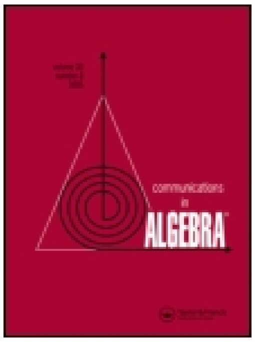

Communications in Algebra

ISSN: 0092-7872 (Print) 1532-4125 (Online) Journal homepage: http://www.tandfonline.com/loi/lagb20

Derived Brauer–Thrall Type Theorem I for Artin Algebras

Chao Zhang

Communications in Algebra®, 44: 3509–3517, 2016

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927872.2015.1085546

DERIVED BRAUER–THRALL TYPE THEOREM I FOR ARTIN ALGEBRAS

Chao Zhang
Department of Mathematics, Guizhou University, Guiyang, China and Faculty of Mathematics, Bielefeld University, Bielefeld, Germany

We define the global cohomological range for artin algebras, and define the derived bounded algebras to be the algebras with finite global cohomological range, then we prove the first Brauer–Thrall type theorem for bounded derived categories of artin algebras, i.e., derived bounded algebras are precisely the derived finite algebras. Moreover, the main theorem establishes that the derived bounded artin algebras are just piecewise hereditary algebras of Dynkin type, and can be also characterized as those artin algebras with derived dimension zero, which can be regarded as a generalization of the results of Han–Zhang [11, Theorem 1] and Chen–Ye–Zhang [4, Theorem] in the context of finite-dimensional algebras over algebraically closed fields, respectively.

Key Words: Derived bounded; Derived finite; Global cohomological width; Piecewise hereditary algebras.

2010 Mathematics Subject Classification: 16E35; 16G60; 16E05; 16G20.

INTRODUCTION

Let R be a commutative artin ring. Throughout this article, all algebras are connected associative artin R-algebras with identity. One of the main topics in representation theory of algebras is to study the indecomposable modules. A famous problem are the Brauer–Thrall conjectures, which were formulated by Jans [12]. The Brauer–Thrall conjecture I says that the algebras of bounded representation type are representation-finite. We say an algebra is *of bounded representation type* if the lengths of all indecomposable modules have a common upper bound, and *of unbounded representation type* otherwise. An algebra is said to be *representation-finite* if there are only finitely many isomorphism classes of indecomposable modules. Brauer–Thrall conjecture I was proved for finite-dimensional algebras over arbitrary fields by Roiter [16]. Moreover, Auslander proved it holds for artin algebras using the irreducible morphisms in the classical AR-theory of module category [1].

In recent years, the bounded derived categories of algebras have been studied widely since Happel's work [5, 6]. The study of the classification and distribution

Received October 14, 2014; Revised June 29, 2015. Communicated by D. Zachaia.
Address correspondence to Dr. Chao Zhang, Department of Mathematics, Guizhou University, Guiyang, 550025, China; E-mail: zhangc@amss.ac.cn

of indecomposable objects in the bounded derived category of an algebra is still an important theme in representation theory of algebras, and it is natural to consider a derived version of Brauer–Thrall conjectures. In order to define "derived bounded algebras," an invariant of a complex analogous to the length of a module in the classical Brauer–Thrall conjectures needed to be introduced. The natural invariant that comes to mind is the cohomology dimension vector. However, the cohomology dimension vectors of a complex X and its shifts X[n] are different. This leads us to study the concept of cohomological range as introduced by Han and Zhang for finite dimensional algebras over algebraically closed fields [11, Definition 3]. The cohomological range naturally leads to the concept of derived bounded algebras, and the Brauer–Thrall type theorem I for bounded derived categories of finite-dimensional algebras over algebraically closed fields can be described as follows: an algebra is derived bounded if and only if it is derived finite, and if and only if it is piecewise hereditary of Dynkin type [11, Theorem 1]. For general artin algebras, the piecewise hereditary algebras of Dynkin type contain not only ones of type $\mathbb{A}, \mathbb{D}, \mathbb{E}$, but also of type $\mathbb{B}, \mathbb{C}, \mathbb{F}, \mathbb{G}$, which are also derived bounded algebras in the sense [11] since there are only finitely many indecomposables in the bounded derived category. This stimulates us to describe all derived bounded artin R-algebras. In this article, we generalize the derived Brauer–Thrall type theorem I from finite-dimensional algebras over algebraically closed fields to the case of arbitrary artin R-algebras with R a commutative artin ring. For this, we provide a characterization of piecewise hereditary artin algebras by the finiteness of global cohomological width (see Definition 1.2). Moreover, Happel and Reiten's classification of piecewise hereditary artin algebras [8, Theorem 3.5] reduces the problem to Auslander's theorem on Brauer–Thrall conjecture I for artin algebras. For this reason, the derived Brauer–Thrall theorem I is a consequence of the classical Brauer–Thrall I.

In the extensive research of bounded derived categories $D^b(A)$ of an algebra A, another important invariant is the *derived dimension*, denoted by $\mathrm{der.dim}(A)$, which is defined to be the Rouquire's dimension of $D^b(A)$ as a triangulated category [17]. Roughly speaking, the Rouquire's dimension measures the minimum steps one required to build the whole triangulated category by a single object. An interesting result is that $\mathrm{der.dim}(A) = 0$ if and only if A is an iterated tilted algebra of Dynkin type for a finite-dimensional algebra A over an algebraically closed field [4, Theorem], in which case, if and only if A is piecewise hereditary of Dynkin type [6, Corollary 5.5]. We will show it holds for general artin R-algebras as well. Indeed, we establish the main theorem as follows.

Theorem. *Let A be an artin R-algebra. Then the following assertions are equivalent*:

(1) *A is derived bounded*;
(2) *A is derived finite*;
(3) $\mathrm{der.dim}(A) = 0$;
(4) *A is piecewise hereditary of Dynkin type, i.e., A is derived equivalent to a hereditary artin R-algebra with the ordinary quiver of Dynkin type.*

The article is organized as follows. In the first section, we introduce some numerical invariants of complexes (algebras) including (global) cohomological length, (global) cohomological width, and (global) cohomological range, and characterize piecewise hereditary algebras as the algebras of finite global cohomological width. Moreover, we establish the finiteness of global cohomological range preserves under the derived equivalences. In Section 2, we will prove the previous theorem.

1. NUMERICAL INVARIANTS FOR ARTIN R-ALGEBRAS

Let A be an artin R-algebra. Denote by mod A the category of all right A-modules of finite length. $C^b(\text{proj}A)$ is the categories of all bounded projective complexes, while $C^{-,b}(\text{proj}A)$ is right bounded projective complexes with bounded cohomology. Denote by $K^b(\text{proj}A)$ and $K^{-,b}(\text{proj}A)$ the homotopy categories of $C^b(\text{proj}A)$ and $C^{-,b}(\text{proj}A)$, respectively. Moreover, $D^b(A)$ is the bounded derived category of mod A.

Now we generalize the definitions of some cohomological invariants in bounded derived category in [11] to the case of artin R-algebras.

Definition 1.1. The cohomological length of a complex $X^\bullet \in D^b(A)$ is

$$\text{hl}(X^\bullet) := \max\{l(H^i(X^\bullet)) \mid i \in \mathbb{Z}\},$$

and the global cohomological length of A is

$$\text{gl.hl}A := \sup\{\text{hl}(X^\bullet) \mid X^\bullet \in D^b(A) \text{ is indecomposable}\}.$$

Note that for a stalk complex corresponding to a module M, its cohomological length is just the length of M. Consider the embedding of mod A into $D^b(A)$ which sends a module to the corresponding stalk complex. Then gl.hl$A < \infty$ implies that A is representation-finite by Auslander's theorem on the Brauer–Thrall conjecture I for artin algebras. Moreover, if A is a finite-dimensional algebra over an algebraically closed field, by a characterization given in [11, Proposition 6], gl.hl$A < \infty$ if and only if A is a derived discrete algebra introduced and classified by Vossieck [19].

Definition 1.2. The cohomological width of a complex $X^\bullet \in D^b(A)$ is

$$\text{hw}(X^\bullet) := \max\{j - i + 1 \mid H^i(X^\bullet) \neq 0 \neq H^j(X^\bullet)\},$$

and the global cohomological width of A is

$$\text{gl.hw}A := \sup\{\text{hw}(X^\bullet) \mid X^\bullet \in D^b(A) \text{ is indecomposable}\}.$$

Let $\mathcal{H}$ be a hereditary abelian category, i.e., $\text{Ext}^2_{\mathcal{H}}(-,-) = 0$. The following lemma implies any indecomposable object in the bounded derived category $D^b(\mathcal{H})$ of $\mathcal{H}$ is a stalk complex. In particular, it holds for the module category of a hereditary artin R-algebra. Therefore, if A is a hereditary artin R-algebra, then gl.hw$A = 1$.

Lemma 1.3 (See Krause [13, Section 1.6]). *Let $\mathcal{H}$ be an a hereditary abelian category and $X \in D^b(\mathcal{H})$. Then $X \cong \coprod_{i\in\mathbb{Z}} H^i(X)[-i]$. In particular, if X is indecomposable, then X is isomorphic to a stalk complex.*

Recall that a complex $X^\bullet = (X^i, d^i) \in C(A)$ is said to be *minimal* if $\mathrm{Im} d^i \subseteq \mathrm{rad} X^{i+1}$ for all $i \in \mathbb{Z}$, see [11] for instance. Note that for any complex $X^\bullet$ in bounded derived category, one can always find a minimal complex quasi-isomorphic to $X^\bullet$. Moreover, for a complex

$$P^\bullet = \cdots \longrightarrow P^{-n-1} \xrightarrow{d^{-n-1}} P^{-n} \xrightarrow{d^{-n}} \cdots \longrightarrow P^{m-1} \xrightarrow{d^{m-1}} P^m \longrightarrow 0,$$

its brutal truncation $\sigma_{\geq -n}(P^\bullet)$ is

$$\sigma_{\geq -n}(P^\bullet) = 0 \longrightarrow P^{-n} \xrightarrow{d^{-n}} \cdots \longrightarrow P^{m-1} \xrightarrow{d^{m-1}} P^m \longrightarrow 0.$$

By a slightly different discussion from that in [11, Corollary 2], we have the following proposition.

Proposition 1.4. *Let A is an artin R-algebra. Then $gl.dimA \leq gl.hwA$.*

Proof. Let M be an indecomposable A-module with a minimal projective resolution

$$P_M^\bullet = \cdots \longrightarrow P^{-n-1} \xrightarrow{d^{-n-1}} P^{-n} \xrightarrow{d^{-n}} \cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \longrightarrow 0,$$

such that $P^{-n-1} \neq 0$. First, we claim that the brutal truncation $\sigma_{\geq -n}(P_M^\bullet)$ is indecomposable. In fact, we assume that $\sigma_{\geq -n}(P_M^\bullet) \cong P_1^\bullet \oplus P_2^\bullet$ in $K^b(\mathrm{proj}A)$. Then we have $\mathrm{H}^i(\sigma_{\geq -n}(P_M^\bullet)) \cong \mathrm{H}^i(P_1^\bullet) \oplus \mathrm{H}^i(P_2^\bullet)$, for any $i \in \mathbb{Z}$. In particular, $M \cong \mathrm{H}^0(P_1^\bullet) \oplus \mathrm{H}^0(P_2^\bullet)$ is indecomposable and $\mathrm{H}^i(P_1^\bullet) = \mathrm{H}^i(P_2^\bullet) = 0$ for $-n+1 \leq i \leq -1$. We assume $H^0(P_1^\bullet) = M$ without loss of generality. Then the minimality of $P_M^\bullet$ implies $\sigma_{\geq -n}(P_M^\bullet) \cong P_1^\bullet$. Thus $\sigma_{\geq -n}(P_M^\bullet)$ is indecomposable. Note that $\mathrm{hw}(\sigma_{\geq -n}(P_M^\bullet)) = n+1$. Hence, for any indecomposable module of projective dimension m, there is an indecomposable complex with cohomological width m. Therefore, if $\mathrm{gl.dim}A < \infty$, then $\mathrm{gl.dim}A \leq \mathrm{gl.hw}A$. If $\mathrm{gl.dim}A = \infty$, then there is a simple A-module S of infinite projective dimension. Then by the claim, we can construct indecomposable objects in $K^b(\mathrm{proj}A)$ of arbitrarily large cohomological width by brutal truncations, and thus $\mathrm{gl.hw}A = \infty$. □

Recall that an artin R-algebra A is said to be *piecewise hereditary* if there is a triangle equivalence $D^b(A) \simeq D^b(\mathcal{H})$ for some hereditary abelian R-category $\mathcal{H}$ ([9]). If R is a field, Happel and Zacharia characterized piecewise hereditary algebras as the algebras of finite strong global dimension [10], where the strong global dimension is defined in [18] for finite-dimensional algebras over fields.

The following proposition characterizes the piecewise hereditary algebras as the algebras of finite global cohomological width, which is a generalization of [11, Proposition 3].

Proposition 1.5. *Let A be an artin R-algebra. Then A is piecewise hereditary if and only if gl.hw$A < \infty$.*

Proof. Let F: $D^b(A) \to D^b(\mathcal{H})$ be an equivalence of triangulated categories, and $P_j (1 \le j \le p)$ be the indecomposable projective A-modules. For any object $X^\bullet \in D^b(A)$, we have

$$H^i(X^\bullet) = \mathrm{Hom}_{D^b(A)}(A, X^\bullet[i]) \cong \mathrm{Hom}_{D^b(\mathcal{H})}(FA, FX^\bullet[i]).$$

Since $\mathcal{H}$ is hereditary abelian category, the indecomposables in $D^b(\mathcal{H})$ are stalk complexes by Lemma 1.3. Thus $FX^\bullet \in \mathcal{H}[r]$ for some $r \in \mathbb{Z}$, and $FA = \bigoplus_{j=1}^p FP_j \in \bigcup_{l=s}^t \mathcal{H}[l]$. Note that $\mathrm{Hom}_{D(\mathcal{H})}(\mathcal{H}[i], \mathcal{H}[j]) \ne 0$ implies $j = i$ or $i + 1$. If $H^i(X^\bullet) \ne 0$, then $s \le r + i \le t + 1$, and thus $s - r \le i \le t - r + 1$. Therefore, $\mathrm{hw}(X^\bullet) \le t - s + 2$. Consequently, gl.hw$A \le t - s + 2 < \infty$.

Conversely, since gl.hw$A < \infty$ and Proposition 1.4, gl.dim$A < \infty$, and then $D^b(A) \simeq K^b(\mathrm{proj}A)$. If A is not a piecewise hereditary artin R-algebra, by [15] and the proof of [10, Corollary 2.3], for any indecomposable object $X^\bullet \in D^b(A)$ and any integer $n > 0$, there exist pairwise non-isomorphic indecomposable objects $X_0^\bullet = X^\bullet[n], X_1^\bullet, \cdots, X_m^\bullet = X^\bullet$, such that $\mathrm{Hom}_{D^b(A)}(X_i^\bullet, X_{i+1}^\bullet) \ne 0$, $0 \le i \le m - 1$. With a discussion similar to [10, Lemma 2.4], there exist an indecomposable object $P^\bullet \in K^b(\mathrm{proj}A)$ and $t \ge 0$, such that

$$\mathrm{Hom}_{D^b(A)}(X^\bullet[n + t], P^\bullet) \ne 0 \ne \mathrm{Hom}_{D^b(A)}(P^\bullet, X^\bullet).$$

Set $X^\bullet = Q$ with Q an indecomposable projective A-module. Then $H^{-n-t}(P^\bullet) \ne 0$ and $\max\{i \mid H^i(P^\bullet) \ne 0\} \ge 0$, which implies $\mathrm{hw}(P^\bullet) > n + t + 1$. Thus there exist indecomposable objects in $D^b(A)$ of arbitrary large cohomological width, which contradicts to the assumption gl.hw$A < \infty$. □

Now we define the global cohomological range for artin R-algebras.

Definition 1.6 ([11, Definition 3]). The *cohomological range* of a complex $X^\bullet \in D^b(A)$ is

$$\mathrm{hr}(X^\bullet) := \mathrm{hl}(X^\bullet) \cdot \mathrm{hw}(X^\bullet),$$

and the *global cohomological range* of A is

$$\mathrm{gl.hr}A := \sup\{\mathrm{hr}(X^\bullet) \mid X^\bullet \in D^b(A) \text{ is indecomposable}\}.$$

The cohomological range will be used to define the derived bounded algebras in the research of a derived version of the Brauer–Thrall theorem I. Moreover, it is invariant under shifts and isomorphisms.

Next we will prove the finiteness of global cohomological range preserves under derived equivalences. Let $S_1, S_2, \cdots, S_n$ be all the simple A-modules. For any $M \in$ mod A, the *length vector* $\mathbf{l}(M) = (m_1, m_2, \cdots, m_n)^T$, where m_i is the number of composition factors of M isomorphic to S_i. Moreover, for any complex $X^\bullet \in D^b(A)$, the *cohomological length vector* $\mathbf{L}(X^\bullet) = (\mathbf{l}(H^j(X^\bullet)))_{j \in \mathbb{Z}}$. Clearly, viewing $\mathbf{L}(X^\bullet)$ as a matrix, we have $\mathrm{hr}(X^\bullet) \ge \sum_{i,j} \mathbf{L}(X^\bullet)_{ij}$.

Proposition 1.7. *Let two artin R-algebras A and B be derived equivalent. Then gl.hrA < ∞ if and only if gl.hrB < ∞.*

Proof. We assume the functor F: $D^b(A) \to D^b(B)$ is an equivalence. It suffices to show that gr.hr$A < \infty$ implies gr.hr$B < \infty$. Note that any indecomposable object $X^\bullet \in D^b(A)$ is generated by the cohomologies via triangles and the cohomologies can be also obtained by triangles with the simples. Since any cohomological functor sends a triangle to a long exact sequence, by the additivity of length functor, we have the following estimate (see also [19]):

$$\mathbf{L}(F(X^\bullet)) \leq \sum_{j\in\mathbb{Z}} \sum_{i=1}^{n} \mathbf{L}(X^\bullet)_{ij}\mathbf{L}(F(S_i[j])).$$

If gl.hr$A = r < \infty$, then $\mathbf{L}(X^\bullet)_{ij} \leq r$ and j runs over at most r integers. Thus for all indecomposable complex $X^\bullet \in D^b(A)$, hr$(F(X^\bullet))$ have a common upper bound. Since F is dense, we have gl.hr$B < \infty$, and the proposition follows. □

2. THE PROOF OF MAIN THEOREM

We first recall the definition of derived bounded algebras.

Definition 2.1 ([11, Definition 4]). An artin R-algebra A is said to be *derived bounded* if gl.hr$A < \infty$, i.e., the cohomological ranges of all indecomposable objects in $D^b(A)$ have a common upper bound.

Recall that an algebra A is said to be *derived finite* if up to shifts and isomorphisms there are only finitely many indecomposable objects in $D^b(A)$ ([3]).

For the proof of the main theorem, we need the following lemma, which was implicitly stated in [8, Introduction], and we provide the proof here for completeness.

Lemma 2.2. *Let A be a piecewise hereditary artin R-algebra. Then A is derived equivalent to a hereditary artin algebra, or a representation-infinite artin algebra.*

Proof. Let F: $D^b(A) \to D^b(\mathcal{H})$ be an equivalence between triangulated categories and $\mathcal{H}$ be a connected Ext-finite hereditary abelian R-category. Then there exists a field k such that $\mathcal{H}$ is a connected Ext-finite hereditary abelian k-category [8, Lemma 1.1]. In fact, by [7, Theorem 1.7], $\mathcal{H}$ has a tilting object T. Set $\Lambda = \mathrm{End}_{D^b(\mathcal{H})}(T)^{\mathrm{op}}$. Then Λ is an indecomposable artin R-algebra and $D^b(\mathcal{H}) \simeq D^b(\Lambda)$ as a triangulated R-category. Note the center $Z(\Lambda)$ of Λ is a commutative local artin ring and there is a canonical ring morphism $\phi : R \to Z(\Lambda)$. Moreover, by the proof of [8, Lemma 1.1], $k = Z(\Lambda)$ is a field, and Λ is a finite-dimensional k-algebra. For any $M, N \in \mathcal{H}$, $i \in \mathbb{Z}$, we have

$$\mathrm{Hom}_{D^b(\mathcal{H})}(M, N[i]) \cong \mathrm{Hom}_{D^b(\Lambda)}(FM, FN[i]).$$

Therefore, $\mathcal{H}$ is a connected Ext-finite hereditary abelian k-category.

By [8, Theorem 3.5], as a triangulated k-category, $D^b(\mathcal{H})$ is equivalent to the bounded derived category of a hereditary k-algebra H, or a representation-infinite k-algebra C. Note that the ring morphism $\phi : R \to k = Z(\Lambda)$ implies that k-algebras H and C are also R-algebras. Assume F: $D^b(\mathcal{H}) \to D^b(A)$ (or $D^b(C)$) be the equivalence of triangulated k-categories. Then F induces a functor F': $D^b(\mathcal{H}) \to D^b(A)$ of triangulated R-categories by defining F' acts on objects and morphisms identical with F. The slight difference between F and F' is that they act on the homomorphism spaces as a k-map and an R-map, respectively. Since the R-module structure of homomorphism spaces is induced by the ring morphism ϕ, F' is also an equivalence. Therefore, $D^b(\mathcal{H})$ is equivalent to the bounded derived category of R-algebra H, or R-algebra C. Note that $D^b(A) \simeq D^b(\mathcal{H})$ as R-categories by assumption. Then it suffices to show H is a hereditary R-algebra and C is a representation-infinite R-algebra. In fact, for any H-modules M, N, $\mathrm{Ext}^2_H(M, N)_k = 0$, and thus $\mathrm{Ext}^2_H(M, N)_R = 0$, so H is hereditary as an artin R-algebra. Moreover, denote by C_k the algebra C as k-algebra and by C_R the algebra C as R-algebra, then for any C-modules M, N, $\mathrm{Hom}_{C_k}(M, N) \cong \mathrm{Hom}_{C_R}(M, N)$ as abelian group. Thus $\mathrm{End}_{C_k}(M)$ is local if and only if $\mathrm{End}_{C_R}(M)$ is local, and $M \cong N$ as C_R-modules if and only if as C_k-modules. Therefore, C is also representation-infinite as an R-algebra. □

Before the proof of our main theorem, we recall the definition of derived dimension from [17]. Let $\mathcal{T}$ be a triangulated R-category with the shift functor [1] and $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{T}$. Denote by $\langle \mathcal{X} \rangle$ the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{X}$ and closed under isomorphisms, finite direct sums, finite direct summands, and shifts. Define $\mathcal{X} \star \mathcal{Y} \subseteq \mathcal{T}$ consisting of objects M if there is a distinguished triangle $X \to M \to Y \to X[1]$ for some $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Now we define $\langle \mathcal{X} \rangle_n$ inductively by $\langle \mathcal{X} \rangle_0 = \{0\}$, $\langle \mathcal{X} \rangle_n = \langle \langle \mathcal{X} \rangle_{n-1} \star \langle \mathcal{X} \rangle \rangle$. Clearly, $\langle \mathcal{X} \rangle_{n-1} \subseteq \langle \mathcal{X} \rangle_n$ and $\langle \mathcal{X} \rangle_\infty := \bigcup_{n \geq 0} \langle \mathcal{X} \rangle_n$ is the smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{X}$. The *Rouquire's dimension* $\dim(\mathcal{T})$ of $\mathcal{T}$ is defined to be the smallest integer d such that there exists an object $M \in \mathcal{T}$ with $\mathcal{T} = \langle M \rangle_{d+1}$, and ∞ if there is no such object M. By definition, the *derived dimension* of an artin R-algebra A is the Rouquire's dimension of triangulated category $D^b(A)$.

Now we are ready to prove the main theorem. The following theorem establishes a derived version of the Brauer–Thrall type theorem and also provides the characterization of the artin R-algebras with derived dimension zero, which can be regarded as a generalization of the results of Han and Zhang [11, Theorem 1] and Chen, Ye, and Zhang [4, Theorem] for finite-dimensional algebras over algebraically closed fields, respectively.

Theorem 2.3. *Let A be an artin R-algebra. Then the following assertions are equivalent*:

(1) *A is derived bounded*;
(2) *A is derived finite*;
(3) $\mathrm{der.\,dim}(A) = 0$;
(4) *A is piecewise hereditary of Dynkin type, i.e., A is derived equivalent to a hereditary R-algebra with the ordinary quiver is of Dynkin type.*

Proof. (1) ⇒ (4): By assumption, $\text{gl.hr}A < \infty$, and thus $\text{gl.hw}A < \infty$. It follows by Proposition 1.5 that A is piecewise hereditary. By Lemma 2.2, A is derived equivalent to a hereditary algebra H or a representation-infinite algebra C. In the first case, we have $\text{gl.hr}H < \infty$ by Proposition 1.7. Moreover, by Auslander's theorem on the Brauer–Thrall conjecture I for artin algebras [1], H is representation-finite, and thus a hereditary algebra of Dynkin type [2, Chapter 8 Theorem 5.4]. In the second case, since C is representation-infinite, $\text{gl.hr}\, C = \infty$ by Auslander's theorem on the Brauer–Thrall conjecture I for artin algebras [1], which is a contradiction to the assumption $\text{gl.hr}A < \infty$ by Proposition 1.7.

(4) ⇒ (3): Let $D^b(A) \simeq D^b(H)$ with H hereditary of Dynkin type. Then by Lemma 1.3, the indecomposables in $D^b(H)$ are stalk complexes corresponding to indecomposable H-modules. Moreover, H is representation-finite by [2, Chapter 8 Theorem 5.4]. Therefore, $D^b(H) = \langle M[0] \rangle$, where M is the direct summand of all indecomposable H-modules. Consequently, $\text{der.}\dim(A) = \text{der.}\dim(H) = 0$.

(3) ⇒ (2): Since $D^b(A)$ is a Krull–Schmidt triangulated category [14, Corollary B], the assumption $\text{der.}\dim(A) = 0$ implies $D^b(A)$ has only finitely many indecomposable objects up to shifts, that is, A is derived finite.

(2) ⇒ (1): Trivial. □

ACKNOWLEDGMENTS

The author would like to thank Yang Han for his suggestion on this topic and constant support, and to thank Yu Zhou for helpful discussions. Moreover, the author is very grateful to the anonymous referees for many helpful comments.

FUNDING

The author is sponsored by Sino-Germany (CSC-DAAD) Postdoc scholarship and Project 11171325 NSFC.

REFERENCES

[1] Auslander, M. (1974). Representation theory of artin algebras II. *Comm. Algebra* 2:269–310.
[2] Auslander, M., Reiten, I., Smalø, S. O. (1997). *Representation Theory of Artin Algebras*. Cambridge University Press.
[3] Bekkert, V., Merklen, H. (2003). Indecomposables in derived categories of gentle algebras. *Alg. Rep. Theory* 6:285–302.
[4] Chen, X. W., Ye, Y., Zhang, P. (2008). Algebras of derived dimension zero. *Comm. Algebra* 36:1–10.
[5] Happel, D. (1987). On the derived category of a finite-dimensional algebra. *Comment. Math. Helv.* 62:339–389.
[6] Happel, D. (1988). Triangulated categories in the representation theory of finite dimensional algebras. *London Math. Soc. Lecture Notes Ser.* 119.
[7] Happel, D., Reiten, I. (1998). Directing objects in hereditary categories. *Contemp. Math.* 229:169–179.
[8] Happel, D., Reiten, I. (2002). Hereditary abelian categories with tilting object over arbitrary base fields. *J. Algebra* 256:414–432.

[9] Happel, D., Reiten, I., Smalø, S. (1996). Piecewise hereditary algebras. *Arch. Math.* 66:182–186.
[10] Happel, D., Zacharia, D. (2008). A homological characterization of piecewise hereditary algebras. *Math. Z.* 260:177–185.
[11] Han, Y., Zhang, C. Brauer-Thrall type theorems for derived category, arXiv:1310.2777.
[12] Jans, J. P. (1957). On the indecomposable representations of algebras. *Ann. Math.* 66:418–429.
[13] Krause, H. (2007). Derived categories, resolutions, and Brown representability. *Contemp. Math.* 436:101–139.
[14] Le, J., Chen, X. W. (2007). Karoubianness of a triangulated category. *J. Algebra* 310:452–457.
[15] Ringel, C. M. Hereditary triangulated categories, to appear in *Compos. Math.*
[16] Roiter, A. V. (1968). The unboundedness of the dimensions of the indecomposable representations of algebras that have an infinite number of indecomposable representations. *Izv. Akad. Nauk SSSR Ser. Math.* 32:1275–1282, English transl.: Math. USSR, Izv. 2 (1968), 1223–1230.
[17] Rouquier, R. (2008). Dimensions of triangulated categories. *Journal of K-theory: K-theory and its Applications to Algebra, Geometry, and Topology* 1:193–256.
[18] Skowroński, A. (1987). On algebras with finite strong global dimension. *Bull. Polish Acad. Sci.* 35:539–547.
[19] Vossieck, D. (2001). The algebras with discrete derived category. *J. Algebra* 243: 168–176.