DERIVED REPRESENTATION TYPE AND FIELD EXTENSIONS

BY

JIE LI (Hefei) and CHAO ZHANG (Guiyang)

Abstract. Let A be a finite-dimensional algebra over a field k. We define A to be **C**-dichotomic if it has the dichotomy property of the representation type on the category of certain bounded complexes of projective A-modules. If k admits a finite separable field extension K/k such that K is algebraically closed (the real number field for example), we prove that A is **C**-dichotomic. As a consequence, the second derived Brauer–Thrall type theorem holds for A, i.e., A is either derived-discrete or strongly derived-unbounded.

1. Introduction. Representation type is an important topic in representation theory of algebras, which studies the classification and distribution of indecomposable modules. The most stimulating problems, the classical Brauer–Thrall conjectures, were formulated for finite-dimensional k-algebras; see [Br41, Th47, Jans57]. The *first Brauer–Thrall conjecture* says that an algebra either is of finite representation type or admits modules of arbitrary large dimensions. The *second Brauer–Thrall conjecture* states that any algebra is either of finite representation type or of strongly unbounded representation type. The two conjectures were proved for algebras over infinite perfect fields; see [ASS06, Aus74, NR75, Ro68, Ro78, Rin80] and references therein.

Brauer–Thrall type theorems are also related to the celebrated tame-wild dichotomy theorem first proved by Drozd [Dro86] for finite-dimensional modules over finite-dimensional K-algebras over an algebraically closed field K. The tame-wild dichotomy theorem was generalized to Cohen-Macaulay modules in [DG92] and to a class of bimodule matrix problems (over not only algebraically closed fields) in [Sim97, Sim05].

For algebras over algebraically closed fields, the derived representation type was pioneered by Vossieck [Vo01]. He defined derived-discrete algebras and classified them into two types: algebras derived equivalent to hereditary algebras of finite type, and gentle one-cycle algebras not satisfying

the *clock condition*. To study Brauer–Thrall type theorems for derived categories, in [ZH16], some numerical invariants were introduced and strongly derived-unbounded algebras were naturally defined. Then in [ZH16] two derived Brauer–Thrall type theorems for algebras over algebraically closed fields were proved. Moreover, the first one was proved by Zhang in [Zh16b] for arbitrary Artin algebras. We mention that the tame-wild dichotomy for bounded derived categories of finite-dimensional algebras was established in [BD03].

The second derived Brauer–Thrall type theorem says that any finite-dimensional algebra A has the dichotomy property at the level of the derived category, i.e., A is either derived-discrete or strongly derived-unbounded. Strongly derived-unbounded algebras have been extensively studied, and dichotomy properties of representation types at the levels of the complex category and homotopy category for finite-dimensional algebras have been obtained [Zh16a]. Note that the dichotomy properties at three levels, including the second derived Brauer–Thrall type theorem, rely heavily on the classification of derived-discrete algebras, where the base field is required to be algebraically closed.

It is natural to ask if the second derived Brauer–Thrall type theorem and other dichotomy properties still hold for algebras over arbitrary infinite fields. Since the classification of derived-discrete algebras over arbitrary fields is unknown, our method is to establish whether the properties are compatible under field extensions. Many properties of module categories, such as representation type [JL82] and Auslander–Reiten theory [Ka00], are compatible under ground field extensions. In [Li19], derived-discreteness is proved to be compatible under finite separable field extensions.

In this paper, we revisit the notions of representation type on complex categories, homotopy categories and derived categories for finite-dimensional algebras over arbitrary fields.

Let A be a finite-dimensional k-algebra. Denote by $\mathbf{C}_m(A\text{-proj})$ the category of homotopy minimal complexes of projective A-modules which are concentrated between degree 0 and m. An algebra A is called $\mathbf{C}$-*dichotomic* if either $\mathbf{C}_m(A\text{-proj})$ is of finite representation type for each $m \geq 1$, or $\mathbf{C}_{m'}(A\text{-proj})$ is of strongly unbounded type for some $m' \geq 1$.

We then prove that $\mathbf{C}$-dichotomic algebras are preserved under finite separable field extensions. Making use of the dichotomy theorem (see [Zh16a, Corollary 2.9]) for algebras over algebraically closed fields, we obtain the following main theorem.

MAIN THEOREM. *Let A be a finite-dimensional k-algebra and K/k be a finite separable field extension such that K is algebraically closed. Then A is $\mathbf{C}$-dichotomic.*

Since the **C**-dichotomy implies the dichotomy properties of representation type at the levels of the homotopy category and derived category (see Proposition 2.10), we have

THE SECOND DERIVED BRAUER–THRALL TYPE THEOREM. *If A is a finite-dimensional k-algebra and K/k is a finite separable extension such that K is algebraically closed, then A is either derived-discrete or strongly derived-unbounded.*

As an example, if k is the real number field $\mathbb{R}$, then the main theorem holds. In particular, the second derived Brauer–Thrall type theorem is true for any finite-dimensional $\mathbb{R}$-algebra.

2. The derived representation type of algebras. In this section, we recall the definitions related to derived representation types, and then introduce **C**-dichotomic algebras.

2.1. Definitions related to derived representation type. Let k be an infinite field and A a finite-dimensional algebra over k. Denote by A-mod the category of all finite-dimensional left A-modules and A-proj its full subcategory consisting of all finitely generated projective left A-modules. Denote by $\mathbf{C}^b(A\text{-mod})$ the category of all bounded complexes of A-mod, and by $\mathbf{C}^b(A\text{-proj})$ its full subcategory consisting of all bounded complexes of A-proj. Denote by $\mathbf{K}^b(A\text{-proj})$ the homotopy category of $\mathbf{C}^b(A\text{-proj})$, and by $\mathbf{D}^b(A\text{-mod})$ the bounded derived category of A-mod.

Recall that a complex $X = (X^i, d^i) \in \mathbf{C}^b(A\text{-proj})$ is said to be *homotopy minimal* if $\operatorname{Im} d^i \subseteq \operatorname{rad} X^{i+1}$ for all $i \in \mathbb{Z}$. For any integer $m \geq 0$, denote by $\mathbf{C}_m(A\text{-proj})$ the full subcategory of $\mathbf{C}^b(A\text{-proj})$ consisting of all homotopy minimal complexes $X = (X^i, d^i)$ such that $X^i = 0$ for any $i \notin \{0, 1, \dots, m\}$.

We recall from [ZH16] the definitions related to finite derived representation type.

DEFINITION 2.1. Let A be a k-algebra.

(1) The category $\mathbf{C}_m(A\text{-proj})$ is defined to be *of finite representation type* if up to isomorphism, there are only finitely many indecomposable objects in $\mathbf{C}_m(A\text{-proj})$.
(2) The category $\mathbf{K}^b(A\text{-proj})$ is defined to be *discrete* if for each cohomology dimension vector, there are only finitely many objects in $\mathbf{K}^b(A\text{-proj})$ (up to isomorphism) with that dimension vector.
(3) The algebra A is defined to be *derived-discrete* if for each cohomology dimension vector, there are only finitely many objects in $\mathbf{D}^b(A\text{-mod})$ (up to isomorphism) with that dimension vector.

The next lemma shows the connections between the above definitions.

LEMMA 2.2. *Let A be a k-algebra. Consider the following statements:*

(1) *for each $m > 0$, $\mathbf{C}_m(A\text{-proj})$ is of finite representation type,*
(2) *the category $\mathbf{K}^b(A\text{-proj})$ is discrete,*
(3) *A is derived-discrete.*

Then (1)⇒(2)⇔(3).

Proof. (1)⇒(2). By [Li19, Lemma 2.5], we only need to prove that, for each dimension vector $(n_i)_{i \in \mathbb{Z}} \in \mathbb{N}^{(\mathbb{Z})}$, the set

$$\{X = (X^i, d^i) \in \mathbf{K}^b(A\text{-proj}) \mid \dim_k X_i = n_i, \ \forall i \in \mathbb{Z}\}$$

has finitely many isomorphism classes. After some shifts, there is an integer $m > 0$ such that $n_i = 0$ for $i < 0$ and $i > m$. By assumption, $\mathbf{C}_m(A\text{-proj})$ has finitely many isomorphism classes. Each object X in the above set is homotopy equivalent to a homotopy minimal complex, denoted by $\bar{X}$, in $\mathbf{C}_m(A\text{-proj})$. Since if $\bar{X}$ is isomorphic to $\bar{Y}$ in $\mathbf{C}_m(A\text{-proj})$ then X is isomorphic to Y in $\mathbf{K}^b(A\text{-proj})$, the above set has finitely many isomorphism classes.

The equivalence (2)⇔(3) follows by applying [Li19, Lemma 2.5]. ∎

Following [ZH16], given a complex X in $\mathbf{C}^b(A\text{-mod})$, the *cohomological range* of X is defined as

$$\mathrm{hr}_k(X) := \mathrm{hl}_k(X) \cdot \mathrm{hw}(X),$$

where

$$\mathrm{hl}_k(X) := \max \{\dim_k H^i(X) \mid i \in \mathbb{Z}\},$$
$$\mathrm{hw}(X) := \max \{j - i + 1 \mid H^i(X) \neq 0 \neq H^j(X)\}.$$

Now we recall from [ZH16] the definitions related to infinite derived representation type. Note that the definition (1) has an equivalent form using the dimension of a complex; see [Zh16a, Lemma 1.6].

DEFINITION 2.3. Let A be a k-algebra.

(1) We say $\mathbf{C}_m(A\text{-proj})$ is *of strongly unbounded type* if there is an increasing sequence $(r_i)_{i \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that for each i, up to isomorphism, there are infinitely many indecomposable objects in $\mathbf{C}_m(A\text{-proj})$ with cohomological range r_i.
(2) We say $\mathbf{K}^b(A\text{-proj})$ is *of strongly unbounded type* if there is an increasing sequence $(r_i)_{i \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that for each i, up to shift and isomorphism, there are infinitely many indecomposable objects in $\mathbf{K}^b(A\text{-proj})$ with cohomological range r_i.
(3) We say A is *strongly derived-unbounded* if there is an increasing sequence $(r_i)_{i \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that for each i, up to shift and isomorphism, there are infinitely many indecomposable objects in $\mathbf{D}^b(A\text{-mod})$ with cohomological range r_i.

EXAMPLE 2.4. (1) Representation-infinite algebras over infinite perfect fields are strongly derived-unbounded, in view of the truth of the classical Brauer–Thrall conjecture II for the module category [NR75, Rin80] and the embedding from the module category to the derived category.

(2) Let k be an algebraically closed field, and A be a gentle algebra with one cycle with clock condition or more than one cycle. Then A is strongly unbounded. Here, the clock condition means that the number of clockwise relations on the cycle equals that of counterclockwise ones. Indeed, A has generalized bands in these cases (see [Rin97] for example), and then one can construct an increasing sequence $(r_i)_{i \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and infinitely many non-isomorphic indecomposables in the derived category for each r_i by the combinatorial description in [BM03].

(3) Algebras which are derived equivalent to strongly derived-unbounded algebras, are also strongly derived-unbounded, since the derived equivalences can be realized as tensor functors by two-sided tilting complexes, under which cohomological ranges can be controlled [ZH16, Prop. 4].

The next lemma shows the connections between the above notions. They were essentially proved in [Bau07, Zh16a], where k was supposed to be algebraically closed. Here we include a proof for an arbitrary infinite field k. The following notion is needed in the proof.

Let $\mathbf{K}^{-,b}(A\text{-proj})$ be the homotopy category consisting of bounded-above complexes with bounded cohomology. There is a well-known triangle equivalence

$$p \colon \mathbf{D}^{b}(A\text{-mod}) \to \mathbf{K}^{-,b}(A\text{-proj}),$$

sending X to its projective resolution pX; see [Wei95]. We can further assume that pX is homotopy minimal. For each P in $\mathbf{K}^{-}(A\text{-proj})$, let $P_{\geq t} \in \mathbf{K}^{b}(A\text{-proj})$ be the brutal truncation of P at degree t.

LEMMA 2.5. *Let A be a k-algebra. Consider the following statements:*

(1) *there is an $m \geq 1$ such that $\mathbf{C}_m(A\text{-proj})$ is of strongly unbounded type,*
(2) *the category $\mathbf{K}^{b}(A\text{-proj})$ is of strongly unbounded type,*
(3) *the algebra A is strongly derived-unbounded.*

Then (1)$\Rightarrow$(2)$\Leftrightarrow$(3).

Proof. (1)$\Rightarrow$(2). By (1), there is an increasing sequence $(r_i)_{i \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that for each i, up to isomorphism, there are infinitely many indecomposable objects in $\mathbf{C}_m(A\text{-proj})$ with cohomological range r_i for some $m \geq 1$. Given two complexes in $\mathbf{C}_m(A\text{-proj})$, the property of homotopy minimality implies that they are isomorphic in $\mathbf{K}^{b}(A\text{-proj})$ if and only if they are isomorphic in $\mathbf{C}_m(A\text{-proj})$. In addition, a complex in $\mathbf{C}_m(A\text{-proj})$ is indecomposable if and only if it is indecomposable as a complex in $\mathbf{K}^{b}(A\text{-proj})$. Hence

for each r_i, there are infinitely many indecomposable objects in $\mathbf{K}^{\mathrm{b}}(A\text{-proj})$ with cohomological range r_i.

(2)$\Leftrightarrow$(3). The "$\Rightarrow$" part holds because $\mathbf{K}^{\mathrm{b}}(A\text{-proj})$ is a full subcategory of $\mathbf{D}^{\mathrm{b}}(A\text{-mod})$ by embedding into $\mathbf{K}^{-,\mathrm{b}}(A\text{-proj})$.

For the "$\Leftarrow$" part, by assumption, there is an increasing sequence $(r_i)_{i\in\mathbb{N}}$ $\in \mathbb{N}^{\mathbb{N}}$ such that for each i, up to shift and isomorphism, there are infinitely many indecomposable objects in $\mathbf{D}^{\mathrm{b}}(A\text{-mod})$ with cohomological range r_i. Let $\mathcal{X}_i$ be the set of complexes in $\mathbf{D}^{\mathrm{b}}(A\text{-mod})$ with cohomological range r_i whose non-zero cohomology concentrates between degrees 1 and r_i. Then up to shift and isomorphism, $\mathcal{X}_i$ has infinitely many objects.

For each $i > 0$ and each X in $\mathcal{X}_i$, denote by pX its homotopy minimal projective resolution and by $(pX)_{\geq 0}$ the brutal truncation at degree 0. Then $(pX)_{\geq 0}$ is an indecomposable object in $\mathbf{C}_{r_i}(A\text{-proj})$ with $\mathrm{hr}((pX)_{\geq 0}) \geq r_i$. By [Li19, Lemma 2.4], the set

$$\{\mathrm{hr}((pX)_{\geq 0}) \mid X \in \mathcal{X}_i\}$$

also has an upper bound. By [Li19, Lemma 2.3], there is a positive number s_i between r_i and the upper bound above such that the set

$$\{(pX)_{\geq 0} \in \mathbf{K}^{\mathrm{b}}(A\text{-proj}) \mid X \in \mathcal{X}_i\}$$

has infinitely many objects up to isomorphism.

Since $(r_i)_{i\in\mathbb{N}}$ is an increasing sequence and $s_i \geq r_i$ for each i, we can inductively pick an increasing subsequence $(s_i')_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ of $(s_i)_{i\in\mathbb{N}}$ such that, up to shift and isomorphism, there are infinitely many indecomposable objects in $\mathbf{K}^{\mathrm{b}}(A\text{-proj})$ with cohomological range s_i'. This completes the proof.

Notice that the proof does not imply the strong unboundedness of $\mathbf{C}_m(A\text{-proj})$ for some m, since different sets $\mathcal{X}_i$ belong to different $\mathbf{C}_m(A\text{-proj})$ and we cannot find a uniform integer m such that for each t_i, up to isomorphism, there are infinitely many indecomposable objects in $\mathbf{C}_m(A\text{-proj})$ with cohomological range t_i for some increasing sequence $(t_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$. ∎

2.2. C-dichotomic algebras. In this subsection, we introduce the definition of $\mathbf{C}$-dichotomic algebras and their relation to the second derived Brauer–Thrall type theorem.

By Lemmas 2.2 and 2.5, we have the following proposition.

PROPOSITION 2.6. *For a finite-dimensional k-algebra A, the following statements are equivalent:*

(1) *the category $\mathbf{K}^{\mathrm{b}}(A\text{-proj})$ is either discrete or of strongly unbounded type,*
(2) *A is either derived-discrete or strongly derived-unbounded.*

As in [ZH16], if one of the above statements holds, we say that *the second derived Brauer–Thrall type theorem* holds for A. The following proposition

of [ZH16] shows that, for any finite-dimensional algebra over an algebraically closed field, the second derived Brauer–Thrall type theorem holds.

PROPOSITION 2.7. *Let A be a finite-dimensional algebra over an algebraically closed field. Then A is either derived-discrete or strongly derived-unbounded.*

DEFINITION 2.8. A finite-dimensional k-algebra A is defined to be **C**-*dichotomic* if either $\mathbf{C}_m(A\text{-proj})$ is of finite representation type for each $m \geq 1$, or $\mathbf{C}_{m'}(A\text{-proj})$ is of strongly unbounded type for some $m' \geq 1$.

REMARK 2.9. The category $\mathbf{C}_m(A\text{-proj})$ is of strongly unbounded type for some integer $m = M \geq 1$, which is equivalent to $\mathbf{C}_m(A\text{-proj})$ being of strongly unbounded type for all $m \geq M$ since $\mathbf{C}_m(A\text{-proj}) \subseteq \mathbf{C}_{m+1}(A\text{-proj})$. Thus a k-algebra A is **C**-dichotomic if either $\mathbf{C}_m(A\text{-proj})$ is of finite representation type for any $m \geq 1$, or $\mathbf{C}_m(A\text{-proj})$ is of strongly unbounded type for almost all positive integers m.

PROPOSITION 2.10. *Assume that A is a **C**-dichotomic finite-dimensional k-algebra.*

(1) *The category $\mathbf{K}^{\mathrm{b}}(A\text{-proj})$ is either discrete or of strongly unbounded type.*
(2) *The second derived Brauer–Thrall type theorem holds for A.*

Proof. (1) By Lemma 2.2, if $\mathbf{K}^{\mathrm{b}}(A\text{-proj})$ is not discrete, then it is not true that $\mathbf{C}_m(A\text{-proj})$ is of finite representation type for each $m \geq 1$. So $\mathbf{C}_{m'}(A\text{-proj})$ is of strongly unbounded type for some $m' \geq 1$ by assumption. By Lemma 2.5, $\mathbf{K}^{\mathrm{b}}(A\text{-proj})$ is of strongly unbounded type.

(2) follows from (1) and Proposition 2.6. ∎

Note that our definitions in this section also make sense for algebras over finite fields. In this case, we give an example which shows that the converse of the proposition may be not true.

EXAMPLE 2.11. Let k be a finite field. For each finite-dimensional k-algebra A, there are finitely many morphisms between projective A-modules. Then by [Li19, Lemma 2.5] (whose proof does not depend on the cardinality of k), $\mathbf{K}^{\mathrm{b}}(A\text{-proj})$ is always discrete.

If A is a hereditary algebra over k which is not of finite representation type, then $\mathbf{C}_1(A\text{-proj})$ is not of finite representation type. However, for any $m \geq 1$ and any $r_i > 0$, there are finitely many objects (X^j, d^j) in $\mathbf{C}_m(A\text{-proj})$ with $\sum_{j=0}^m X^j = r_i$. By [Zh16a, Lemma 1.6] (whose proof does not depend on the cardinality of k), $\mathbf{C}_m(A\text{-proj})$ is not of strongly unbounded type for any $m \geq 1$. Hence A is not **C**-dichotomic.

The **C**-dichotomy implies not only the second derived Brauer–Thrall type theorem, but also the equivalence of discreteness and strongly unbounded properties at three levels, as in the following corollary.

COROLLARY 2.12. *Assume that A is a $\mathbf{C}$-dichotomic finite-dimensional k-algebra.*

(a) *The following three conditions are equivalent:*

(a1) *A is derived-discrete,*
(a2) *the category $\mathbf{K}^{\mathrm{b}}(A\text{-proj})$ is discrete,*
(a3) *the category $\mathbf{C}_m(A\text{-proj})$ is of finite representation type for each $m \geq 1$.*

(b) *The following three conditions are equivalent:*

(b1) *A is strongly derived-unbounded,*
(b2) *the category $\mathbf{K}^{\mathrm{b}}(A\text{-proj})$ is of strongly unbounded type,*
(b3) *the category $\mathbf{C}_m(A\text{-proj})$ is of strongly unbounded type for some $m \geq 1$.*

Proof. We only prove (a), and (b) can be proved in a similar way. By Lemma 2.5, to prove the equivalence of the three statements, it suffices to show that the discreteness of $\mathbf{D}^{\mathrm{b}}(A\text{-mod})$ implies the finiteness of $\mathbf{C}_m(A\text{-proj})$ for any $m > 1$. If not, then $\mathbf{C}_M(A\text{-proj})$ is of strongly unbounded type for some $M > 1$ since A is a $\mathbf{C}$-dichotomic k-algebra. Therefore A is strongly derived-unbounded by Lemma 2.5. This is absurd since no algebra can be derived-discrete and strongly derived-unbounded by definition. ∎

REMARK 2.13. Assume that A is a finite-dimensional k-algebra.

(1) If the field k is algebraically closed, then A is $\mathbf{C}$-dichotomic; see [Zh16a, Corollary 2.9].
(2) We do not know whether or not A is $\mathbf{C}$-dichotomic if the field k is not algebraically closed.

3. Base field extensions. In this section, we mainly explore the $\mathbf{C}$-dichotomy of a k-algebra with k admitting a finite separable extension K/k such that K is an algebraically closed field.

Let K/k be a finite separable field extension. It is well known that the algebra extension $A \to A \otimes_k K$ induces an adjoint pair $(- \otimes_k K, F)$ between A-mod and $A \otimes_k K$-mod, where

$$F \colon A \otimes_k K\text{-mod} \to A\text{-mod}$$

is the restriction functor. These two functors are both exact, mapping projective modules to projective modules and radicals to radicals. So they extend in a natural manner to adjoint pairs between $\mathbf{C}_m(A\text{-proj})$ and $\mathbf{C}_m(A \otimes_k K\text{-proj})$. We still denote them by $(- \otimes_k K, F)$ for convenience.

Since both $- \otimes_k K$ and F are separable functors, each complex X is a direct summand of $F(X \otimes_k K)$ in $\mathbf{C}_m(A\text{-proj})$, and each complex Y is a

direct summand of $F(Y) \otimes_k K$ in $\mathbf{C}_m(A \otimes_k K\text{-proj})$; see [Li19]. So we have the following lemmas.

LEMMA 3.1.

(1) *For each indecomposable object X in $\mathbf{C}_m(A\text{-proj})$, there is an indecomposable direct summand Y of $X \otimes_k K$ in $\mathbf{C}_m(A \otimes_k K\text{-proj})$ such that X is a direct summand of $F(Y)$.*
(2) *For each indecomposable object Y in $\mathbf{C}_m(A \otimes_k K\text{-proj})$, there is an indecomposable direct summand X of $F(Y)$ in $\mathbf{C}_m(A\text{-proj})$ such that Y is a direct summand of $X \otimes_k K$.*

LEMMA 3.2. *Let K/k be a field extension of degree l, and A be a k-algebra.*

(1) *For each indecomposable object X in $\mathbf{C}_m(A\text{-proj})$, $X \otimes_k K$ has at most l indecomposable direct summands up to isomorphism.*
(2) *For each indecomposable object Y in $\mathbf{C}_m(A \otimes_k K\text{-proj})$, $F(Y)$ has at most l indecomposable direct summands up to isomorphism.*

Proof. (1) Let $\{\alpha_1, \ldots, \alpha_l\}$ be a k-basis of K. We have an isomorphism in $\mathbf{C}_m(A\text{-proj})$,

$$F(X \otimes_k K) \simeq X^{\oplus l}, \quad x \otimes \lambda \mapsto (\lambda_i x)_{i=1}^{l},$$

where $X^{\oplus l}$ is the direct sum of l copies of X and the λ_i are elements in k such that $\lambda = \sum_{i=1}^{l} \lambda_i \alpha_i$. So our statement holds since F is an additive functor.

(2) For each indecomposable object Y in $\mathbf{C}_m(A \otimes_k K\text{-proj})$, there is an indecomposable direct summand X of $F(Y)$ in $\mathbf{C}_m(A\text{-proj})$ such that Y is a direct summand of $X \otimes_k K$ (Lemma 3.1). Hence $F(Y)$ is a direct summand of $F(X \otimes_k K)$ in $\mathbf{C}_m(A\text{-proj})$. The isomorphism $F(X \otimes_k K) \simeq X^{\oplus l}$ then implies that $F(Y)$ has at most l indecomposable direct summands. ∎

PROPOSITION 3.3. *Let A be a k-algebra and K/k be a finite separable field extension.*

(1) *For each $m > 0$, $\mathbf{C}_m(A\text{-proj})$ is of finite representation type if and only if so is $\mathbf{C}_m(A \otimes_k K\text{-proj})$.*
(2) *For each $m > 0$, $\mathbf{C}_m(A\text{-proj})$ is of strongly unbounded type if and only if so is $\mathbf{C}_m(A \otimes_k K\text{-proj})$.*

As a consequence, A is $\mathbf{C}$-dichotomic if and only if so is $A \otimes_k K$.

Proof. Let l be the degree of the extension K/k.

(1) *The "if" part.* For each $m \geq 1$, up to isomorphism, let $Y_1, \ldots, Y_n$ be all the indecomposable objects in $\mathbf{C}_m(A \otimes_k K\text{-proj})$. For each indecomposable X in $\mathbf{C}_m(A\text{-proj})$, X is a direct summand of $F(X \otimes_k K)$. By Lemma 3.1, there is an indecomposable object, say Y_i for some $i \in \{1, \ldots, n\}$, which is a direct summand of $X \otimes_k K$ such that X is a direct summand of $F(Y_i)$.

Since up to isomorphism, there are only finitely many indecomposable direct summands of $\bigoplus_{i=1}^{n} F(Y_i)$, $\mathbf{C}_m(A\text{-proj})$ is of finite representation type.

The "only if" part can be proved similarly.

(2) For the "if" part, let $(r_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ be an increasing sequence such that for each i, the set

$$\mathcal{X}_i = \{X \in \mathbf{C}_m(A\text{-proj}) \mid X \text{ is indecomposable with } \mathrm{hr}_k(X) = r_i\}$$

has infinitely many objects up to isomorphism (i.e. $\mathcal{X}_i$ has infinitely many isomorphism classes in $\mathbf{C}_m(A\text{-proj})$). Denote by $\mathcal{Y}_i$ all the indecomposable objects Y in $\mathbf{C}_m(A \otimes_k K\text{-proj})$ such that Y is a direct summand of $X \otimes_k K$ and $F(Y)$ contains X as a direct summand for some X in $\mathcal{X}_i$. Because $\mathcal{X}_i$ contains infinitely many objects up to isomorphism, so does $\mathcal{Y}_i$.

By the exactness of $-\otimes_k K$, we have $\mathrm{hr}_K(X\otimes_k K) = r_i$. By Lemma 3.2(1),

$$\{\mathrm{hr}_K(Y) \mid Y \in \mathcal{Y}_i\} \subseteq [r_i/l, r_i].$$

So there is an integer s_i between r_i/l and r_i such that, up to isomorphism, there are infinitely many indecomposable objects in $\mathbf{C}_m(A \otimes_k K\text{-proj})$ with cohomological range s_i.

Since $(r_i)_{i\in\mathbb{N}}$ is an increasing sequence, for each i there is a larger j such that $r_j > r_i/l$. Because $r_i \geq s_i \geq r_i/l$, we can pick inductively an increasing subsequence $(s_i')_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ of $(s_i)_{i\in\mathbb{N}}$ such that, up to isomorphism, there are infinitely many indecomposable objects in $\mathbf{C}_m(A \otimes_k K\text{-proj})$ with cohomological range s_i'. Therefore $\mathbf{C}_m(A \otimes_k K\text{-proj})$ is of strongly unbounded type.

The "only if" part. Let $(r_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ be an increasing sequence such that for each r_i, the set

$$\mathcal{Y}_i = \{Y \in \mathbf{C}_m(A \otimes_k K\text{-proj}) \mid Y \text{ is indecomposable with } \mathrm{hr}_K(Y) = r_i\}$$

has infinitely many objects up to isomorphism. We denote by $\mathcal{X}_i$ all the indecomposable objects X in $\mathbf{C}_m(A\text{-proj})$ such that X is a direct summand of $F(Y)$ and $X \otimes_k K$ contains Y as a direct summand for some Y in $\mathcal{Y}_i$. Then $\mathcal{X}_i$ has infinitely many objects up to isomorphism.

Since F is exact, $\mathrm{hr}_k(F(Y)) = l \cdot r_i$. Here, notice that the cohomological range is defined by the dimension over k. By Lemma 3.2(2),

$$\{\mathrm{hr}_k(X) \mid X \in \mathcal{X}_i\} \subseteq [r_i, \, l \cdot r_i].$$

Then there is an integer s_i between r_i and $l\cdot r_i$ such that, up to isomorphism, there are infinitely many indecomposable objects in $\mathbf{C}_m(A\text{-proj})$ with cohomological range s_i.

Since $(r_i)_{i\in\mathbb{N}}$ is an increasing sequence, for each i there is a larger j such that $r_j > l \cdot r_i$. So we can find inductively an increasing subsequence $(s_i')_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ of $(s_i)_{i\in\mathbb{N}}$ such that, up to isomorphism, there are infinitely many indecomposable objects in $\mathbf{C}_m(A\text{-proj})$ with cohomological range s_i'. Therefore $\mathbf{C}_m(A\text{-proj})$ is of strongly unbounded type. ∎

REMARK 3.4. The adjoint pairs $(- \otimes_k K, F)$ exist at the levels of the homotopy category and the derived category. By the argument used in the proof of Proposition 3.3, we can prove that the second derived Brauer–Thrall type theorem holds for A if and only if it holds for $A \otimes_k K$.

If K is algebraically closed, then by [Zh16a, Corollary 2.9] and the above proposition, we have

THEOREM 3.5. *Let A be a k-algebra and K/k be a finite separable extension such that K is algebraically closed. Then A is $\mathbf{C}$-dichotomic.*

In view of Proposition 2.10 and Corollary 2.12, we deduce from Theorem 3.5 the following useful corollary.

COROLLARY 3.6. *Let A be a k-algebra and K/k be a finite separable extension such that K is algebraically closed.*

(1) *The second Brauer–Thrall type theorem holds for A.*
(2) *The category $\mathbf{K}^{\mathrm{b}}(A\text{-proj})$ is either discrete or of strongly unbounded type.*
(3) *The algebra A is derived-discrete if and only if $\mathbf{K}^{\mathrm{b}}(A\text{-proj})$ is discrete if and only if $\mathbf{C}_m(A\text{-proj})$ is of finite representation type for each $m \geq 1$.*
(4) *The algebra A is strongly derived-unbounded if and only if $\mathbf{K}^{\mathrm{b}}(A\text{-proj})$ is of strongly unbounded type if and only if $\mathbf{C}_m(A\text{-proj})$ is of strongly unbounded type for some $m \geq 1$.*

COROLLARY 3.7. *Let A be a finite-dimensional algebra over the real number field. Then the second derived Brauer–Thrall type theorem holds for A.*

REMARK 3.8. (1) In Theorem 3.5, the condition that K/k is separable is necessary; see the example in [JL82, Remark 3.4].

(2) It is still unknown whether Theorem 3.5 is true or not if K/k is a MacLane-separable infinite field extension; see [JL82].

(3) Let k be a finite field with K its algebraic closure and Q be a Kronecker quiver. Then kQ is derived-discrete while KQ is not.

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Jie Li
Department of Mathematics
University of Science and Technology of China
Hefei, Anhui, P.R. China
E-mail: lijie0@mail.ustc.edu.cn

Chao Zhang (corresponding author)
Department of Mathematics
School of Mathematics and Statistics
Guizhou University
550025, Guiyang, Guizhou, P.R. China
E-mail: zhangc@amss.ac.cn