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Indecomposables with smaller cohomological length in the derived category of gentle algebras

Dedicated to Professor Yingbo Zhang on the Occasion of Her 70th Birthday

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Abstract Bongartz (2013) and Ringel (2011) proved that there is no gaps in the sequence of lengths of indecomposable modules for the finite-dimensional algebras over algebraically closed fields. The present paper mainly studies this "no gaps" theorem as to cohomological length for the bounded derived category $D^b(A)$ of a gentle algebra A: if there is an indecomposable object in $D^b(A)$ of cohomological length $l > 1$, then there exists an indecomposable with cohomological length $l - 1$.

Keywords cohomological length, generalized string (band), derived discrete algebras

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1 Introduction

Throughout this paper, k is an algebraically closed field, all algebras are connected, basic, finite-dimensional, associative k-algebras with identity, and all modules are finite-dimensional right modules, unless stated otherwise. During the study of the representation theory of finite-dimensional algebras, the classification and distribution of indecomposable modules play a significant role. Besides the famous Brauer-Thrall conjectures [1,6,7,9,10], Bongartz [4] and Ringel [8] proved the following elegant theorem.

Theorem 1.1. *Let A be a finite-dimensional algebra. If there is an indecomposable A-module of length $n > 1$, then there exists an indecomposable A-module of length $n - 1$.*

Since Happel [5], the bounded derived categories of finite-dimensional algebras have been studied widely. The classification and distribution of indecomposable objects in the bounded derived category is still an important theme in representation theory of algebras. In this context, the definitive work was due to Vossieck [11]. He classified a class of algebras, *derived discrete algebras,* i.e., with only finitely many indecomposables distributed in each cohomology dimension vector in their bounded derived category. In the research of Brauer-Thrall type theorems for the bounded derived category of an algebra [12], some numerical invariants, i.e., the cohomological length, width, and range of a complex in bounded derived category are introduced: let A be a finite-dimensional algebra with $D^b(A)$ the bounded derived module category, the *cohomological length, cohomological width, cohomological range* of a complex $X^\bullet \in D^b(A)$ are

$$\begin{aligned}&\mathrm{hl}(X^\bullet) := \max\{\dim H^i(X^\bullet) \mid i \in \mathbb{Z}\},\\ &\mathrm{hw}(X^\bullet) := \max\{j - i + 1 \mid H^i(X^\bullet) \neq 0 \neq H^j(X^\bullet)\},\\ &\mathrm{hr}(X^\bullet) := \mathrm{hl}(X^\bullet) \cdot \mathrm{hw}(X^\bullet),\end{aligned}$$

respectively. Moreover, the derived Brauer-Thrall type theorems are established in [12] with cohomological range to be the replacement of length of modules in classical Brauer-Thrall conjectures. Note that there is a full embedding of mod$A$ into $D^b(A)$, which sends any A-module to the corresponding stalk complex. Obviously, the dimension of an A-module M is equal to the cohomological length and the cohomological range of the stalk complex M. As pointed out as a question in [12], it is natural to consider the derived version of Bongartz-Ringel's theorem and ask whether there are no gaps in the sequence of cohomological lengths (ranges) of indecomposable objects in $D^b(A)$.

Question I. *Is there an indecomposable object in $D^b(A)$ of cohomological length $l-1$ if there is one of cohomological length $l \geqslant 2$?*

Question II. *Is there an indecomposable object in $D^b(A)$ of cohomological range $r-1$ if there is one of cohomological range $r \geqslant 2$?*

Evidently, the questions have positive answers for representation-infinite algebras by Bongartz-Ringel's theorem for the module category of algebras. However, it seems difficult to give answers for general finite-dimensional algebras to the above questions since we know little about the description of indecomposables in the bounded derived category.

In this paper, we prove that for gentle algebras, the answer to Question I is positive, but the answer to Question II is negative. To be precise, there is no gaps in the sequence of cohomological lengths of indecomposables in the bounded derived category of gentle algebras. In addition, we construct a gentle algebra A_0 such that there is an indecomposable object in $D^b(A_0)$ of cohomological range r_0 but no indecomposable object with cohomological range $r_0 - 1$. Our result relies on the constructions of indecomposables in the bounded derived category of gentle algebras due to Bekkert and Merklen [2].

The paper is organized as follows. In Section 2, we recall the constructions of indecomposable objects in the bounded derived category of gentle algebras. In Section 3, we prove the main theorem of this paper. Finally, we produce a gentle algebra which demonstrates that Question II has a negative answer.

2 Indecomposables in bounded derived category of gentle algebras

In this section, we mainly recall the description of the indecomposable objects in the bounded derived category of gentle algebras from [2].

Let A be an algebra admitting a presentation kQ/I, where Q is a finite quiver with vertex set Q_0 and arrow set Q_1, and where I is an admissible ideal of kQ. Throughout this paper, we write the path in kQ/I from left to right. Recall that $A = kQ/I$ is a *gentle algebra* if

(1) the number of arrows with a given source (resp. target) is at most two;

(2) for any arrow $\alpha \in Q_1$, there is at most one arrow $\beta \in Q_1$ such that $s(\alpha) = t(\beta)$ (resp. $t(\alpha) = s(\beta)$) and $\beta\alpha \in I$ (resp. $\alpha\beta \in I$);

(3) for any arrow $\alpha \in Q_1$, there is at most one arrow $\gamma \in Q_1$ such that $s(\alpha) = t(\gamma)$ (resp. $t(\alpha) = s(\gamma)$) and $\gamma\alpha \notin I$ (resp. $\alpha\gamma \notin I$);

(4) I is generated by a set of paths of length two.

Let $A = kQ/I$ be a gentle algebra. We need to recall some notation. For a path $p = \alpha_1\alpha_2\cdots\alpha_r$ with $\alpha_i \in Q_1$, we say its *length* $l(p) = r$. Let $\mathbf{Pa}_{\geqslant 1}$ be the set of all paths in kQ/I of length greater than 1. For any arrow $\alpha \in Q_1$, we denote by α^{-1} its formal *inverse* with $s(\alpha^{-1}) = t(\alpha)$ and $t(\alpha^{-1}) = s(\alpha)$. For a path $p = \alpha_1\alpha_2\cdots\alpha_r$, its *inverse* $p^{-1} = \alpha_r^{-1}\alpha_{r-1}^{-1}\cdots\alpha_1^{-1}$. A sequence $w = w_1w_2\cdots w_n$ is a *walk* (resp.

a *generalized walk*) if each w_i is of form p or p^{-1} with $p \in Q_1$ (resp., $p \in \mathbf{Pa}_{\geqslant 1}$), and $s(w_{i+1}) = t(w_i)$ for $i = 1, 2, \ldots, n-1$.

We denote by $\mathbf{St}$ the set of all walks $w = w_1 w_2 \cdots w_n$ such that $w_{i+1} \neq w_i^{-1}$ for each $1 \leqslant i < n$ and no subword of w or w^{-1} lies in I. We call an element in $\mathbf{St}$ a *string*. By $\overline{\mathbf{Gst}}$ we denote the set of all generalized walks such that

(1) $w_i w_{i+1} \in I$ if $w_i, w_{i+1} \in \mathbf{Pa}_{\geqslant 1}$;

(2) $w_{i+1}^{-1} w_i^{-1} \in I$ if $w_i^{-1}, w_{i+1}^{-1} \in \mathbf{Pa}_{\geqslant 1}$;

(3) $w_i w_{i+1} \in \mathbf{St}$ otherwise.

We write $\mathbf{Gst}$ the set consisting of all trivial paths and the representatives of $\overline{\mathbf{Gst}}$ modulo the relation $w \sim w^{-1}$. An element $w = w_1 w_2 \cdots w_n$ in $\mathbf{Gst}$ is called a *generalized string* of width n.

Generalized bands are special generalized strings. Before its definition, we need the following notation. Let $w = w_1 w_2 \cdots w_n$ be a generalized string. Set $\mu_w(0) = 0$, $\mu_w(i) = \mu_w(i-1) - 1$ if $w_i \in \mathbf{Pa}_{\geqslant 1}$ and $\mu_w(i) = \mu_w(i-1) + 1$ otherwise. Suppose $\overline{\mathbf{GBa}}$ is the set of all generalized walks $w = w_1 w_2 \cdots w_n$ such that

(1) $s(w_1) = t(w_n)$;

(2) $\mu_w(n) = \mu_w(0) = 0$;

(3) $w^2 = w_1 w_2 \cdots w_n w_1 w_2 \cdots w_n \in \overline{\mathbf{Gst}}$.

We denote by $\mathbf{Gba}$ the set consisting of the representatives of $\overline{\mathbf{Gba}}$ modulo the relation $w \sim w^{-1}$ and $w_1 w_2 \cdots w_n \sim w_2 \cdots w_n w_1$. We call an element in $\mathbf{Gba}$ a *generalized band*.

By the description of Bekkert and Merklen [2], a generalized string in $A = kQ/I$ corresponds to a unique indecomposable object of bounded homotopy category $K^b(\mathrm{proj} A)$ up to shift, while a generalized band w corresponds to a unique family of indecomposables $\{P^{\bullet}_{w,\lambda} \mid \lambda \in k^*,\, d > 0\}$ in $K^b(\mathrm{proj} A)$ up to shift, in which $P^{\bullet}_{w,\lambda}$ and $P^{\bullet}_{w,\lambda'}$ have the same cohomology dimension vector for any λ, λ'. Thus A is derived discrete if and only if A contains no generalized bands (see [2, 11]).

Let α be a path in $\mathbf{Pa}_{\geqslant 1}$. Then it induces a morphism $P(\alpha)$ from $P_{t(\alpha)}$ to $P_{s(\alpha)}$ by left multiplication, where P_i is the indecomposable projective right A-module $e_i A$ associated to vertex i. More precisely, $P(\alpha)(u) = \alpha u$ for any $u \in kQ/I$.

Definition 2.1. Let $w = w_1 w_2 \cdots w_n$ be a generalized string. Then the complex of projective modules $P^{\bullet}_w = \cdots \xrightarrow{d_w^{i-1}} P_w^i \xrightarrow{d_w^i} P_w^{i+1} \xrightarrow{d_w^{i+1}} \cdots$ is defined as follows. The module on the i-th component

$$P_w^i = \bigoplus_{j=0}^{n} \delta(\mu_w(j), i) P_{c(j)},$$

where δ is the Kronecker sign, $c(j) = s(w_{j+1})$ for $j < n$ and $c(n) = t(w_n)$. The differential d_w^i is given by the matrix $(d_{j,k}^i)$ with entries, where

$$d_{j,k}^i = \begin{cases} P(w_j), & \text{if } w_j \in \mathbf{Pa}_{\geqslant 1}, \quad \mu_w(j) = i, \quad k = j-1, \\ P(w_{j+1}^{-1}), & \text{if } w_{j+1}^{-1} \in \mathbf{Pa}_{\geqslant 1}, \quad \mu_w(j) = i, \quad k = j+1, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.2. Let $w = w_1 w_2 \cdots w_n$ be a generalized band. Then for any $\lambda \in k^*$, $d > 0$, the complex of projective modules

$$P^{\bullet}_{w,\lambda} = \cdots \xrightarrow{d_w^{i-1}} P_{w,\lambda}^i \xrightarrow{d_w^i} P_{w,\lambda}^{i+1} \xrightarrow{d_w^{i+1}} \cdots$$

is defined as follows. The module on the i-th component

$$P_{w,\lambda}^i = \bigoplus_{j=0}^{n-1} \delta(\mu_w(j), i) P_{c(j)}^d.$$

The differential $d_w^i = (d_{j,k}^i)$ and

$$d_{j,k}^i = \begin{cases} P(w_j)\mathbf{Id}_d, & \text{if } w_j \in \mathbf{Pa}_{\geqslant 1}, \quad \mu_w(j) = i, \quad k = j-1, \\ P(w_{j+1}^{-1})\mathbf{Id}_d, & \text{if } w_{j+1}^{-1} \in \mathbf{Pa}_{\geqslant 1}, \quad \mu_w(j) = i, \quad k = j+1, \\ P(w_n)J_{\lambda,d}, & \text{if } w_n \in \mathbf{Pa}_{\geqslant 1}, \quad \mu_w(n) = 0 = i, \quad k = n-1, \\ P(w_n^{-1})J_{\lambda,d}, & \text{if } w_n^{-1} \in \mathbf{Pa}_{\geqslant 1}, \quad \mu_w(n-1) = i, \quad k = 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $J_{\lambda,d}$ is the upper triangular $d \times d$ Jordan block with eigenvalue $\lambda \in k^*$.

Note that the definitions above are slightly different from the ones in [2] since we consider right projective modules throughout this paper.

Recall that a complex $X^\bullet = (X^i, d^i) \in C(A)$ is said to be *minimal* if $\mathrm{Im} d^i \subseteq \mathrm{rad} X^{i+1}$ for all $i \in \mathbb{Z}$. For a complex $P^\bullet$ in $C^{-,b}(\mathrm{proj}A)$ of the form

$$P^\bullet = \cdots \longrightarrow P^{-n-1} \overset{d^{-n-1}}{\longrightarrow} P^{-n} \overset{d^{-n}}{\longrightarrow} \cdots \longrightarrow P^{m-1} \overset{d^{m-1}}{\longrightarrow} P^m \longrightarrow 0,$$

its *brutal truncation* $\sigma_{\geqslant -n}(P^\bullet)$ is

$$\sigma_{\geqslant -n}(P^\bullet) = 0 \longrightarrow P^{-n} \overset{d^{-n}}{\longrightarrow} \cdots \longrightarrow P^{m-1} \overset{d^{m-1}}{\longrightarrow} P^m \longrightarrow 0.$$

The following lemma due to [12, Proposition 2] sets up the connection between the indecomposable objects in $K^b(\mathrm{proj}A)$ and those in $K^{-,b}(\mathrm{proj}A)$.

Lemma 2.3. *Let $P^\bullet \in K^{-,b}(\mathrm{proj}A)$ be a minimal complex and $-n := \min\{i \in \mathbb{Z} \mid H^i(P^\bullet) \neq 0\}$. Then $P^\bullet$ is indecomposable if and only if so is the brutal truncation $\sigma_{\geqslant j}(P^\bullet) \in K^b(\mathrm{proj}A)$ for some $j < -n$ or for all some $j < -n$.*

Let A be a finite-dimensional algebra and $P^\bullet \in K^b(\mathrm{proj}A)$ an indecomposable minimal complex of the form

$$P^\bullet = 0 \longrightarrow P^{-n} \overset{d^{-n}}{\longrightarrow} P^{-n+1} \overset{d^{-n+1}}{\longrightarrow} \cdots \overset{d^{-2}}{\longrightarrow} P^{-1} \overset{d^{-1}}{\longrightarrow} P^0 \longrightarrow 0.$$

Now we can construct a minimal object in $D^b(A)$ by eliminating the cohomology of minimal degree. Suppose $H^{-n}(P^\bullet) \cong \mathrm{Ker} d^{-n}$. We take a minimal projective resolution of $\mathrm{Ker} d^{-n}$, say

$$P'^\bullet = \cdots \longrightarrow P^{-n-2} \overset{d^{-n-2}}{\longrightarrow} P^{-n-1} \longrightarrow 0.$$

Gluing $P'^\bullet$ and $P^\bullet$ together, we get a minimal complex

$$\beta(P^\bullet) = \cdots \longrightarrow P^{-n-2} \overset{d^{-n-2}}{\longrightarrow} P^{-n-1} \overset{d^{-n-1}}{\longrightarrow} P^{-n} \overset{d^{-n}}{\longrightarrow} \cdots \overset{d^{-1}}{\longrightarrow} P^0 \longrightarrow 0,$$

where d^{-n-1} is the composition $P^{-n-1} \twoheadrightarrow \mathrm{Ker} d^{-n} \hookrightarrow P^{-n}$. Note that $H^{-n}(\beta(P^\bullet)) = 0$, and $H^j(\beta(P^\bullet)) = H^j(P^\bullet)$ for $j \neq -n$.

Lemma 2.4. *Keep the notation as above. Then $\beta(P^\bullet)$ is indecomposable.*

Proof. If $H^{-n}(P^\bullet) = 0$, then $\beta(P^\bullet) = P^\bullet$ and the statement follows. Now suppose $H^{-n}(P^\bullet) \neq 0$. Since $P^\bullet$ is the brutal truncation $\sigma_{\geqslant -n}(\beta(P^\bullet))$, which is indecomposable and $H^i(\beta(P^\bullet)) = 0$ for all $i \leqslant -n$, $\beta(P^\bullet)$ is indecomposable by Lemma 2.3. □

The following theorem from [2, Theorem 3] provides an explicit description of the indecomposables in the bounded derived category $D^b(A)$.

Theorem 2.5. *Let $A = kQ/I$ be a gentle algebra with $[-1]$ the shift functor in $D^b(A)$. Then the set of indecomposable objects in $K^b(\mathrm{proj}A)$ is*

$$\{P_w^\bullet[i] \mid w \in \mathbf{Gst}, i \in \mathbb{Z}\} \cup \{P_{w,\lambda}^\bullet[i] \mid w \in \mathbf{Gba},\ \lambda \in k^*,\ d > 0,\ i \in \mathbb{Z}\}.$$

*Moreover, the indecomposables in $K^{-,b}(\mathrm{proj}A) \backslash K^b(\mathrm{proj}A)$ is of the form $\beta(P_w^\bullet)$ for $w \in$ **Gst** with certain conditions.*

3 Question I for gentle algebras

In this section, we will discuss the cohomological lengths of the indecomposables in the bounded derived category of gentle algebras. Indeed, we prove the following theorem.

Theorem 3.1. *Let A be a gentle algebra. If there is an indecomposable object in $D^b(A)$ of cohomological length $l>1$, then there exists an indecomposable with cohomological length $l-1$.*

Before the proof, we need some preparations. First, we recall the definitions of some numerical invariants for finite-dimensional algebras introduced in [12].

Definition 3.2. Let A be a finite-dimensional algebra with $D^b(A)$ the bounded derived category. The *cohomological length* of a complex $X^\bullet \in D^b(A)$ is

$$\mathrm{hl}(X^\bullet) := \max\{\dim H^i(X^\bullet) \mid i \in \mathbb{Z}\}.$$

As well known, there is a full embedding of mod$A$ into $D^b(A)$, which sends an A-module M to the corresponding stalk complex and the cohomological length of the stalk complex M equals dimension of M. If A is representation-infinite, i.e., there exist indecomposable A-modules of arbitrary large dimensions, then the *global cohomological length* of A

$$\mathrm{gl.hl}A := \sup\{\mathrm{hl}(X^\bullet) \mid X^\bullet \in D^b(A) \text{ is indecomposable}\}$$

is infinite. Moreover, by the Bongartz and Ringel's theorem, Theorem 3.1 also holds for representation-infinite algebras since the Brauer-Trall conjecture I holds in this case [1, 9].

Definition 3.3. The *cohomological width* of a complex $X^\bullet \in D^b(A)$ is

$$\mathrm{hw}(X^\bullet) := \max\{j-i+1 \mid H^i(X^\bullet) \neq 0 \neq H^j(X^\bullet)\},$$

and the *cohomological range* of $X^\bullet$ is

$$\mathrm{hr}(X^\bullet) := \mathrm{hl}(X^\bullet)\cdot \mathrm{hw}(X^\bullet).$$

Since the cohomological width of a stalk complex is one, the cohomological range of a stalk complex is precisely the cohomological length. Thus, there is also no gaps in the sequence of cohomological ranges of indecomposable objects in $D^b(A)$ if A is representation-infinite. Moreover, the cohomological length, width and range are invariant under shifts and isomorphisms.

Let A be a gentle algebra. By Theorem 2.5, any indecomposable complex $P^\bullet \in D^b(A)$ is of the form $P^\bullet_w$ determined by a generalized string w, or of the form $\beta(P^\bullet_w)$ for some generalized string w, or of the form $P^\bullet = P^\bullet_{w,\lambda}$ determined by a generalized band w. Thus we divide the proof of Theorem 3.1 into three theorems as follows and their proofs depend strongly on the description of the indecomposables in the bounded derived category of gentle algebras due to Bekkert and Merklen [2].

We should recall more notation for a gentle algebras $A = kQ/I$ from [2, 3], some of which are slightly different for our convenience. For any $p \in \mathbf{Pa}_{\geqslant 1}$, there is a unique maximal path $\tilde{p} = p\hat{p}$ starting with p. Besides the path $\tilde{p}$, there may be another maximal path, say $\check{p}$, beginning with the starting point $s(p)$ of p. If this is not the case, we write $l(\check{p}) = 0$. For any walk $p = p_1p_2\cdots p_l$ and any $j < l$, we write $\kappa^+_j(p) = p_{j+1}p_{j+2}\cdots p_l$ for the walk truncating the first j arrows from the path p along the positive direction. Similarly, we write $\kappa^-_j(p) = p_1p_2\cdots p_{l-j}$ for the walk truncating the last j arrows from path p along the negative direction. Moreover, for a path α, we denote by $\overline{\alpha}$ the generalized string $\alpha\alpha_1\alpha_2\cdots$ of maximal width with $\alpha_i \in Q_1$. Note that $\alpha\alpha_1 \in I$, $\alpha_i\alpha_{i+1} \in I$ for $i \geqslant 1$, and $\overline{\alpha} = \alpha$ if there is no such arrow α_1 that $\alpha\alpha_1 \in I$.

Now we are ready for the following theorem.

Theorem 3.4. *Let A be a gentle algebra. If there is an indecomposable $P^\bullet_w \in K^b(\mathrm{proj}A)$ determined by a generalized string w such that $\mathrm{hl}(P^\bullet) = l > 1$, then there is an indecomposable $P'^\bullet \in D^b(A)$ with $\mathrm{hl}(P'^\bullet) = l-1$.*

Proof. We shall divide the proof into two cases.

Case 1. Let $w = w_1w_2\cdots w_n$ be a one-sided generalized string, i.e., $w_i \in \mathbf{Pa}_{\geqslant 1}$ for all $1 \leqslant i \leqslant n$, or $w_i^{-1} \in \mathbf{Pa}_{\geqslant 1}$ for all $1 \leqslant i \leqslant n$. Without loss of generality, we assume $w_i \in \mathbf{Pa}_{\geqslant 1}$ for all $1 \leqslant i \leqslant n$ (otherwise, we can consider the generalized string w^{-1}, and they determine the same complex). Let $P^\bullet$ be the complex determined by w of the form

$$P^\bullet_w = \ 0 \longrightarrow P_{t(w_n)} \xrightarrow{P(w_n)} P_{t(w_{n-1})} \xrightarrow{P(w_{n-1})} \cdots \xrightarrow{P(w_2)} P_{t(w_1)} \xrightarrow{P(w_1)} P_{s(w_1)} \longrightarrow 0,$$

where $P_{s(w_1)}$ lies in the 0-th component. Thus,

$$\begin{aligned}\dim H^0(P^\bullet_w) &= \dim P_{s(w_1)} - \dim \mathrm{Im} P(w_1)\\ &= \dim P_{s(w_1)} - \dim w_1 P_{t(w_1)}\\ &= (l(\widetilde{w_1}) + l(\check{w}_1) + 1) - (l(\widehat{w_1}) + 1)\\ &= l(w_1) + l(\check{w}_1).\end{aligned}$$

For any $1 \leqslant i \leqslant n-1$,

$$\begin{aligned}\dim H^{-i}(P^\bullet_w) &= \dim \mathrm{Ker} P(w_i) - \dim \mathrm{Im} P(w_{i+1})\\ &= l(\widetilde{w_{i+1}}) - (l(\widehat{w_{i+1}}) + 1)\\ &= l(w_{i+1}) - 1.\end{aligned}$$

Similarly,

$$\begin{aligned}\dim H^{-n}(P^\bullet_w) &= \dim \mathrm{Ker} P(w_n) = \#\{p \in \mathbf{Pa}_{\geqslant 1} \,|\, w_n p = 0\}\\ &= \begin{cases} 0, & \text{if there is no arrows } \alpha \text{ such that } w_n\alpha = 0,\\ l(\tilde{\alpha}), & \text{if there is an arrow } \alpha \text{ such that } w_n\alpha = 0.\end{cases}\end{aligned}$$

Now we suppose

$$i = \max\{j \,|\, \dim H^{-j}(P^\bullet_w) = \mathrm{hl}(P^\bullet_w); 0 \leqslant j \leqslant n\}.$$

We consider the possible values of i in each case.

(1) If $i = 0$, then $\dim H^j(P^\bullet_w) < \dim H^0(P^\bullet_w)$ for any $j \neq 0$. Now we want to obtain a generalized string, which determines a projective complex whose cohomological length equals to $\dim H^0(P^\bullet_w) - 1 = l(w_1) + l(\check{w}_1) - 1$.

If $l(\check{w}_1) = 0$, namely, $\tilde{w_1}$ is the unique maximal path starting from $s(w_1)$, then we get a generalized string $w' = \kappa_1^+(w_1)w_2\cdots w_n$ by the truncating from positive direction. Now if there is a unique maximal path beginning with $s(w') = s(\kappa_1^+(w_1))$, then

$$\dim H^0(P^\bullet_{w'}) = l(\kappa_1^+(w_1)) = l(w_1) - 1 = \dim H^0(P^\bullet_w) - 1,$$

and the cohomologies of other degrees remain unchanged. Thus $P'^\bullet = P^\bullet_{w'}$ is as required with $\mathrm{hl}(P'^\bullet) = l - 1$. If there is another arrow p starting from $s(w')$ besides w', then we set $w'' = \overline{p}^{-1}\kappa_1^+(w_1)w_2\cdots w_n$. Indeed, the complex $P^\bullet_{w''}$ determined by w'' can be illustrated as follows:

$$\begin{array}{ccccccc} 0 \longrightarrow P_{t(w_n)} \xrightarrow{P(w_n)} P_{t(w_{n-1})} \xrightarrow{P(w_{n-1})} \cdots \xrightarrow{P(w_2)} P_{t(w_1)} \xrightarrow{P(\kappa_1^+(w_1))} & P_{s(\kappa_1^+(w_1))} & \longrightarrow 0\\ & \nearrow{\scriptstyle P(p)} & \\ \cdots \xrightarrow{P(p_1)} P_{t(p)} & & \end{array}$$

with $P_{s(\kappa_1^+(w_1))}$ on the 0-th component. Now we calculate the dimension of cohomologies of $P^\bullet_{w''}$:

$$\dim H^0(P^\bullet_{w''}) = \dim P_{s(\kappa_1^+(w_1))} - \dim \mathrm{Im}(P(\kappa_1^+(w_1)), P(p))$$

$$
\begin{aligned}
&= l(\widetilde{\kappa_1^+(w_1)}) + l(\widetilde{p}) + 1 - (l(\widehat{\kappa_1^+(w_1)}) + 1) - (l(\widehat{p}) + 1) \\
&= l(\kappa_1^+(w_1)) + l(p) - 1 = l(\kappa_1^+(w_1)) \\
&= l(w_1) - 1 = \dim H^0(P_w^\bullet) - 1. \qquad (*)
\end{aligned}
$$

Moreover, the cohomologies of other degrees remain unchanged since $p_i \in Q_1$. Note that if $\overline{p}^{-1}$ is a walk of infinite length, then $P_{w''}^\bullet$ is of the form $\beta(P_u^\bullet)$, where u is a generalized string obtained by truncation of w'' at certain position. So $P_{w''}^\bullet$ is indecomposable. Thus $P'^\bullet = P_{w''}^\bullet$ is as required with $\mathrm{hl}(P'^\bullet) = l - 1$.

If $l(\breve{w}_1) = a > 0$, then we set $w' = \overline{\breve{w}_1}^{-1} w_1 w_2 \cdots w_n$. By the calculation as in the equation $(*)$, $\dim H^0(P_{w'}^\bullet) = l(w_1) + l(\breve{w}_1) - 1 = \dim H^0(P_w^\bullet) - 1$, and the cohomologies of other degrees remain unchanged. Thus $P'^\bullet = P_{w''}^\bullet$ is the complex as required.

(2) If $1 \leqslant i \leqslant n-1$, since $\dim H^i(P_w^\bullet) = l(w_{i+1}) - 1 = \mathrm{hl}(P_w^\bullet)$, we only need to consider the case $l(w_{i+1}) > 2$. We set the generalized string $w' = \kappa_2^+(w_{i+1}) w_{i+2} \cdots w_n$ obtained by truncating from the positive direction. Similar to the discussion in Case (1), if $\kappa_2^+(w_{i+1})$ is the unique maximal path beginning with $s(\kappa_2^+(w_{i+1}))$, then w' determines an indecomposable $P_{w'}^\bullet$ such that

$$
\begin{aligned}
\dim H^{-i}(P_{w'}^\bullet[-i]) &= \dim H^0(P_{w'}^\bullet) = l(\kappa_2^+(w_{i+1})) \\
&= l(w_{i+1}) - 2 = \dim H^{-i}(P_w^\bullet) - 1 = \mathrm{hl}(P_w^\bullet) - 1,
\end{aligned}
$$

and $\dim H^{-j}(P_{w'}^\bullet[-i]) = 0$ for any $j < i$, $\dim H^{-j}(P_{w'}^\bullet[-i]) \leqslant \dim H^{-j}(P_w^\bullet) < \dim H^{-i}(P_w^\bullet)$ for any $j > i$. So $P_{w'}^\bullet[-i]$ is the complex as required in this case. If there is another arrow p beginning with $s(\kappa_2^+(w_{i+1}))$, then we set $w'' = \overline{p}^{-1}\kappa_2^+(w_{i+1}) w_{i+2} \cdots w_n$. By a similar calculation to that in Case (1), $P'^\bullet = P_{w''}^\bullet$ satisfies $\mathrm{hl}(P'^\bullet) = \mathrm{hl}(P_w^\bullet) - 1$.

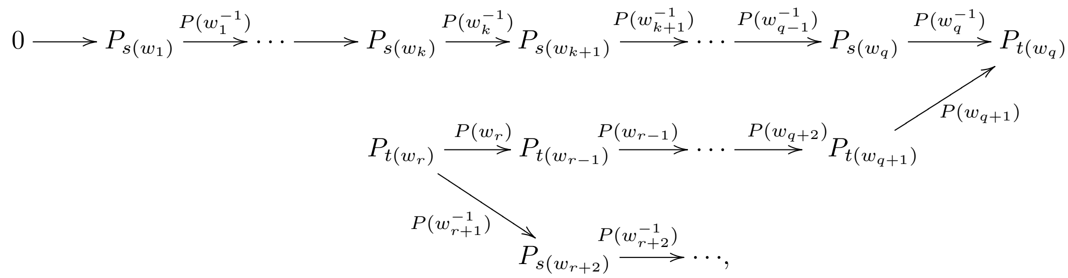
(3) Finally, for the case $i = n$, if there is no arrow α such that $w_n\alpha = 0$, then $\mathrm{hl}(P_w^\bullet) = 0$, which is impossible. Let α be such an arrow that $w_n\alpha = 0$ and $l(\tilde{\alpha}) > 1$. Then we choose the generalized string $w' = \kappa_1^+(\tilde{\alpha})$. With a similar discussion to the above, if there is a unique path beginning with $s(w')$, then w' determines the indecomposable object $P_{w'}^\bullet$. Set the indecomposable object $P'^\bullet = \beta(P_{w'}^\bullet)$. Then we have $\dim H^{-n}(P'^\bullet[-n]) = \dim H^0(P_{w'}^\bullet) = l(\tilde{\alpha}) - 1 = \dim H^{-n}(P_w^\bullet) - 1 = \mathrm{hl}(P_w^\bullet)$, and the cohomologies of other degrees vanish. Therefore, $\mathrm{hl}(P'^\bullet) = \mathrm{hl}(P^\bullet) - 1$. If there is another arrow p beginning with the starting point of w', then set $w'' = p^{-1}w' = p^{-1}\kappa_1^+(\tilde{\alpha})$ and $P'^\bullet = \beta(P_{w''}^\bullet)$. Thus $\dim H^{-n}(P'^\bullet[-n]) = \dim H^0(P_{w''}^\bullet) = l(\kappa_1^+(\tilde{\alpha})) + l(p) - 1 = l(\tilde{\alpha}) - 1 = \dim H^{-n}(P_w^\bullet) - 1 = \mathrm{hl}(P_w^\bullet)$, and the cohomologies of other degrees vanish.

In the above three cases, the construction of the indecomposable object $P'^\bullet$ is based on the generalized string obtained via truncation from the positive direction. Indeed, in each case, we can also obtain another indecomposable object by truncating the generalized strings from the negative direction. We shall take Case (2) above for example. First, we set

$$
i = \min\{j \mid \dim H^{-j}(P_w^\bullet) = \mathrm{hl}(P_w^\bullet); 0 \leqslant j \leqslant n\}.
$$

Now, we need to reduce the dimension of i-th cohomology by 1 and eliminate the j-th cohomology for $j < -i$. We get a generalized string $w' = w_1 \cdots w_i \kappa_1^-(w_{i+1})$ by truncation from the negative direction. As in Case (1), we glue w' and a generalized string together if needed to eliminate the cohomology at certain degree. To be precise, if there is no arrow α such that $\kappa_1^-(w_{i+1})\alpha \in I$, then $P'^\bullet = P_{w'}^\bullet$ is also an indecomposable object with $\mathrm{hl}(P'^\bullet) = \mathrm{hl}(P^\bullet) - 1$ as required. If there is an arrow α with $\kappa_1^-(w_{i+1})\alpha \in I$, then we set $w'' = w_1 \cdots w_i \kappa_1^-(w_{i+1})\overline{\alpha}$. Then by a similar calculation, $P'^\bullet = P_{w''}^\bullet$ is also an indecomposable object with $\mathrm{hl}(P'^\bullet) = \mathrm{hl}(P^\bullet) - 1$ as required. Note that in this case, $P'^\bullet = P_{w''}^\bullet = \beta(P_{w'}^\bullet)$.

Case 2. Let $w = w_1 w_2 \cdots w_n$ be a generalized string. Without loss of generality, assume that $w_1^{-1}, w_2^{-1}, \ldots, w_q^{-1} \in \mathbf{Pa}_{\geqslant 1}$ and $w_{q+1}, w_{q+2}, \ldots, w_r \in \mathbf{Pa}_{\geqslant 1}$, while $w_{r+1}^{-1} \in \mathbf{Pa}_{\geqslant 1}$. Then w determines the indecomposable object $P_w^\bullet$ of the form



where $P_{s(w_1)}$ lies in the 0-th component.

As illustrated above, there may be more than one indecomposable projective direct summands at a component. Note that at each component, we can order these indecomposable projective direct summands *which have nonzero cohomology* along the generalized string w. For example, in the above diagram, suppose the projective module $P_{s(w_{k+1})}$ lies in the i-th component. Then we write $P_w^i = P_1^i \oplus P_2^i \oplus P_3^i \oplus \cdots$, where $P_1^i = P_{s(w_{k+1})}$, $P_2^i = P_{t(w_{r-1})}, \ldots$ since the cohomologies are nontrivial at these direct summands. Then the cohomology of the degree i is the direct summand of cohomologies at these projective direct summands.

Now, as in Case 1, we want to construct an indecomposable object $P'^\bullet$ such that $\mathrm{hl}(P'^\bullet) = \mathrm{hl}(P_w^\bullet) - 1$. In order to reduce the dimension of cohomologies of i-th degree by 1, it suffices to reduce the dimension of cohomologies at the first projective direct summand of i-th degree. Indeed, we need to find a unique projective direct summand Q satisfying

(1) it is the first direct projective summand of its component under the ordering as above;

(2) it lies in the j-th component such that $\dim H^j(P^\bullet) = \mathrm{hl}(P^\bullet)$;

(3) it is the closest one from the starting point along the generalized string among those satisfying (1) and (2).

To construct an indecomposable object $P'^\bullet$ such that $\mathrm{hl}(P'^\bullet) = \mathrm{hl}(P_w^\bullet) - 1$, we only need to construct such $P'^\bullet$ by reducing the dimension of cohomology at Q by 1. By the analysis in Case 1, we can manage this via truncating the generalized string from positive or negative side and gluing suitable generalized string of the form $\overline{p}^{-1}$ or $\overline{p}$ if needed, except the following two case:

(1) Q is the *backward turning points* as $P_{t(w_q)}$, i.e., $Q = P_{t(w_i)}$ for some i such that $w_i^{-1}, w_{i+1} \in \mathbf{Pa}_{\geqslant 1}$. Let $Q = P_{t(w_i)}$ be a backward turning point. Then the dimension of cohomology at this point Q, write $H^{t(w_i)}(P_w^\bullet)$ (it is unnecessarily the whole cohomology group at this degree):

$$\begin{aligned}\dim H^{t(w_i)}(P_w^\bullet) &= \dim P(t(w_i)) - \dim \mathrm{Im}(P(w_i^{-1}), P(w_{i+1}))\\ &= l(\widetilde{w_{i+1}}) + l(\widetilde{w_i^{-1}}) + 1 - (l(\widehat{w_{i+1}}) + 1) - (l(\widehat{w_i^{-1}}) + 1)\\ &= l(w_{i+1}) + l(w_i^{-1}) - 1.\end{aligned}$$

Set $w' = \kappa_1^+(w_i)w_{i+1}\cdots w_n$. As in Case 1(1), if there is an arrow p such that $\kappa_1^+(w_i)p \in I$, then we write $w'' = \overline{p}^{-1}\kappa_1^+(w_i)w_{i+1}\cdots w_n$, and $w'' = w'$ otherwise. We have $\dim H^{t(w_i)}(P_{w''}^\bullet) = \dim H^{t(w_i)}(P_w^\bullet) - 1$ and then $\mathrm{hl}(P_{w''}^\bullet) = \mathrm{hl}(P_w^\bullet) - 1$.

(2) Q is the *forward turning point* as $P_{t(w_r)}$, i.e., $Q = P_{t(w_j)}$ for some j such that $w_j, w_{j+1}^{-1} \in \mathbf{Pa}_{\geqslant 1}$. Similarly let $Q = P_{t(w_j)}$ be a forward turning point. Then the dimension of cohomology at this point

$$\begin{aligned}\dim H^{t(w_j)}(P_w^\bullet) &= \dim \mathrm{Ker}(P(w_j), P(w_{j+1}^{-1}))^{\mathrm{T}}\\ &= \dim(\mathrm{Ker}P(w_j) \cap \mathrm{Ker}P(w_{j+1}^{-1}))\\ &= 0,\end{aligned}$$

which is impossible by the choice of Q. □

Now we consider the indecomposable objects in $K^{-,b}(\mathrm{proj}A) \setminus K^b(\mathrm{proj}A)$.

Theorem 3.5. *Let A be a gentle algebra. If there is an indecomposable $P^\bullet \in K^{-,b}(\mathrm{proj}A)\setminus K^b(\mathrm{proj}A)$ such that $\mathrm{hl}(P^\bullet) = l > 1$, then there is an indecomposable $P'^\bullet \in D^b(A)$ with $\mathrm{hl}(P'^\bullet) = l-1$.*

Proof. Since $P^\bullet \in K^{-,b}(\mathrm{proj}A)\setminus K^b(\mathrm{proj}A)$ is indecomposable, by Theorem 2.5, the brutal truncation $\sigma_{\geqslant j}(P^\bullet) \in K^b(\mathrm{proj}A)$ is indecomposable for some $j \ll 0$, and $\sigma_{\geqslant j}(P^\bullet) = P^\bullet_w$ for some generalized string w. Now we can consider the complex $P^\bullet_w$ using the similar argument as Theorem 3.4. If $\dim H^j(P^\bullet_w) \leqslant l$, then $\mathrm{hl}(P^\bullet_w) = l$ and the statement from the previous theorem holds. Suppose $\dim H^j(P^\bullet_w) > l$. By a similar analysis to that in the proof of the previous theorem, we can find a unique projective direct summand Q which satisfies the following: it is the first direct projective summand, it lies in m-th component such that $\dim H^m(P^\bullet_w) = l$ and it is the closest one from the starting point along w. Then we can construct $P^\bullet_{w'}$ by reducing the dimension of cohomology at Q by 1. Note that $\dim H^j(P^\bullet_{w'})$ may have the maximal dimension among the cohomologies of all degrees. If this is the case, then we have an indecomposable object $P^\bullet_{w''}$ obtained by gluing a generalized string to w' to eliminate the cohomology of j-th degree as in the proof of the previous theorem and we are done. □

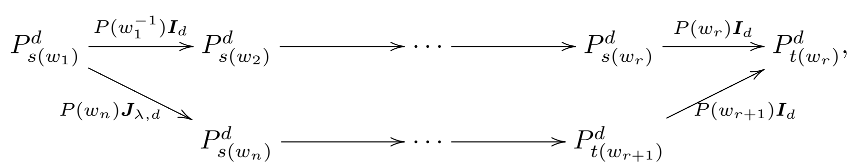
To finish the proof of Theorem 3.1, we only need to prove the last case, i.e., for the indecomposable objects determined by generalized bands.

Theorem 3.6. *Let A be a gentle algebra. If there is an indecomposable $P^\bullet \in K^b(\mathrm{proj}A)$ determined by a generalized band w such that $\mathrm{hl}(P^\bullet) = l > 1$, then there is an indecomposable $P'^\bullet \in D^b(A)$ with $\mathrm{hl}(P'^\bullet) = l-1$.*

Proof. Let $w = w_1w_2\cdots w_n$ be a generalized band. We assume without loss of generality that $w_1^{-1}, w_n \in \mathbf{Pa}_{\geqslant 1}$ and

$$\mu(0) = \mu(n) = \min\{\mu(i)\,|\,0 \leqslant i \leqslant n\}.$$

Then w determines a family of indecomposable objects $\{P^\bullet_{w,\lambda}\,|\,w \in \mathbf{Gba}, \lambda \in k^*, d > 0, i \in \mathbb{Z}\}$, where $P^\bullet_{w,\lambda}$ has the form of



where $P_{s(w_1)}$ lies in the 0-th component.

By the previous two theorems, it is sufficient to find a generalized string w' such that $\mathrm{hl}(\beta(P^\bullet_{w'})) = \mathrm{hl}(P^\bullet_{w,\lambda})$. We claim the generalized string $w' = (w_1w_2\cdots w_n)^d$ is the one as required. Roughly speaking, the complex $P^\bullet_{w'}$ can be seen as the one untying the "band complex" $P^\bullet_{w,\lambda}$ into a "string complex". Let $P^\bullet_w$ be the indecomposable object determined by $w = w_1w_2\cdots w_n$ viewed as a generalized string. Then for any $i \in \mathbb{Z}$ except $i = 0$,

$$\dim H^i(P^\bullet_{w,\lambda}) = d\cdot \dim H^i(P^\bullet_w) = \dim H^i(\beta(P^\bullet_{w'})).$$

Moreover, if $i = 0$, then

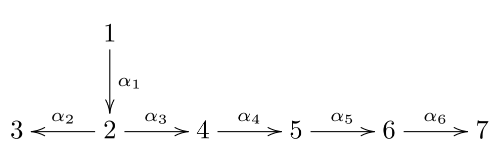
$$\dim H^0(P^\bullet_{w,\lambda}) = \dim(\mathrm{Ker}P(w_1^{-1})\boldsymbol{I}_d \cap \mathrm{Ker}P(w_n)\boldsymbol{J}_{\lambda,d}) = 0 = \dim H^0(\beta(P^\bullet_{w'})).$$

Therefore, $\mathrm{hl}(\beta(P^\bullet_{w'})) = \mathrm{hl}(P^\bullet_{w,\lambda})$ as claimed. □

4 A negative answer to Question II

In this section, we will construct a gentle algebra which provides a negative answer to Question II.

Let $A_0 = kQ/I$ be the gentle algebra defined by the quiver



and the admissible ideal generated by $\alpha_1\alpha_3$. Now we consider the indecomposable object $P_w^\bullet$ determined by generalized string $w = \alpha_1$, where

$$P_w^\bullet = \ 0 \longrightarrow P_2 \xrightarrow{P(w)} P_1 \longrightarrow 0$$

with P_1 in the 0-th component. Clearly, $\dim H^{-1}(P_w^\bullet) = 4$ and $\dim H^0(P_w^\bullet) = 1$. So $\mathrm{hr}(P_w^\bullet) = \mathrm{hl}(P_w^\bullet) \cdot \mathrm{hw}(P_w^\bullet) = 8$.

Next, we claim that there is no indecomposable object in $D^b(A_0)$ with cohomological range 7. Assume to the contrary that there is an indecomposable $P^\bullet \in K^b(\mathrm{proj}A_0)$ with $\mathrm{hr}(P^\bullet) = 7$. Then $\mathrm{hw}(P^\bullet) = 7$ or $\mathrm{hl}(P^\bullet) = 7$. We shall show they are impossible. Indeed, by the description due to [2], the indecomposables in the $D^b(A_0)$ are determined by the generalized strings in A_0. Since the indecomposables in $D^b(A_0)$ are determined by the generalized strings, we have

$$\mathrm{gl.hw}A_0 := \sup\{\mathrm{hw}(X^\bullet) \mid X^\bullet \in D^b(A_0) \text{ is indecomposable}\} = 3.$$

Moreover, since any generalized string in A_0 is one-sided, each component of the indecomposable object $P_w^\bullet \in K^b(\mathrm{proj}A_0)$ is indecomposable, and then

$$\mathrm{gl.hl}A_0 := \sup\{\mathrm{hl}(X^\bullet) \mid X^\bullet \in D^b(A_0) \text{ is indecomposable}\} \leqslant \dim P_2 = 6.$$

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