem].

Lemma 2.6. *Let A be a bounded category. Then the following statements are equivalent:*

(1) *$\hat{A}$ is of discrete representation type;*
(2) *A is derived discrete;*
(3) *$K^b(\text{proj}\,A)$ is discrete;*
(4) *A is piecewise hereditary of Dynkin type or admits a presentation kQ/I with Q one-cycle gentle quiver such that the numbers of clockwise and of counterclockwise paths of length two which belongs to I are different.*

Definition 2.7. Let A be a bounded category. The category $K^b(\text{proj}\,A)$ is said to be *of strongly unbounded type* if there is an increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ such that for each r_i, up to shifts and isomorphisms, there are infinitely many indecomposable objects in $K^b(\text{proj}\,A)$ of cohomological range r_i.

Now we can prove the main theorem.

Theorem 2.8. *Let A be a bounded category. Then the following statements are equivalent:*

(1) *A is strongly derived unbounded;*
(2) *There exists an integer $m \geq 1$, such that the category $C_m(\text{proj}\,A)$ is of strongly unbounded type;*
(3) *$K^b(\text{proj}\,A)$ is of strongly unbounded type;*
(4) *$\hat{A}$ is of strongly unbounded representation type.*

Proof. (1) $\Rightarrow$ (2): We assume for any integer $m > 0$, $C_m(\text{proj}\,A)$ is not of strongly unbounded type. Then A is representation-finite by Lemma 2.1. Thus for any object $a \in A$, we have $A(a,a)$ is a uniserial local algebra, and thus $A(a,a) \cong k$ or $A(a,a) \cong k[x]/(x^l)$ with $l \geq 2$. Moreover, we will exclude the possibility that $A(a,a) \cong k[x]/(x^l)$, $l \geq 3$. Indeed, we consider the functor $F : A^l_{3l} \to A$ given by $F(i) = a$ and $F(\alpha_i) = x$, where A^l_{3l} is the bounded category defined by the quiver

$$3l \xrightarrow{\alpha_{3l-1}} 3l-1 \xrightarrow{\alpha_{3l-2}} \cdots \xrightarrow{\alpha_2} 2 \xrightarrow{\alpha_1} 1,$$

and the admissible ideal generated by all paths of length l. Note that F is a cleaving functor. If $l \geq 3$, then we assume $w_1 = \alpha_{3l-1}$, $w_2 = \alpha_{3l-2} \cdots \alpha_{2l}$, $w_3 = \alpha_{2l-1} \cdots \alpha_{l+1}$, $w_4 = \alpha_l \cdots \alpha_2$, $w_5 = \alpha_1$, $w_1' = \alpha_{3l-1} \cdots \alpha_{2l+1}$, $w_2' = \alpha_{2l}$, $w_3' = w_3$, $w_4' = \alpha_l$, and $w_5' = \alpha_{l-1} \cdots \alpha_1$. By the construction in [16, Lemma 4], there exists a family of complexes $\{P^\bullet_{\lambda,d} \mid \lambda \in k,\ d \geq 1\}$ in $D^b(A^l_{3l})$, where

$$P^\bullet_{\lambda,d} := 0 \longrightarrow P_1^d \xrightarrow{\delta^0} P_l^d \oplus P_2^d \xrightarrow{\delta^1} P_{l+1}^d \oplus P_{l+1}^d \xrightarrow{\delta^2} P_{2l}^d \oplus P_{2l}^d \xrightarrow{\delta^3} P_{2l+1}^d \oplus P_{3l-1}^d \xrightarrow{\delta^4} P_{3l}^d \longrightarrow 0$$

with the differential maps

$$\delta^0 := \begin{pmatrix} P(w_5')\mathbf{I}_d \\ P(w_5)\mathbf{J}_{\lambda,d} \end{pmatrix}, \quad \delta^i := \begin{pmatrix} P(w_{5-i}')\mathbf{I}_d & 0 \\ 0 & P(w_{5-i})\mathbf{I}_d \end{pmatrix}, \quad \text{for } i = 1,2,3,$$

and $\delta^4 := (P(w_1')\mathbf{I}_d, P(w_1)\mathbf{I}_d)$. Here $\mathbf{J}_{\lambda,d}$ denotes the $d \times d$ Jordan block with eigenvalue $\lambda \in k$, and the map $P(u)$ from $P_{t(u)}$ to $P_{s(u)}$ is the left multiplication by the path u. It is straightforward to check that $\{P^\bullet_{\lambda,d} \mid \lambda \in k,\ d \geq 1\}$ are pairwise non-isomorphic indecomposables, and $\dim P^\bullet_{\lambda,d} = \dim P^\bullet_{\lambda',d'}$ if and only if $d = d'$. Then $C_5(\operatorname{proj} A^l_{3l})$ is strongly unbounded, and thus $C_5(\operatorname{proj} A)$ is of strongly unbounded type by Proposition 2.4, which is a contradiction. Therefore, for any $a \in A$, $A(a,a) \cong k$ or $A(a,a) \cong k[x]/(x^2)$. By [7, Section 9], A is standard since A contains no Riedtmann contours.

If A is simply connected, then A is not strongly unbounded by Corollary 2.3. Assume A is not simply connected, then there is a Galois covering $\pi : \tilde{A} \to A$ with non-trivial free Galois group G and $\tilde{A}$ simply connected [10,13]. Now we consider any finite full convex subcategory B of $\tilde{A}$. Then B is also simply connected. Since the composition of the embedding $i : B \hookrightarrow \tilde{A}$ and π is cleaving functor, B is not strongly derived unbounded by Corollary 2.3. Thus B is piecewise hereditary of Dynkin type [16, Lemma 2]. Then B is piecewise hereditary of type $\mathbf{A}$ with the same argument as that in the proof of [26, Lemma 4.4] and $\tilde{A}$ admits a presentation given by a gentle quiver (Q, I) (Ref. [1, Theorem]), and so does A. By Bekkert and Merklen's classification on the indecomposable objects in the derived category of a gentle algebra [8], if A contains a generalized band w then we can construct a family of pairwise non-isomorphic indecomposables $P^\bullet_{w,f}$ for $f = (x - \lambda)^d \in k[x]$ in $C_m(\operatorname{proj} A)$ for some integer m, such that $P^\bullet_{w,f}$ and $P^\bullet_{w,f'}$ have the same dimension if and only if $\deg(f) = \deg(f')$, where $\lambda \in k \setminus \{0\}$ and $d > 0$. Then $C_m(\operatorname{proj} A)$ is of strongly unbounded type, which is a contradiction to the assumption. Thus A contains no generalized bands and then A is derived discrete by [8, Theorem 4]. It contradicts to the strongly derived unboundedness of A and hence $C_m(\operatorname{proj} A)$ is of strongly unbounded type for some $m > 0$.

(2) ⇒ (3): Suppose that there exists an integer $m \geq 1$, such that $C_m(\operatorname{proj} A)$ is of strongly unbounded type. Then by Lemma 1.6, there exist an increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and pairwise non-isomorphic objects $\{P^\bullet_{ij} \mid i,j \in \mathbb{N}\}$ in $C_m(\operatorname{proj} A)$ such that $\operatorname{hr}(P^\bullet_{ij}) = r_i$. Since the elements in $\{P^\bullet_{ij} \mid i,j \in \mathbb{N}\}$, seen as objects in $K^b(\operatorname{proj} A)$, are also pairwise non-isomorphic indecomposables, $K^b(\operatorname{proj} A)$ is strongly unbounded.

(3) ⇒ (1): Trivial.

(2) ⇒ (4): If $C_m(\operatorname{proj} A)$ is of strongly unbounded type, then there exist an increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and pairwise non-isomorphic complexes $\{P^\bullet_{ij} \mid i,j \in \mathbb{N}\}$ in $C_m(\operatorname{proj} A)$ such that $\operatorname{hr}(P^\bullet_{ij}) = r_i$ by Lemma 1.6. Note that $\{P^\bullet_{ij} \mid i,j \in \mathbb{N}\}$ are pairwise non-isomorphic indecomposables viewed as objects in $D^b(A)$ by Lemma 1.5. Assume $\{S_a \mid a \in A\}$ and $\{S_h \mid h \in \hat{A}\}$ are the sets of all simple A-modules and $\hat{A}$-modules respectively.

Now we consider the full embedding $F : D^b(A) \to \underline{\operatorname{mod}}\, \hat{A}$. Note that $P^\bullet_{ij}$ is generated by the cohomologies via triangles and the cohomologies can be also obtained by triangles with the simples. Since F sends a triangle in $D^b(A)$ to a triangle in $\underline{\operatorname{mod}}\, \hat{A}$, by the additivity of dimension functor $\mathbf{dim}(-)$ in $\underline{\operatorname{mod}}\, \hat{A}$, we have the following estimate for any $i,j \in \mathbb{N}$ (see also [26])

$$\mathbf{dim}F(P^\bullet_{ij}) \leq \sum_{a\in A}\sum_{l=0}^{m} \dim H^l(P^\bullet_{ij})(a)\cdot \mathbf{dim}F(S_a[l]) \leq \mathrm{hr}(P^\bullet_{ij})\cdot \sum_{a\in A}\sum_{l=0}^{m}\mathbf{dim}F(S_a[l]).$$

Set $\mathbf{d} = \sum_{a\in A}\sum_{l=0}^{m}\mathbf{dim}F(S_a[l])$. Then $\mathbf{dim}F(P^\bullet_{ij}) \leq r_i\cdot \mathbf{d}$.

On the other hand, for any $i, j \in \mathbb{N}$, we have

$$\begin{aligned}
\mathrm{hr}(P^\bullet_{ij}) &\leq \mathrm{hw}(P^\bullet_{ij})\cdot \sum_{l\in\mathbb{Z}} \dim H^l(P^\bullet_{ij}) \leq (m+1)\cdot \sum_{l=0}^{m} \dim \mathrm{Hom}_{D^b(A)}(A[-l], P^\bullet_{ij})\\
&= (m+1)\cdot \sum_{l=0}^{m} \dim \mathrm{Hom}_{\hat{A}}(F(A[-l]), F(P^\bullet_{ij}))\\
&\leq (m+1)\cdot \sum_{l=0}^{m}\sum_{h\in\hat{A}} c_h(F(A[-l]))\dim \mathrm{Hom}_{\hat{A}}(S_h, F(P^\bullet_{ij}))\\
&\leq (m+1)\cdot \sum_{l=0}^{m}\sum_{h\in\hat{A}} c_h(F(A[-l]))\dim \mathrm{Hom}_{\hat{A}}(P_h, F(P^\bullet_{ij}))\\
&\leq (m+1)\cdot \sum_{l=0}^{m}\sum_{h\in\hat{A}} c_h(F(A[-l]))\dim F(P^\bullet_{ij})(h),
\end{aligned}$$

where $c_h(F(A[-l]))$ denotes the number of composition factors of $F(A[-l])$ isomorphic to S_h, and P_h is the indecomposable projective $\hat{A}$-module associated with $h \in \hat{A}$. Set $c = \sup\{c_h(F(A[-l])) \mid 0 \leq l \leq m,\ h \in \hat{A}\}$. Then for any $i, j \in \mathbb{N}$, $\mathrm{hr}(P^\bullet_{ij}) \leq c\cdot (m+1)^2 \cdot \dim F(P^\bullet_{ij})$.

To prove $\hat{A}$ is of strongly unbounded representation type, we shall find inductively infinitely many vectors $\{\mathbf{d_i} \mid i \in \mathbb{N}\}$ and infinitely many indecomposable objects $\{M^\bullet_{ij} \in \mathrm{mod}\,\hat{A} \mid i, j \in \mathbb{N}\}$ which are pairwise different up to isomorphisms such that $\mathbf{dim}M^\bullet_{ij} = \mathbf{d_i}$ for all $j \in \mathbb{N}$. For $i = 1$, we have $0 < \mathbf{dim}F(P^\bullet_{1j}) \leq r_1\cdot\mathbf{d}$. Then there exist $\mathbf{d_1} \in \mathbb{N}^{|\hat{A}|}$ with $0 < \mathbf{d_1} \leq r_1\cdot\mathbf{d}$ and infinitely many indecomposable objects $\{M_{1j} \mid j \in \mathbb{N}\} \subseteq \{F(P^\bullet_{1j}) \mid j \in \mathbb{N}\}$ of dimension vector $\mathbf{d_1}$. Assume that we have done for i. Set $d_i = \sum_{j\in\mathbb{Z}}(\mathbf{d_i})_j$. Then we can choose r_l with $r_l > c(m+1)^2\cdot d_i$, and thus $d_i < \frac{1}{c(m+1)^2}\cdot \mathrm{hr}(P^\bullet_{lj}) \leq \dim(F(P^\bullet_{lj}))$. Since $\mathbf{dim}F(P^\bullet_{lj}) \leq r_l\cdot\mathbf{d}$, we can choose a vector $\mathbf{d_{i+1}}$, which is different from $\{\mathbf{d_s} \mid s = 1, 2, \cdots, i\}$, such that $\mathbf{d_{i+1}} \leq r_l\cdot\mathbf{d}$, and infinitely many pairwise non-isomorphism indecomposable objects $\{M_{i+1,j} \mid j \in \mathbb{N}\} \subseteq \{F(P^\bullet_{lj}) \mid j \in \mathbb{N}\}$ with $\mathbf{dim}M_{i+1,j} = \mathbf{d_{i+1}}$ for all $j \in \mathbb{N}$.

(4) ⇒ (1): If A is not strongly derived unbounded, then by [16, Theorem 2], A is derived discrete. Thus $\hat{A}$ is representation discrete by Lemma 2.6, which is a contradiction with the assumption. □

Recall that for an algebra A and a fixed integer m, the category $C_m(\mathrm{proj}\,A)$ is said to be *of finite representation type* if $C_m(\mathrm{proj}\,A)$ contains only finitely many indecomposables up to isomorphisms [6]. As a corollary of the previous theorem, we obtain the dichotomy on the representation type of $C_m(\mathrm{proj}\,A)$, $K^b(\mathrm{proj}\,A)$ and also the repetitive algebra $\hat{A}$.

Corollary 2.9. *Let A be a finite-dimensional algebra. Then we have*

(1) $C_m(\mathrm{proj}\,A)$ *is either of finite representation type for any m, or of strongly unbounded type for all but finitely many m;*
(2) $K^b(\mathrm{proj}\,A)$ *is either discrete or of strongly unbounded type;*
(3) *The repetitive algebra $\hat{A}$ is either of discrete representation type or of strongly unbounded representation type.*

Proof. By [6, Theorem 2.4(1)], A is derived discrete if and only if $C_m(\text{proj}\, A)$ is of finite representation type for any m. Moreover, A is strongly derived unbounded if and only if $C_m(\text{proj}\, A)$ is of strongly unbounded type for some integer $m \geq 1$ by the previous theorem, which is also equivalent to that $C_m(\text{proj}\, A)$ is of strongly unbounded type for almost all m. Since any algebra A is either derived discrete or strongly derived unbounded by [16, Theorem 2], the statement (1) follows. Similarly, the statements (2) and (3) hold by Lemma 2.6 and the previous theorem. □

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