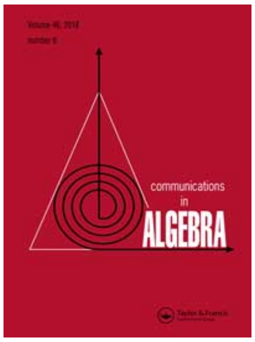

Communications in Algebra

ISSN: 0092-7872 (Print) 1532-4125 (Online) Journal homepage: http://www.tandfonline.com/loi/lagb20

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To cite this article: Chao Zhang (2018) Derived representation type and cleaving functors, Communications in Algebra, 46:6, 2696-2701, DOI: 10.1080/00927872.2017.1399404

To link to this article: https://doi.org/10.1080/00927872.2017.1399404

Published online: 15 Dec 2017.

COMMUNICATIONS IN ALGEBRA®
2018, VOL. 46, NO. 6, 2696–2701
https://doi.org/10.1080/00927872.2017.1399404

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Derived representation type and cleaving functors

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ABSTRACT
We introduce the notation of weakly derived tameness, and establish the equivalence of derived tameness and weakly derived tameness for algebras of finite global dimension. Moreover, we observe the relation between derived representation type and cleaving functors, and obtain a method to judge an algebra to be derived wild. As an application, we determine the derived representation type of self-injective Nakayama algebras.

ARTICLE HISTORY
Received 29 September 2014
Communicated by Q. Wu

KEYWORDS
Cleaving functor; derived tame; derived wild; self-injective Nakayama algebras

2010 MATHEMATICS SUBJECT CLASSIFICATION
16E35; 16G60; 16E05; 16G20

1. Introduction

Throughout this paper, we denote by k an algebraically closed field. We assume that the algebras are associative finite-dimensional connected basic k-algebras with identity. During the research of representation theory of algebras, one main topic is the study of representation type. The representation-finite algebras are well-understood, for example, hereditary algebras of Dynkin type, Nakayama algebras and so on, see [3, Chapter VI]. In the intensive research of indecomposables in module category of representation-infinite algebras, Drozd proved his celebrated Tame and Wild theorem, that is, any finite-dimensional algebra is either tame or wild, but not both, where tameness means that all indecomposables with fixed dimension can be parameterized by only one continuous parameter [13].

In the study of representation type of finite-dimensional algebras, one of the most effective techniques is observing the algebras connected by cleaving functors, which was first adapted in [5] for the study of the representation-finite algebras. In the context of cleaving functors, we should view the bounded quiver algebras as bounded categories by seeing the vertices as objects and the combinations of paths as morphisms. A functor $F : B \to A$ is a cleaving functor, roughly speaking, if F acts on the morphisms with natural retractions. The authors established in [5] that A representation-finite implies B representation-finite. Moreover, Geiss explored a sufficient condition to judge an algebra to be wild in terms of cleaving functors: if there is a cleaving functor from a wild locally bounded category to another locally bounded category, say A, then A is also of wild type [15].

The bounded derived categories of finite-dimensional algebras have been studied widely by Happel [17, 18]. The study of the derived representation type of an algebra becomes an important theme in representation theory of algebras. In particular, the notations of derived tameness and derived wildness were introduced [12, 16], and the tame-wild dichotomy for bounded derived categories of finite-dimensional algebras is established [7], see also [4, Theorem 2.4]. The derived representation type of an algebra measures the complexity of its bounded derived category on the level of indecomposable objects. In compare with the module category, the bounded derived category is more complicated and the derived representation type is determined just for hereditary algebras [2], radical square zero algebras [6, 8], local and two-point algebras [9], gentle algebras [10], tree algebras [11], tubular algebras [19].

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In the present paper, we mainly observe the relation between derived representation type and cleaving functors and obtain a method to judge a finite-dimensional k-algebra to be derived wild:

Theorem. *Let $F : B \to A$ be a cleaving functor between bounded categories with gl.dim$B < \infty$. Then B derived wild implies A derived wild.*

As an application, we determine the derived representation type of self-injective Nakayama algebras, or equivalently, m-truncated cycle algebras, from which we find a class of representation-finite algebras with very complicated bounded derived categories on the level of indecomposable objects. This result follows also from [4, Corollary 2.6] and [22].

Proposition. *An m-truncated cycle algebras is derived tame if and only if $m = 2$.*

This paper is organized as follows. In the first section, we introduce the notation of weakly derived tameness and prove that an algebra of finite global dimension is weakly derived tame if and only if it is derived tame. In Section 2, we obtain a method to judge an algebra to be derived wild by using cleaving functors. In the last section, we determine the derived representation type of self-injective Nakayama algebras as an application.

2. Weakly derived tameness

Throughout this article, bounded quiver algebras are viewed as bounded categories [14]. Recall that a *locally bounded category* is a k-linear category A satisfying:
(1) different objects in A are not isomorphic;
(2) the endomorphism algebra $A(a, a)$ is local for all $a \in A$;
(3) $\dim_k \sum_{x\in A} A(a, x) < \infty$ and $\dim_k \sum_{x\in A} A(x, a) < \infty$ for all $a \in A$.
A *bounded category* is a locally bounded category having only finitely many objects. Note that a bounded quiver algebra $A = kQ/I$ with Q a finite quiver and I an admissible ideal can be viewed as a bounded category by taking the vertices in Q_0 as objects and the k-linear combinations of paths in kQ/I as morphisms. Conversely, a bounded category A admits a presentation $A \cong kQ/I$ for a finite quiver Q and an admissible ideal I.

Let A be a locally bounded category and Λ be a k-algebra. A Λ-A-*bimodule* M is a contravariant k-linear functor from A to the category of left Λ-modules. If $\Lambda = k$, Λ-A-bimodules are just right A-modules. We denote Mod$A$ the category of all right A-modules and by mod$A$ the full subcategory of Mod$A$ consisting of all A-modules satisfying $\dim M(x) < \infty$, for any $x \in A$.

Recall that for any $M \in$ mod$A$, the *dimension vector* of M is the vector $\mathbf{dim}M = (\dim M(x))_{x\in A}$. A locally bounded category A is called *tame* if for any $\mathbf{d} \in \mathbb{N}^{|A|}$, there are finitely many $k[x]$-A-bimodules $M_1, M_2, \cdots, M_r$ which are $k[x]$-free modules of finite rank, such that any indecomposable right A-module with dimension vector $\mathbf{d}$ is of form $S \otimes_{k[x]} M_i$, for some $1 \le i \le r$ and some simple $k[x]$-module S, $\lambda \in k$. A is called *wild* if there is a $k\langle x, y\rangle$-A-bimodule M which is $k\langle x, y\rangle$-free module of finite rank, such that the functor $- \otimes_{k\langle x,y\rangle} M : \text{mod}k\langle x, y\rangle \to \text{mod}A$ preserves indecomposability and isomorphism classes, see [15] for example.

In [15], Geiss introduced the definition of weakly tameness and proved the equivalence of tameness and weakly tameness for locally bounded category by using geometric method. A locally bounded category A is called *weakly tame* if for any $\mathbf{d} \in \mathbb{N}^{|A|}$, there are finitely many $k[x]$-A-bimodules M_1, $M_2, \cdots, M_r$ which are $k[x]$-free modules of finite rank, such that each indecomposable right A-module with dimension vector $\mathbf{d}$ is a direct summand of $S \otimes_{k[x]} M_i$, for some simple $k[x]$-module S.

Let A be a bounded category. We denote by $K^b(\text{proj}A)$ the homotopy category of bounded complexes of projective right A-modules and by $D^b(A)$ the bounded derived category of mod$A$. For any $X^\bullet \in D^b(A)$, the *cohomology dimension vector* of $X^\bullet$ is defined to be the vector $\mathbf{Dim}(X^\bullet) = (\dim_k H^n(X^\bullet))_{n\in\mathbb{Z}}$.

Recall from [16] that a bounded category A is *derived tame* if for any $\mathbf{h} \in \mathbb{N}^{(\mathbb{Z})}$, there exist a localization $R = k[x]_f$ with respect to some $f \in k[x]$ and bounded complexes $X_1^\bullet, X_2^\bullet, \cdots, X_r^\bullet$ of R-A bimodules which are R-free of finite rank, such that almost all indecomposable objects of cohomology dimension vector $\mathbf{h}$ in $D^b(A)$ are of form $S \otimes_R X_i^\bullet$ for some i and simple R-module S. Note that for any $X_i^\bullet$ as above, we can find a bounded complex $Y_i^\bullet$ of $k[x]$-A bimodules such that $S \otimes_R X_i^\bullet \cong S \otimes_{k[x]} Y_i^\bullet$ for all simple R-module S. So equivalently, A is derived tame if for any $\mathbf{h} \in \mathbb{N}^{(\mathbb{Z})}$, there exist bounded complexes $X_1^\bullet, X_2^\bullet, \cdots, X_r^\bullet$ of $k[x]$-A bimodules which are $k[x]$-free of finite rank, such that almost all indecomposable objects of cohomology dimension vector $\mathbf{h}$ in $D^b(A)$ are of form $S \otimes_{k[x]} X_i^\bullet$ for simple $k[x]$-module S, see [6, 8, 10] for details. Moreover, A is *derived wild* if there exists a bounded complex $M^\bullet$ of $k\langle x, y\rangle$-A-modules which are $k\langle x, y\rangle$-free of finite rank, such that the functor $- \otimes_{k\langle x,y\rangle} M^\bullet$: $\text{mod}\, k\langle x, y\rangle \to \text{mod}A$ preserves indecomposability and isomorphism classes.

Now we define weakly derived tameness for bounded categories.

Definition 2.1. A bounded category A is *weakly derived tame* if for any $\mathbf{h} \in \mathbb{N}^{(\mathbb{Z})}$, there exist a localization $R = k[x]_f$ with respect to some $f \in k[x]$ and bounded complexes $X_1^\bullet, X_2^\bullet, \cdots, X_r^\bullet$ of R-A bimodules which are R-free of finite rank, such that almost all indecomposable $X^\bullet$ of cohomology dimension vector $\mathbf{h}$ in $D^b(A)$ is a direct summand of $S \otimes_R X_i^\bullet$, for some $1 \leq i \leq r$ and some simple R-module S.

Note that in the definition of weakly derived tameness, as in the definition of derived tameness, the bounded complexes of $k[x]_f$-A bimodules can be replaced with bounded complexes of $k[x]$-A bimodules. Moreover, derived tameness implies weakly tameness obviously. We will prove the converse is true for these bounded categories with finite global dimension. For this we need some preparations.

Let A be a bounded category. Recall that the repetitive category $\hat{A}$ of A has the pairs (a, i) as objects, where $a \in A$ and $i \in \mathbb{Z}$, while the morphisms from (a, i) to (b, i) and $(b, i + 1)$ are determined by $A(a, b)$ and $A(b, a)$, respectively, and zero else [21]. Note that $\hat{A}$ is self-injective locally bounded category. Moreover, there is a full embedding triangulated functor $F : D^b(A) \to \underline{\text{mod}}\hat{A}$.

Lemma 2.2. *Let A be a bounded category with finite global dimension. Then A is weakly derived tame if and only if $\hat{A}$ is weakly tame.*

Proof. It is well known that the functor $F : D^b(A) \to \underline{\text{mod}}\hat{A}$ is an equivalence in the case of gl.dim$A < \infty$. We assume $\hat{A}$ is weakly tame. For any $\mathbf{h} \in \mathbb{N}^{(\mathbb{Z})}$ and any indecomposable complex $X^\bullet \in D^b(A)$ of cohomology dimension vector $\mathbf{h}$, $F(X^\bullet) \in \underline{\text{mod}}\hat{A}$ has a common upper bound in $\mathbb{N}^{(A\times\mathbb{Z})}$ by Lemma 4.6 in [16], then there exist finitely many $k[x]$-$\hat{A}$-bimodules $M_1, M_2, \cdots, M_l$ which are $k[x]$-free of finite rank, such that any indecomposable complex $X^\bullet \in D^b(A)$ of cohomology dimension vector $\mathbf{h}$ satisfies $F(X^\bullet) \oplus M \cong S \otimes_{k[x]} M_j$ for some j, S and $M \in \underline{\text{mod}}\hat{A}$. Note that $M \cong F(Y^\bullet)$ for some $Y^\bullet \in D^b(A)$ by the denseness of F. Since by Proposition 5.2(2) in [16], we can find a localization $R = k[x]_f$ and bounded complexes $Y_j^\bullet$ of R-A bimodules which are R-free of finite rank such that $F(S \otimes_R Y_j^\bullet) \cong S \otimes_R (M_j)_f \cong S \otimes_{k[x]} M_j$ for $1 \leq j \leq l$ and all simple S. Thus $F(X^\bullet \oplus Y^\bullet) \cong F(X^\bullet) \oplus F(Y^\bullet) \cong F(S \otimes_R Y_j^\bullet)$, and thus each indecomposable complex $X^\bullet \in D^b(A)$ of cohomology dimension vector $\mathbf{h}$ is a direct summand of $S \otimes_R Y_j^\bullet$ for some $1 \leq j \leq l$ and simple module S. Hence A is weakly derived tame.

Conversely, we assume A is weakly derived tame. For any $\mathbf{d} \in \mathbb{N}^{(A\times\mathbb{Z})}$ and non-projective indecomposable $\hat{A}$-module M, $M \cong F(Y^\bullet)$ for some indecomposable object $Y^\bullet \in D^b(A)$. Moreover, by Lemma 4.7 in [16], the cohomology dimension vector of $Y^\bullet$ has a common upper bound, and thus there exist bounded complexes $Y_1^\bullet, Y_2^\bullet, \cdots, Y_l^\bullet$ of $k[x]$-A bimodules which are $k[x]$-free of finite rank, such that almost all non-projective indecomposable $\hat{A}$-module M of dimension vector $\mathbf{d}$ satisfies $M \cong F(Y^\bullet)$ and is a direct summand of $F(S \otimes_{k[x]} Y_j^\bullet)$ for some $1 \leq j \leq l$ and some simple module S. Moreover, one can construct appropriate localization $R = k[x]_h$ and R-$\hat{A}$-bimodules $M_1, M_2, \cdots, M_l$ which are R-free of finite rank such that M is a direct summand of $S \otimes_R M_j$ for some j and simple R-module S by Proposition 5.2(1) in [16]. Therefore, $\hat{A}$ is weakly tame. □

Now we are ready to prove the equivalence of derived tameness and weakly derive tameness for bounded categories of finite global dimension.

Proposition 2.3. *Let A be a bounded category with gl.dim$A < \infty$. Then A is weakly derived tame if and only if A is derived tame.*

Proof. If A is derived tame, then clearly A is weakly derived tame by definition. Now assume A is weakly derived tame, we have $\hat{A}$ is weakly tame by previous lemma. Note that $\hat{A}$ is locally bounded, then $\hat{A}$ is tame by [15]. Since $\hat{A}$ is tame if and only if A is derived tame for bounded categories of finite global dimension [16], A is derived tame. □

3. Cleaving functors and derived representation type

Recall that to a k-linear functor $F : B \to A$ between bounded categories, we associate a *restriction functor* $F_* : \text{mod}A \to \text{mod}B$, which is given by $F_*(M) = M \circ F$ and exact. The restriction functor F_* admits a left adjoint functor F^*, called the *extension functor*, which sends a projective B-module $B(b, -)$ to a projective A-module $A(Fb, -)$. Moreover, if gl.dim$B < \infty$ then F_* extends naturally to a derived functor $F_* : D^b(A) \to D^b(B)$, which has a left adjoint $\mathbf{L}F^* : D^b(B) \to D^b(A)$. Note that $\mathbf{L}F^*$ is the left derived functor associated with F^* and maps $K^b(\text{proj}B)$ into $K^b(\text{proj}A)$. We refer to [23] for the definition of derived functors.

A k-linear functor $F : B \to A$ with gl.dim$B < \infty$ between bounded categories is called a *cleaving functor* [5, 22] if it satisfies the following equivalent conditions:
(1) The linear map $B(b, b') \to A(Fb, Fb')$ associated with F admits a natural retraction for all $b, b' \in B$;
(2) The adjunction morphism $\phi_M : M \to (F_* \circ F^*)(M)$ admits a natural retraction for all $M \in \text{mod}B$;
(3) The adjunction morphism $\Phi_{X^\bullet} : X^\bullet \to (F_* \circ \mathbf{L}F^*)(X^\bullet)$ admits a natural retraction for all $X^\bullet \in D^b(B)$.

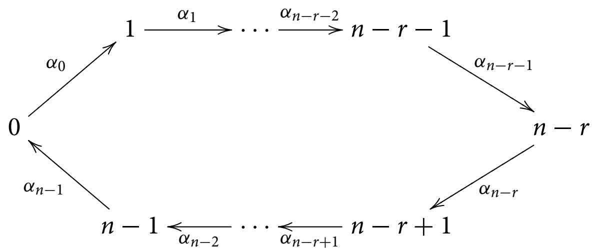
The following proposition provides us a method to determine a bounded category to be derived wild by cleaving functors.

Theorem 3.1. *Let $F : B \to A$ be a cleaving functor between bounded categories with gl.dim$B < \infty$. Then B derived wild implies A derived wild.*

Proof. We assume A is not derived wild and then A is derived tame [7]. Fix $\mathbf{d} \in \mathbb{N}^{(\mathbb{Z})}$, then for any indecomposable complex $X^\bullet \in D^b(B)$ with $\mathbf{Dim}(X^\bullet) = \mathbf{d}$, we have $\mathbf{Dim}(\mathbf{L}F^*(X^\bullet)) \in \mathbb{N}^{(\mathbb{Z})}$ has a common upper bound by the estimate given in [22], and thus so is any indecomposable direct summand $Z^\bullet$ of $\mathbf{L}F^*(X^\bullet)$. Therefore we can choose a localization $R = k[x]_f$ and finitely many bounded complexes $Y_1^\bullet, Y_2^\bullet, \cdots, Y_l^\bullet$ of R-A-bimodules which are R-free of finite rank, such that almost all indecomposable direct summand $Z^\bullet$ of these $\mathbf{L}F^*(X^\bullet)$'s has the form of $S \otimes_R Y_j^\bullet$, for some j and simple module S. Since F is a cleaving functor, for almost all indecomposable complex $X^\bullet$ of cohomological dimension $\mathbf{d}$, we can find a direct summand $Z^\bullet$ of $\mathbf{L}F^*(X^\bullet)$, such that $X^\bullet$ is a direct summand of $F_*(Z^\bullet) = F_*(S \otimes_R Y_j^\bullet) = S \otimes_R F_*(Y_j^\bullet)$, for some j and simple R-module S, which implies that B is weakly derived tame. Thus B is derived tame by Proposition 2.3, which contradicts to the derived wildness of B. □

4. The application to self-injective nakayama algebras

Let $A(n,m)(m \geq 2)$ is the bounded category determined by the quiver



and the relation of all paths of length m. Note that the $A(n,m)$'s, viewed as finite-dimensional algebras, are the so-called m-truncated cycle algebras, and are the unique class of self-injective Nakayama algebras, see [1, Chapter V Proposition 3.8] for details.

As an application of Theorem 3.1, we determine the derived representation type of $A(n,m)$, which coincides with that in [4, Corollary 2.6] and [22]. Note that $A(n,m)$ is an representation-finite algebra, and the following proposition implies that the bounded derived category of $A(n,m)$ is very complicated if $m > 2$.

Proposition 4.1. *$A(n,m)$ is derived tame if and only if $m = 2$.*

Proof. If $m = 2$, then $A(n,m)$ is a gentle algebra and hence derived tame [10]. Now assume $m \geq 3$. We consider the bounded category A_l^p defined by the quiver

$$0 \xrightarrow{\alpha_0} 1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{l-3}} 1-2 \xrightarrow{\alpha_{l-2}} l-1 \ ,$$

and the admissible ideal generated by paths of length p, where $l > p$. It is well known that the functor $F : A_{jn}^m \to A(n,m)$ such that $F(i) = \bar{i}$ and $F(\alpha_i) = \alpha_{\bar{i}}$ is a cleaving functor, where $\bar{i}$ satisfies $0 \leq \bar{i} \leq n-1$ and stands for the representation element of i in the residue class ring $\mathbb{Z}_n$. Moreover, A_l^p is a tree algebra, and thus it is derived tame if and only if its Euler form is non-negative [11]. Note that $A_{m+2}^m (m \geq 8)$ is piecewise hereditary of wild type [20] and thus derived wild, and $A_{11}^3, A_{10}^4, A_{11}^5, A_{11}^6, A_{10}^7$ have negative Euler forms. Therefore, for any $m \geq 3$, A_{4m}^m is derived wild since it contains a derived wild full subcategory, and thus $A(n,m)$ is wild as well by Theorem 3.1. □

Remark 4.2. If $n = 1$, then the proposition implies that $k[x]/(x^m)(m \geq 2)$ is derived tame if and only if $m = 2$, which coincides with the result in [9].

Acknowledgments

The author would like to thank Yang Han, Henning Krause Yu Zhou for helpful discussions, and thank Raymundo Bautista for his kind e-mail to explain the notation of derived tameness. Moreover, the author is very grateful to the anonymous referees for many helpful comments.

Funding

The author is supported by the National Natural Science Foundation of China (Grant No. 11601098 and 11171325) and Natural Science Foundation of Guizhou Province (Grant No. QSF[2016]1038).

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