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On the representation type of subcategories of derived categories

Chao Zhang
Department of Mathematics, Guizhou University
Guiyang 550025, Guizhou, P. R. China
zhangc@amss.ac.cn

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Let A be a finite-dimensional k-algebra. In this paper, we mainly study the representation type of subcategories of the bounded derived category $D^b(A)$. First, we define the representation type and some homological invariants including cohomological length, width, range for subcategories. In this framework, we provide a characterization for derived discrete algebras. Moreover, for a finite-dimensional algebra A, we establish the first Brauer–Thrall type theorem of certain contravariantly finite subcategories $\mathcal{C}$ of $D^b(A)$, that is, $\mathcal{C}$ is of finite type if and only if its cohomological range is finite.

Keywords: Cohomological range; derived discrete algebras; contravariantly finite subcatgories.

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1. Introduction

Throughout this paper, k is an algebraically closed field, all algebras are connected basic finite-dimensional associative k-algebras with identity, and all modules are finite-dimensional right modules. The bounded derived categories of finite-dimensional algebras have been studied widely since Happel [9]. Derived representation type, including the classification and distribution of indecomposable objects in the bounded derived category, is still an important theme in representation theory of algebras. Vossieck defined the *derived discrete algebras* using the cohomology dimension vector of objects in the bounded derived category and classified the derived discrete algebras: piecewise hereditary algebras of Dynkin type and a special class of gentle algebras, in his paper [12]. In [13], the cohomological range of a bounded complex is introduced, which leads to the concepts of derived bounded algebras and strongly derived unbounded algebras naturally. Moreover, Brauer–Thrall type theorems for derived categories are established, in which the cohomological range of complexes plays a similar role as the dimension of modules in the classical Brauer–Thrall theorems. The first Brauer–Thrall theorem for the derived category states that the finiteness of the global cohomological range of an algebra implies the derived finiteness of this algebra, while the second one says that any algebra is either derived discrete or strongly derived unbounded.

Certain subcategories of the bounded derived category, for instance, the bounded homotopy category of projective modules, play an important role in the research of finite dimensional algebras. The representation type of these subcategories of derived category is an interesting theme. In this paper, we mainly study the representation type of the subcategories of the bounded derived category. We define the subcategories of finite and discrete type. Then we provide a characterization of derived discrete algebras by the dimension of the Hom-spaces between indecomposables in the bounded derived category. Moreover, we obtain an alternative definition of the cohomological width for *subcategories closed under gluing and cutting of objects*, i.e. subcategories whose objects closed under taking projective resolutions and certain permissible brutal truncations, see Sec. 3 for the precise definition. Finally, we consider what kind of conditions ensure that the first Brauer–Thrall type theorem for subcategories of the bounded derived category holds. By using the method of Auslander's classical proof for the first Brauer–Thrall conjecture for module categories, we establish the first Brauer–Thrall type theorem for those cohomology-homogeneous contravariantly finite subcategories $\mathcal{C}$ closed under gluing and cutting of objects, that is, $\mathcal{C}$ is of finite type if and only if its cohomological range is finite. The theorem can be regarded as a generalization of [13, Theorem 1].

The paper is organized as follows: in Sec. 2, we introduce the definition of cohomological invariants and representation type of subcategories, and then characterize the discreteness of the bounded derived category itself. In Sec. 3, we prove that the width equals the cohomological width plus one for subcategories closed under gluing and cutting of objects. The last section of this paper proves the first Brauer–Thrall type theorem for those contravariantly finite and cohomology-homogeneous subcategories closed under gluing and cutting of objects.

2. Some Definitions for Subcategories

Let A be a finite-dimensional algebra, and mod$A$ be the category of all finite-dimensional right A-modules and proj$A$ be the full subcategory of mod$A$ consisting of all projective right A-modules. Assume $C(A)$ is the category of all complexes of finite-dimensional right A-modules. Denote by $C^b(A)$ and $C^{-,b}(A)$ its full subcategories consisting of all bounded complexes and right bounded complexes with bounded cohomology respectively, by $C^b(\text{proj}A)$ and $C^{-,b}(\text{proj}A)$ the full subcategories of $C^b(A)$ and $C^{-,b}(A)$ respectively consisting of all complexes of projective modules. Moreover, $K(A)$, $K^b(\text{proj}A)$ and $K^{-,b}(\text{proj}A)$ are the homotopy categories of $C(A)$, $C^b(\text{proj}A)$ and $C^{-,b}(\text{proj}A)$ respectively, and $D^b(A)$ is the bounded derived category of mod$A$ with [1] the shift functor.

In [13], for any complex $X^\bullet \in D^b(A)$, the *cohomological length* is defined to be

$$\mathrm{hl}(X^\bullet) := \max\{\dim H^i(X^\bullet) \mid i \in \mathbb{Z}\},$$

the *cohomological width* of $X^\bullet$ is

$$\mathrm{hw}(X^\bullet) := \max\{j - i + 1 \mid H^i(X^\bullet) \neq 0 \neq H^j(X^\bullet)\},$$

and the *cohomological range* of $X^\bullet$ is defined as

$$\mathrm{hr}(X^\bullet) := \mathrm{hl}(X^\bullet) \cdot \mathrm{hw}(X^\bullet).$$

Note that these numerical invariants are preserved under shifts and isomorphism. Thanks to the well-known equivalence $D^b(A) \simeq K^{-,b}(\mathrm{proj}A)$, we always view any subcategory of $D^b(A)$ as a subcategory of $K^{-,b}(\mathrm{proj}A)$. Moreover, throughout this paper, by a subcategory of $K^{-,b}(\mathrm{proj}A)$ we always mean a full subcategory closed under isomorphisms and summands.

Definition 2.1. Let $\mathcal{C}$ be a subcategory of $K^{-,b}(\mathrm{proj}A)$. Then

(1) the cohomological length of $\mathcal{C}$ is

$$\mathrm{hl}(\mathcal{C}) := \sup\{\mathrm{hl}(X^\bullet) \mid X^\bullet \in \mathcal{C} \text{ is indecomposable}\};$$

(2) the cohomological width of $\mathcal{C}$ is

$$\mathrm{hw}(\mathcal{C}) := \sup\{\mathrm{hw}(X^\bullet) \mid X^\bullet \in \mathcal{C} \text{ is indecomposable}\};$$

(3) the cohomological range of $\mathcal{C}$ is

$$\mathrm{hr}(\mathcal{C}) := \sup\{\mathrm{hr}(X^\bullet) \mid X^\bullet \in \mathcal{C} \text{ is indecomposable}\}.$$

If the subcategory $\mathcal{C}$ does only contain the zero object then we set $\mathrm{hl}(\mathcal{C}) = \mathrm{hw}(\mathcal{C}) = \mathrm{hr}(\mathcal{C}) = 0$. and we formally define the cohomological length, width and range to be ∞ if the corresponding supremum does not exist. Moreover, it is obvious that if we take $\mathcal{C} = K^{-,b}(\mathrm{proj}A)$ then the definitions of these invariants are exactly the global cohomological length, width, range of A originally introduced in [13] respectively.

Now we define the representation type of the subcategories of $K^{-,b}(\mathrm{proj}A)$.

Definition 2.2. Let $\mathcal{C}$ be a subcategory of $K^{-,b}(\mathrm{proj}A)$. Then

(1) we say $\mathcal{C}$ is of *finite type* if there are only finitely many indecomposables in $\mathcal{C}$ up to isomorphism and shifts;
(2) we say $\mathcal{C}$ is of *discrete type* if for any integer r, there are only finitely many indecomposable objects in $\mathcal{C}$ with cohomological range r up to isomorphism and shifts.

Recall that in [12], an algebra A is said to be *derived discrete* if for any $\mathbf{d} = (d_i)_{i\in\mathbb{Z}} \in \mathbb{N}^{\mathbb{Z}}$, there are only finitely many indecomposables in $D^b(A)$ with cohomology dimension vector $\mathbf{d}$.

Lemma 2.3. *Suppose that A is a finite-dimensional k-algebra. Then $K^{-,b}(\text{proj}A)$ is of discrete type if and only if A is derived discrete.*

Proof. If $K^{-,b}(\text{proj}A)$ is of discrete type, then A is derived discrete obviously. For the other part, it is suffice to notice that for any $r \in \mathbb{N}$, only finitely many cohomology dimension vectors $\mathbf{d}$ take the value r as the cohomological range up to shifts. □

The following proposition characterizes finite-dimensional derived discrete algebras in terms of the dimension of Hom-spaces between indecomposables in the derived category. However, this is not true in general for derived-discrete abelian categories and the reference [8] gives an explicit example.

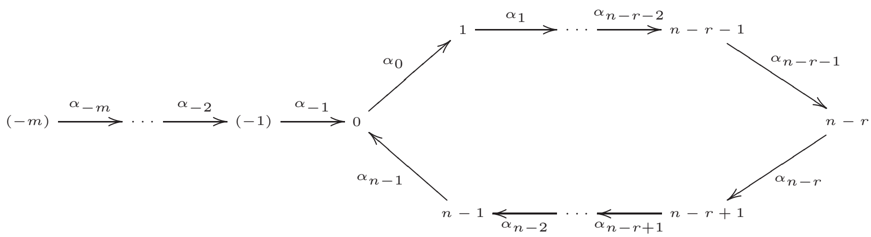
Theorem 2.4. *Let A be a finite-dimensional k-algebra. Then the following assertions are equivalent*:

(1) $\text{hl}(K^{-,b}(\text{proj}A)) < \infty$;
(2) $K^{-,b}(\text{proj}A)$ *is of discrete type*;
(3) $\dim \text{Hom}_{D^b(A)}(X^\bullet, Y^\bullet) \leq 6$ *for any indecomposables* $X^\bullet, Y^\bullet \in D^b(A)$;
(4) $\dim \text{Hom}_{D^b(A)}(X^\bullet, Y^\bullet) \leq l$ *for any indecomposable objects* $X^\bullet, Y^\bullet \in D^b(A)$ *and some integer* $l \geq 6$.

Moreover, $\text{hw}(K^{-,b}(\text{proj}A)) < \infty$ *if and only if A is a piecewise hereditary algebra*; $\text{hr}(K^{-,b}(\text{proj}A)) < \infty$ *if and only if A is piecewise hereditary of Dynkin type.*

Proof. The equivalence of (1) and (2) follows from [13, Proposition 6].

(2) ⇒ (3): Since A is derived discrete, A is either piecewise hereditary of Dynkin type or derived equivalent to $D^b(\Lambda(r,n,m))$ by Bobiński–Geiss–Skowroński's classification of derived discrete algebras [6, Theorem A], where $\Lambda(r,n,m)$ ($n \geq r \geq 1$, $m \geq 0$) is given by the quiver



with the relations $\alpha_{n-1}\alpha_0, \alpha_{n-2}\alpha_{n-1}, \ldots, \alpha_{n-r}\alpha_{n-r+1}$.

Let $F : D^b(A) \to D^b(B)$ be an equivalence as triangulated categories, then for any indecomposables $X^\bullet, Y^\bullet$ in $D^b(A)$,

$$\text{dimHom}_{D^b(A)}(X^\bullet, Y^\bullet) = \text{dimHom}_{D^b(B)}(FX^\bullet, FY^\bullet).$$

If $B = \Lambda(r,n,m)$ for some r, m, n, then by [7, Theorem 6.1], $\text{dimHom}_{D^b(A)}(X^\bullet, Y^\bullet) = \text{dimHom}_{D^b(B)}(FX^\bullet, FY^\bullet) \leq 2$. If $B = kQ$ for a Dynkin quiver Q, then we have $\mathrm{Hom}_{D^b(A)}(X^\bullet, Y^\bullet) \cong \mathrm{Hom}_{D^b(kQ)}(FX^\bullet, FY^\bullet)$, where $FX^\bullet, FY^\bullet$ are indecomposable stalk complexes. By [1, Chap. 4, Corollaries 2.14, 2.15], for any kQ-modules N and non-projective kQ-module M, we have

$$\mathrm{Hom}_{kQ}(M, N) \cong \mathrm{Hom}_{kQ}(\tau M, \tau N),$$

$$\mathrm{Ext}^1_{kQ}(N, M) \cong D\mathrm{Hom}_{kQ}(M, \tau N) \cong D\mathrm{Hom}_{kQ}(\tau M, \tau^2 N).$$

So without loss of generality, we can assume that M is an indecomposable projective kQ-module P_i. Then the dimension of the Hom-spaces or Ext-spaces is the ith component of the dimension vectors of some indecomposable kQ-module. Since Q is of Dynkin type, $\dim\mathrm{Hom}_{D^b(A)}(X^\bullet, Y^\bullet) = \dim\mathrm{Hom}_{D^b(B)}(FX^\bullet, FY^\bullet) \leq 6$.

(3) $\Rightarrow$ (4): It is clear.

(4) $\Rightarrow$ (1): Take $X^\bullet$ to be an indecomposable projective stalk complex $P_i[j]$, then for any indecomposable $Y^\bullet$ in $D^b(A)$, we have $\dim H^j(Y^\bullet) \leq nl$, where n is the number of simple A-modules. It follows that $\mathrm{hl}(K^{-,b}(\mathrm{proj}A)) \leq nl < \infty$.

The rest of statement was originally proved in [13]. □

3. Homological Invariants of Subcategories

In this section, we will study the homological invariants of subcategories introduced in the above section. Throughout this paper, we assume that all the complexes $(P^\bullet, d^\bullet)$ are *minimal* unless stated otherwise, that is, $\mathrm{Im}d^i \subseteq \mathrm{rad}P^{i+1}$ for any i.

The following result due to [13, Proposition 2] sets up the connection between the indecomposable objects in $K^b(\mathrm{proj}A)$ and those in $K^{-,b}(\mathrm{proj}A)$.

Proposition 3.1. *Let $P^\bullet \in K^{-,b}(\mathrm{proj}A)$ be a complex and $-n := \min\{i \in \mathbb{Z} \mid H^i(P^\bullet) \neq 0\}$. Then $P^\bullet$ is indecomposable if and only if so is the brutal truncation $\sigma_{\geq j}(P^\bullet) \in K^b(\mathrm{proj}A)$ for some (equivalently, all) $j < -n$.*

To be precise, the above proposition provides two natural methods as follows to construct indecomposable objects from known ones:

(1) (Gluing) If $Q^\bullet \in K^b(\mathrm{proj}A)$ is a indecomposable complex of the form

$$Q^\bullet = 0 \longrightarrow Q^{-n} \xrightarrow{\delta^{-n}} Q^{-n+1} \xrightarrow{\delta^{-n+1}} \cdots \xrightarrow{\delta^{-2}} Q^{-1} \xrightarrow{\delta^{-1}} Q^0 \longrightarrow 0$$

with $H^{-n}(Q^\bullet) = \mathrm{Ker}\delta^{-n} \neq 0$. We take a minimal projective resolution of $\mathrm{Ker}\delta^{-n}$, say

$$Q'^\bullet = \cdots \longrightarrow Q^{-n-2} \xrightarrow{\delta^{-n-2}} Q^{-n-1} \longrightarrow 0.$$

Gluing $Q'^\bullet$ and $Q^\bullet$ together, we get a minimal complex

$$Q''^\bullet = \cdots \longrightarrow Q^{-n-2} \xrightarrow{\delta^{-n-2}} Q^{-n-1} \xrightarrow{\delta^{-n-1}} Q^{-n} \xrightarrow{\delta^{-n}} \cdots \xrightarrow{\delta^{-1}} Q^0 \longrightarrow 0,$$

where δ^{-n-1} is the composition $Q^{-n-1} \twoheadrightarrow \mathrm{Ker}\delta^{-n} \hookrightarrow Q^{-n}$. Then by Proposition 3.1, $Q''^\bullet$ is indecomposable.

(2) (Cutting) If $P^\bullet \in K^{-,b}(\text{proj}A)$ is a minimal indecomposable complex of the form

$$P^\bullet = \cdots \longrightarrow P^{-n-2} \xrightarrow{d^{-n-2}} P^{-n-1} \xrightarrow{d^{-n-1}} P^{-n} \xrightarrow{d^{-n}} \cdots \xrightarrow{d^{-1}} P^0 \longrightarrow 0,$$

where $H^i(P^\bullet) = 0$ for $i < -n$. Then the *brutal truncation*

$$\sigma_{\geq j}(P^\bullet) = 0 \longrightarrow P^j \xrightarrow{d^j} \cdots \xrightarrow{d^{-1}} P^0 \longrightarrow 0$$

is indecomposable for $j < -n$.

Definition 3.2. Let $P^\bullet$ be a minimal indecomposable complex in $K^{-,b}(\text{proj}A)$. If another indecomposable complex $Q^\bullet$ can be obtained from $P^\bullet$ via the two methods above, then we say $Q^\bullet$ is *generated from $P^\bullet$ via gluing and cutting.*

Moreover, let $\mathcal{C} \subseteq K^{-,b}(\text{proj}A)$ be a subcategory, we say $\mathcal{C}$ is *closed under gluing and cutting of objects* if for any indecomposable object $P^\bullet$ in $\mathcal{C}$, the objects generated from $P^\bullet$ via gluing and cutting also lie in $\mathcal{C}$.

Proposition 3.3. *Let $\mathcal{C}$ be a subcategory of $K^{-,b}(\text{proj}A)$ closed under gluing and cutting of objects. Then*

(1) *for any indecomposable complex $P^\bullet \in \mathcal{C}$ such that $P^{-n-1} = 0$ and $H^{-n}(P^\bullet) \neq 0$, we have*

$$\text{hw}(\mathcal{C}) > \text{pd}H^{-n}(P^\bullet).$$

(2) $\text{hw}\,(\mathcal{C}) \geq \sup\{\text{pd}M \mid M$ *is an indecomposable module in* $\mathcal{C}\}$.

Proof. Note that $P^\bullet$ is of the form

$$0 \longrightarrow P^{-n} \xrightarrow{d^{-n}} \cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \longrightarrow 0$$

with $H^{-n}(P^\bullet) \neq 0$. We assume the $\text{pd}H^{-n}(P^\bullet) = m$, by gluing a minimal resolution of $H^{-n}(P^\bullet)$ to $P^\bullet$, we get an indecomposable object $Q^\bullet \in \mathcal{C}$ of the form

$$0 \longrightarrow P^{-n-m-1} \longrightarrow \cdots \longrightarrow P^{-n-1} \xrightarrow{d^{-n-1}} P^{-n} \xrightarrow{d^{-n}} \cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \longrightarrow 0.$$

Then $\sigma_{\geq -n-m}Q^\bullet$ is also indecomposable in $\mathcal{C}$ with $H^{-n-m}(\sigma_{\geq -n-m}Q^\bullet) \neq 0 \neq H^0(\sigma_{\geq -n-m}Q^\bullet)$, so its cohomological width is $n+m+1$. The first assertion follows. The second one follows by a similar argument on the brutal truncations of the minimal resolution of M. □

Definition 3.4. Let $P^\bullet \in K^{-,b}(\text{proj}A)$ be a minimal complex. We define its *width*

$$\text{width}(P^\bullet) := \begin{cases} \max\{j - i + 1 \mid P^i \neq 0 \neq P^j\}, & \text{if } P^\bullet \in K^b(\text{proj}A); \\ \infty, & \text{if } P^\bullet \notin K^b(\text{proj}A). \end{cases}$$

Moreover, for any subcategory $\mathcal{C} \subseteq K^{-,b}(\text{proj}A)$, we define the width of $\mathcal{C}$ to be

$$\text{width}(\mathcal{C}) = \sup\{\text{width}(P^\bullet) \mid P^\bullet \text{ is indecomposable in } \mathcal{C}\}.$$

The following proposition implies that for those full subcategories closed under gluing and cutting of objects, the width equals the cohomological width plus one, though the width and the cohomological width of any complex $P^\bullet$ have no evident relation except $\mathrm{width}(P^\bullet) \geq \mathrm{hw}(P^\bullet)$.

Proposition 3.5. *Let $\mathcal{C}$ be a full subcategories of $K^{-,b}(\mathrm{proj}A)$ closed under gluing and cutting of objects. If $\mathrm{hw}(\mathcal{C}) < \infty$, then $\mathrm{width}(\mathcal{C}) = \mathrm{hw}(\mathcal{C}) + 1$.*

Proof. To prove the statement, it suffices to show $\mathrm{width}(\mathcal{C}) \geq \mathrm{hw}(\mathcal{C}) + 1$ and $\mathrm{width}(\mathcal{C}) \leq \mathrm{hw}(\mathcal{C}) + 1$.

First, we prove $\mathrm{width}(\mathcal{C}) \geq \mathrm{hw}(\mathcal{C})+1$. It is enough to prove that for any indecomposable object $P^\bullet \in \mathcal{C}$ with $\mathrm{hw}(P^\bullet) = n$, there is an indecomposable object $Q^\bullet \in \mathcal{C}$ such that $\mathrm{width}(Q^\bullet) \geq n+1$. It is clear that $\mathrm{width}(P^\bullet) \geq n$. If $\mathrm{width}(P^\bullet) \geq n+1$, then $Q^\bullet = P^\bullet$ is the complex as required. If $\mathrm{width}(P^\bullet) = n$, then $P^\bullet$ is of the form

$$0 \longrightarrow P^{-n+1} \xrightarrow{d^{-n+1}} \cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \longrightarrow 0$$

such that $H^{-n+1}(P^\bullet) = \mathrm{Ker}d^{-n+1} \neq 0$ and $H^0(P^\bullet) \neq 0$. Now by gluing the minimal resolution of $\mathrm{Ker}d^{-n+1}$, we can construct an indecomposable object $Q^\bullet \in \mathcal{C}$ with $\mathrm{width}(Q^\bullet) \geq n+1$, since $\mathcal{C}$ is closed under gluing on objects.

Next, we will show $\mathrm{width}(\mathcal{C}) \leq \mathrm{hw}(\mathcal{C}) + 1$. If there is an indecomposable object $P^\bullet \in K^{-,b}(\mathrm{proj}A)\backslash K^b(\mathrm{proj}A)$, then $\mathrm{width}(\mathcal{C}) = \infty$. By Proposition 3.1, the complex $\sigma_{\geq -j}P^\bullet$ is indecomposable object in $\mathcal{C}$ for any $j >> 0$. Then we have $\mathrm{hw}(\mathcal{C}) = \infty$ since $\mathrm{hw}(\sigma_{\geq -j}P^\bullet) = j+1$. Now it is sufficient to show that for any minimal indecomposable complex $P^\bullet \in K^b(\mathrm{proj}A)$ with $\mathrm{width}(P^\bullet) = n+1$, there is a minimal indecomposable complex $Q^\bullet \in K^b(\mathrm{proj}A)$ such that $\mathrm{hw}(Q^\bullet) \geq n$. Note that $H^0(P^\bullet) \neq 0$ since $P^\bullet$ is minimal. If either $H^{-n}(P^\bullet)$ or $H^{-n+1}(P^\bullet)$ is nonzero, then $Q^\bullet = P^\bullet$ is the complex as required. We assume $H^{-n}(P^\bullet) = H^{-n+1}(P^\bullet) = 0$. By Proposition 3.1, $\sigma_{\geq -n+1}(P^\bullet)$ is indecomposable with cohomological width n. Thus the assertion follows since $\mathcal{C}$ is closed under gluing and cutting of objects. □

Remark 3.6. By the proof of previous proposition, if $\mathcal{C}$ is a subcategory of $K^{-,b}(\mathrm{proj}A)$ closed under gluing and cutting of objects with $\mathrm{hw}(\mathcal{C}) = n$, then

(1) $\mathcal{C} \subseteq K^b(\mathrm{proj}A)$;
(2) there is an indecomposable complex $P^\bullet$ in $\mathcal{C}$ with $\mathrm{width}(P^\bullet) = n+1$ and $\mathrm{hw}(P^\bullet) = n$.

The following corollary is a direct consequence of the previous proposition, which implies that the cohomological width of Hom-complexes between the indecomposables in the subcategory are controlled by the cohomological width of this subcategory.

Corollary 3.7. *If $\mathcal{C}$ is a subcategory of $K^{-,b}(\mathrm{proj}A)$ closed under gluing and cutting of objects, then $\mathrm{hw}(\mathcal{C}) < \infty$ if and only if $\mathrm{width}(\mathcal{C}) < \infty$. Moreover, if $\mathrm{hw}(\mathcal{C}) = n$, then for any objects $X^\bullet, Y^\bullet \in \mathcal{C}$, we have $\mathrm{hw}(\mathrm{Hom}^\bullet_A(X^\bullet, Y^\bullet[i])) \leq 2n+1$.*

Proof. The first part is clear. For the second one, it is known that

$$H^i\mathrm{Hom}_A^\bullet(X^\bullet, Y^\bullet) \cong \mathrm{Hom}_{D^b(A)}(X^\bullet, Y^\bullet[i]).$$

Moreover, by the assumption, the widths of $X^\bullet$ and $Y^\bullet$ are not larger than $n+1$. The corollary follows. □

4. The First Brauer–Thrall Type Theorem for Subcategories

Recall that from [13], the first Brauer–Thrall type theorem for the bounded derived category is that $\mathrm{hr}(K^{-,b}(\mathrm{proj}A)) < \infty$ if and only if $K^{-,b}(\mathrm{proj}A)$ is of finite type. In this section, we try to explore the conditions to ensure the truth of first Brauer–Thrall type theorem for subcategories, i.e. $\mathrm{hr}(\mathcal{C}) < \infty$ if and only if $\mathcal{C}$ is of finite type.

Definition 4.1. Let $X^\bullet$ be a complex in $C^{-,b}(A)$. Then we define the *dimension* of $X^\bullet$ is

$$\mathrm{dim}X^\bullet = \sum_{i\in\mathbb{Z}} \mathrm{dim}X^i.$$

If a complex $P^\bullet$ in $K^b(\mathrm{proj}A)$ satisfies $P^i \neq 0$ if and only if $n \leq i \leq m$, then we say $P^\bullet$ *lies in the interval* $[n, m]$. Similarly if $P^\bullet$ satisfies $H^i(P^\bullet) \neq 0$ if and only if $n \leq i \leq m$ then we say *the cohomology of* $P^\bullet$ *lies in the interval* $[n, m]$. The following lemma implies that the dimension of the minimal complex of fixed width can be controlled by the cohomological length.

Lemma 4.2 (see [4, Lemma 2.2]). *Let $P^\bullet$ be a complex in $K^b(\mathrm{proj}A)$ lies in the interval $[0, m]$ with $\mathrm{hl}(P^\bullet) = c$, and $\mathrm{dim}A = d$. Then for any $0 \leq i \leq m$,*

$$\mathrm{dim}P^i \leq c(d + d^2 + \cdots + d^{m-i+1}).$$

Proof. Since $P^\bullet = (P^i, d^i)$ is a minimal complex, i.e. $\mathrm{Im}d^i \subseteq \mathrm{rad}P^{i+1}$, then for any $i \in [0, m]$,

$$\begin{aligned}
\mathrm{dim}P^i &\leq \mathrm{dim}A \cdot \mathrm{dim}(P^i/\mathrm{rad}P^i)\\
&\leq \mathrm{dim}A \cdot \mathrm{dim}(P^i/\mathrm{Im}d^{i-1})\\
&= \mathrm{dim}A \cdot \big(\mathrm{dim}(P^i/\mathrm{Ker}d^i) + \mathrm{dim}(\mathrm{Ker}d^i/\mathrm{Im}d^{i-1})\big)\\
&= \mathrm{dim}A \cdot \big(\mathrm{dim}\mathrm{Im}d^i + \mathrm{dim}H^i(P^\bullet)\big)\\
&\leq \mathrm{dim}A \cdot \big(\mathrm{dim}P^{i+1} + \mathrm{dim}H^i(P^\bullet)\big)\\
&\leq d \cdot \big(\mathrm{dim}P^{i+1} + c\big).
\end{aligned}$$

Thus we can get the inequality as required recursively. □

Proposition 4.3. *Let $\mathcal{C}$ be a full subcategories of $K^{-,b}(\mathrm{proj}A)$ closed under gluing and cutting of objects. If $\mathrm{hr}(\mathcal{C}) = n < \infty$, then there is an integer $f(n)$, such that $\mathrm{dim}P^\bullet < f(n)$ for any indecomposable object $P^\bullet \in \mathcal{C}$.*

Proof. By assumption that $\mathrm{hr}(\mathcal{C}) = n$, we know $\mathrm{hw}(\mathcal{C}) \leq n$ and thus $\mathrm{width}(\mathcal{C}) \leq n+1$, see Proposition 3.5. Since $\mathrm{hl}(\mathcal{C}) \leq n$, by the previous lemma, the dimension of the indecomposable objects in $\mathcal{C}$ have a common bound, and the proposition follows. □

We need the following classical lemma, see [3, Corollary VI. 1.3] for the proof, which is also valid since $C^b(A)$ is also a length category.

Lemma 4.4 (Harada-Sai). *Let A be a finite-dimensional algebra and l be an integer. Then the composition of $X_1^\bullet \to X_2^\bullet \to X_3^\bullet \to \cdots \to X_{2^l}^\bullet$ of non-isomorphisms between indecomposable objects in $C^b(A)$ of dimension at most l is zero.*

Recall that a subcategory $\mathcal{C}$ of $K^{-,b}(\mathrm{proj}A)$ is *contravariantly finite* if any object $X^\bullet \in K^{-,b}(\mathrm{proj}A)$ admits a *right $\mathcal{C}$-approximation*, that is, a map $f_{X^\bullet} : C^\bullet \to X^\bullet$ with $C^\bullet \in \mathcal{C}$ such that any morphism $C'^\bullet \to X^\bullet$ in $K^{-,b}(\mathrm{proj}A)$ with $C'^\bullet \in \mathcal{C}$ factors through $f_{X^\bullet}$. We define *the orbit with respect to the shift functor* of $X^\bullet \in \mathcal{C}$ to be

$$\mathcal{O}_{X^\bullet} = \{X^\bullet[i] \in \mathcal{C} \mid i \in \mathbb{Z}\}.$$

The subcategory $\mathcal{C}$ is said to be *cohomologically homogeneous* if there are two integers s, t such that for any $X^\bullet \in \mathcal{C}$, we can find an object $X^\bullet[i] \in \mathcal{O}_{X^\bullet}$, whose maximal degree of nonzero cohomology lies in the interval $[s,t]$.

Now we are ready to prove the following theorem. The idea of the proof is essentially due to Auslander's classical argument on the first Brauer–Thrall conjecture, see [2, 3, 11].

Theorem 4.5. *Let A be a finite-dimensional algebra, and $\mathcal{C}$ be a contravariantly finite and cohomologically homogeneous subcategory closed under gluing and cutting of objects. Then $\mathrm{hr}(\mathcal{C}) < \infty$ if and only if $\mathcal{C}$ is of finite type.*

Proof. If $\mathcal{C}$ is of finite type then $\mathrm{hr}(\mathcal{C}) < \infty$ obviously.

Now we prove $\mathrm{hr}(\mathcal{C}) < \infty$ implies that $\mathcal{C}$ is of finite type. The strategy of the proof is to construct finitely many indecomposable objects $\{P_i^\bullet \in \mathcal{C} \mid i \in I\}$ with I a finite set, such that any indecomposable object $P^\bullet$ in $\mathcal{C}$ satisfies $P^\bullet[j] \cong P_i^\bullet$ for some $P^\bullet[j] \in \mathcal{O}_{P^\bullet}$ and $i \in I$.

Since $\mathcal{C}$ is cohomologically homogeneous, there is a finite interval $[s,t]$ such that any $P^\bullet \in \mathcal{C}$ admits a shift $P^\bullet[j] \in \mathcal{C}$, whose maximal degree of nonzero cohomology, say m, lies in $[s,t]$. Then we can choose a simple A-module S, for example, a direct summand of the top of $H^m(P^\bullet)$, such that $f : P^\bullet[j] \to S[m]$ is nonzero. We take a right $\mathcal{C}$-approximation of $C_1^\bullet \to S[m]$ and then f is the composition $P^\bullet[j] \to C_1^\bullet \to S[m]$. Thus we can take an indecomposable direct summand $P_1^\bullet$ of $C_1^\bullet$ such that $P^\bullet[j] \xrightarrow{g} P_1^\bullet \to S[m]$ is nonzero. If g is an isomorphism then stop. Otherwise, since [9, Sec. 4.5], any object in $\mathcal{C} \subseteq K^b(\mathrm{proj}A)$ admits a right almost split map, and then we can take a right almost split map $Q_1^\bullet \to P_1^\bullet$. Compose it with a right $\mathcal{C}$-approximation $C_2^\bullet \to Q_1^\bullet$ with $C_2^\bullet \in \mathcal{C}$ and then g is the composition $P^\bullet[j] \to C_2^\bullet \to P_1^\bullet$. Take an indecomposable direct summand $P_2^\bullet$ of $C_2^\bullet$ such that the composition $P^\bullet[j] \xrightarrow{h} P_2^\bullet \to P_1^\bullet$ is nonzero. If h is an isomorphism then stop. Otherwise, we repeat the argument as above and then obtain a sequence of indecomposable objects in $\mathcal{C}$

$$P_r^\bullet \to \cdots \to P_2^\bullet \to P_1^\bullet$$

with nonzero composition. Since $\mathrm{hr}(\mathcal{C}) = n < \infty$, the dimension of the indecomposable objects has a common bound $f(n)$ by Proposition 4.3. Moreover, by the Harada-Sai lemma, the composition of $2^{f(n)}$ non-invertible maps in $C^b(A)$ is zero and so is in $K^b(\mathrm{proj}A)$. Therefore, the above argument stops in finitely many steps, i.e. $P^\bullet[j]$ is isomorphic to some $P_i^\bullet$.

Next, it suffices to show that there are only finitely many indecomposable objects $P_i^\bullet$ possibly appearing in these sequences. Indeed, we start from the simple A-modules with only finitely many shifts $S[m]$ with $s \le m \le t$, and these $P_i^\bullet$'s in the sequences are indecomposable direct summands of the $\mathcal{C}$-approximations within at most $2^{f(n)}$ steps, which are only finitely many clearly. □

Example 4.6. Let A be a finite-dimensional algebra of finite global dimension, $S = \{P_1^\bullet, P_2^\bullet, \dots, P_n^\bullet\}$ be a finite set of indecomposable objects in $K^b(\mathrm{proj}A)$, and $\mathcal{C}_S$ be the full subcategory determined by S. The closure with respect to gluing and cutting of objects $\mathcal{C}$ of $\mathcal{C}_S$, that is, the minimal full subcategory closed under gluing and cutting of objects, consists of only finitely many indecomposable objects. Indeed, the objects in $\mathcal{C}$ generated by the ones in S by taking the projective resolutions, and then taking certain brutal truncations. Then $\mathcal{C}$ is contravariantly finite since for any object $X^\bullet \in D^b(A)$, a natural $\mathcal{C}$-approximation is the direct sum of all indecomposables in $\mathcal{C}$. Moreover, it is obvious that $\mathcal{C}$ is cohomologically homogeneous. In this setting, $\mathrm{hr}(\mathcal{C}) < \infty$ and $\mathcal{C}$ has only finitely many indecomposable objects.

Example 4.7. Let A be a finite-dimensional algebra and $D^b(A)$ be the bounded derived category. Recall that from [5], a *t-structure* of $D^b(A)$ is a pair of isomorphism closed full subcategories $(\mathcal{C}^{\le 0}, \mathcal{C}^{\ge 0})$ satisfying

(1) $\mathcal{C}^{\le 0}$ is closed under $[-1]$, and $\mathcal{C}^{\ge 0}$ is closed under $[1]$;
(2) $\mathrm{Hom}_{D^b(A)}(X^\bullet, Y^\bullet[1]) = 0$, for any $X^\bullet \in \mathcal{C}^{\le 0}$, $Y^\bullet \in \mathcal{C}^{\ge 0}$;
(3) For any $X^\bullet \in D^b(A)$, we have a triangle

$$X_1^\bullet \to X^\bullet \to X_2^\bullet[1] \to X_1^\bullet[-1]$$

with $X_1^\bullet \in \mathcal{C}^{\le 0}$, $X_2^\bullet \in \mathcal{C}^{\ge 0}$.

Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathrm{mod}A$, one can define two subcategories of $D^b(A)$ as follows

$$\mathcal{D}^{\le 0} = \{X^\bullet \in D^b(A) \mid H^i(X^\bullet) = 0, i > 0;\ H^0(X^\bullet) \in \mathcal{T}\};$$

$$\mathcal{D}^{\ge 0} = \{X^\bullet \in D^b(A) \mid H^i(X^\bullet) = 0, i < -1;\ H^{-1}(X^\bullet) \in \mathcal{F}\}.$$

Then $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure of $D^b(A)$ [10, Propositon 2.1], we call it *the t-structure induced by* $(\mathcal{T}, \mathcal{F})$. Note that if we take $(\mathcal{T}, \mathcal{F}) = (\mathrm{mod}A, 0)$ then $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is the standard t-structure.

Let A be a finite-dimensional algebra, and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be the t-structure induced by a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathrm{mod}A$. It is clear that $\mathcal{D}^{\leq 0}$ is a contravariantly finite and cohomologically homogeneous subcategory closed under gluing and cutting of objects. Then $\mathrm{hr}(\mathcal{D}^{\leq 0}) < \infty$ if and only if $\mathcal{D}^{\leq 0}$ is of finite type.

For a finite-dimensional algebra and we take the subcategory $\mathcal{C}$ in the theorem to be $K^{-,b}(\mathrm{proj}A)$, which is equivalent to $D^b(A)$. Then we recover the first Brauer–Thrall type theorem of derived category established in [13] with a totally different method.

Corollary 4.8. *Let A be a finite-dimensional algebra. Then* $\mathrm{hr}(D^b(A)) < \infty$ *if and only if $D^b(A)$ is of finite type.*

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