Volume 55 Number 6 June 2012 1099–1320

ISSN 1674-7283 CN 11-5837/O1

中国科学：数学

SCIENCE CHINA Mathematics

Sponsored by

CHINESE ACADEMY OF SCIENCES
NATIONAL NATURAL SCIENCE FOUNDATION OF CHINA

math.scichina.com
www.springer.com/scp
www.springerlink.com

SCIENCE CHINA PRESS

Springer

SCIENCE CHINA
Mathematics

• ARTICLES •

June 2012 Vol. 55 No. 6: 1153–1170
doi: 10.1007/s11425-012-4388-9

Hochschild cohomology of Beilinson algebra of exterior algebra

XU YunGe[1,*], ZHANG Chao[1,2], MA XiaoJing[1] & HU QingFeng[1]

[1]*School of Mathematics and Computer Science, Hubei University, Wuhan* 430062*, China;*
[2]*Academy of Mathematics and System Science, Chinese Academy of Science, Beijing* 100190*, China*
Email: xuy@hubu.edu.cn, zhangc@amss.ac.cn, 46904229*@qq.com, qingfenghu@yahoo.com.cn*

Received May 5, 2010; accepted January 9, 2012

Abstract Let Λ_n be the Beilinson algebra of exterior algebra of an n-dimensional vector space, which is derived equivalent to the endomorphism algebra $\mathrm{End}_{\mathcal{O}_X}(T)$ of a tilting complex $T = \Pi_{i=0}^{n}\mathcal{O}_X(i)$ of coherent $\mathcal{O}_X$-modules over a projective scheme $X = P_k^n$. In this paper we first construct a minimal projective bimodule resolution of Λ_n, and then apply it to calculate k-dimensions of the Hochschild cohomology groups of Λ_n in terms of parallel paths. Finally, we give an explicit description of the cup product and obtain a Gabriel presentation of Hochschild cohomology ring of Λ_n. As a consequence, we provide a class of algebras of finite global dimension whose Hochschild cohomology rings have non-trivial multiplicative structures.

Keywords Beilinson algebra, finite global dimension, Hochschild cohomology ring, parallel path

MSC(2010) 16E40, 16G10

Citation: Xu Y G, Zhang C, Ma X J, et al. Hochschild cohomology of Beilinson algebra of exterior algebra. Sci China Math, 2012, 55(6): 1153–1170, doi: 10.1007/s11425-012-4388-9

1 Introduction

Representation dimensions of artin algebras were introduced by Auslander to measure homologically how far an artin algebra is from being of finite representation type [2]. However, during the last thirty years, there was not much progress in this subject [28, 29]. Until recently, Iyama [15] showed that the representation dimensions of artin algebras are always finite. Then many examples concerning the representation dimensions of artin algebras having upper bounds were founded, but little is known about the lower bounds and even less is known if there exist artin algebras having the representation dimensions larger than 3. Until 2005, by studying the triangulated category, Rouquier [25] showed that the representation dimension of exterior algebra of an n-dimensional vector space is $n+1$, thus gave the first example of an algebra known to have representation dimension strictly larger than 3. Then Krause and Kussin [16] showed that the representation dimension of the endomorphism algebra $\Lambda_n = \mathrm{End}_{\mathcal{O}_X}(T)$ of a tilting complex $T = \prod_{i=0}^{n}\mathcal{O}_X(i)$ of coherent $\mathcal{O}_X$-modules over a projective scheme $X = P_k^n$ is at least n, thus simplified Rouquier's original proof concerning the representation dimension which has no boundary. The algebra Λ_n is derived equivalent to the Beilinson algebra $b(A)$ of the exterior algebra A of an n-dimensional vector space (with the usual grading), which appeared in Beilinson's study on the bounded derived category of projective spaces, see [4, 10]. We also call the algebra Λ_n Beilinson algebra. The purpose of this paper is to give a further investigation on the Hochschild cohomology behavior of

*Corresponding author

Λ_n. Note that Λ_n is a triangular algebra (or directed in some literature), all the Hochschild homology groups of non-zero order are zero.

Hochschild cohomology was introduced by Hochschild in 1945, and was developed and improved by Cartan and Eilenberg [9, 14]. In recent years, the Hochschild cohomology and Hochschild cohomology rings have been studied extensively [5, 6, 13, 23, 30–32], and have played an important role in many branches of mathematics and physics. For example, the Hochschild cohomology is a subtle invariance (such as Morita equivalent invariance, tilting equivalent invariance and derived equivalent invariance) of associative algebras [13,24]; Hochschild cohomology is closely related to simple connectedness, separability and deformation theory [1, 17, 27]. It is well known that $\mathrm{HH}^*(\Lambda) = \oplus_{i=0}^{\infty}\mathrm{HH}^i(\Lambda)$ is endowed with the so-called Gerstenhaber algebra structure under the cup product and Gerstenhaber Lie bracket [18]. However, for most finite dimensional algebras, little is known about the Hochschild cohomology groups and even less is known about the Hochschild cohomology rings. As Green and Solberg pointed out in [22], "the ring structure of $\mathrm{HH}^*(\Lambda)$ has often been observed to be trivial", such as algebras of radical square zero whose ordinary quivers are not oriented cycles [11], quadratic triangular string algebras [7], Fibonacci algebras [12] and so on. Although "one knows that for many self-injective rings there are non-zero products in $\mathrm{HH}^*(\Lambda)$", there are rather few known examples of algebras having finite global dimensions such that the ring structures of $\mathrm{HH}^*(\Lambda)$ are non-trivial. And based on this, Bustamante gave the following conjecture in [7]: *Let $\Lambda = kQ/I$ be a monomial triangular algebra. Then the ring structure of $HH^*(\Lambda)$ is trivial.* Note also that Green et al. introduced a method of constructing non-selfinjective algebras (possibly, of infinite global dimension) with non-trivial ring structure on the Hochschild cohomology ring by means of one-point extensions [21].

In this paper, we will provide a class of algebras of finite global dimension whose Hochschild cohomology rings have non-trivial cup products by studying the Hochschild cohomology of the Beilinson algebra Λ_n. We first construct a minimal projective bimodule resolution of the Beilinson algebra Λ_n in Section 2, and then calculate k-dimensions of all the Hochschild cohomology spaces of Λ_n in terms of combinatorics in Section 3. Furthermore, in the final section, we find k-base of Hochschild cohomology spaces of Λ_n, and give an explicit description of the cup product in terms of parallel paths by showing that the cup product is essentially juxtaposition of parallel paths up to sign. Based on this description, we obtain a presentation of Hochschild cohomology ring of the Beilinson algebra Λ_n. This shows that Hochschild cohomology rings of Beilinson algebras have non-trivial cup products (note that the global dimension of Λ_n is finite).

Throughout the paper, we always fix a field k, and write the composition of arrows from left to right, but for the composition of maps, from right to left.

2 Minimal projective bimodule resolutions

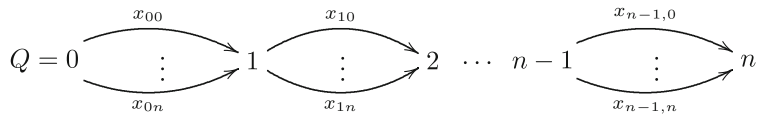
Let Λ be a finite-dimensional k-algebra (associative with identity). Denote by Λ^e the enveloping algebra of Λ, i.e., the tensor product $\Lambda\otimes_k\Lambda^{\mathrm{op}}$ of the algebra Λ and its opposite Λ^{op}. Then by Cartan-Eilenberg [9] the m-th Hochschild cohomology of Λ is identified with k-space

$$\mathrm{HH}^m(\Lambda) = \mathrm{Ext}^m_{\Lambda^e}(\Lambda, \Lambda).$$

So the first step for computing the Hochschild cohomology of Λ is to seek a minimal projective resolution of Λ over its enveloping algebra Λ^e.

It follows from the proof of the main theorem in [16] (see also [4]) that the Beilinson algebra Λ_n is isomorphic to the quotient algebra kQ/I, where Q is the following finite quiver with $n+1$ vertices and

$n(n+1)$ arrows $(n>1)$:



and I is the admissible ideal of kQ generated by $R=\{x_{ti}x_{t+1,j}-x_{tj}x_{t+1,i}\mid t=0,1,\dots,n-2;i,j=0,1,\dots,n\}$. Throughout the paper we always fix the positive integer n and denote by Λ the Beilinson algebra Λ_n. Let $e_0,e_1,\dots,e_n$ be the complete set of primitive orthogonal idempotents in Λ which are viewed as paths of length 0. Order the paths in Q by *left length lexicographic order* by choosing $e_0<e_1<\cdots<e_n$ and $x_{ij}<x_{kl}$ if $i<k$, or $i=k$ but $j<l$. Clearly, R is a reduced Gröbner basis of I consisting of quadratic elements and hence Λ is a Koszul algebra [20]. Furthermore, if we identify the paths in Q with their images in Λ, then by [3] Λ has a multiplicative basis $\mathcal{B}$ consisting of paths. Denote by $\mathcal{B}_m$ the subset of $\mathcal{B}$ consisting of all paths of length m and for $j=0,1,\dots,n-m$,

$$\mathcal{B}_m^j=\{b^{m,j}_{i_1i_2\cdots i_m}=x_{ji_1}x_{j+1,i_2}\cdots x_{j+m-1,i_m}\mid n\geqslant i_1\geqslant i_2\geqslant\cdots\geqslant i_m\geqslant 0\}$$

is the subset of $\mathcal{B}_m$ consisting of all paths of origin j. It is not difficult to check that $|\mathcal{B}_m^j|$, the cardinary of $\mathcal{B}_m^j$, is the number of non-negative integral solutions of the linear Diophantine equation $x_0+x_1+\cdots+x_n=m$, so $|\mathcal{B}_m^j|=\binom{n+m}{n}$. Set $\mathcal{B}_m=\bigcup_{j=0}^{n-m}\mathcal{B}_m^j$. Then $|\mathcal{B}_m|=(n+1-m)\binom{n+m}{n}$.

Next we will construct a minimal projective bimodule resolution of Λ over Λ^e. For each $m\geqslant 0$, we firstly construct elements $\{\overline{f}^{m,j}_{i_1i_2\cdots i_m}\mid n\geqslant i_1>i_2>\cdots>i_m\geqslant 0, j=0,1,\dots,n-m\}$. Let $\overline{f}^{0,j}=e_j$, $j=0,1,\dots,n$, and $\overline{f}^{1,j}_{i_1}=x_{ji_1}$, $j=0,1,\dots,n-1$. For $m\geqslant 2$ and $0\leqslant j\leqslant n-m$, one defines $\overline{f}^{m,j}_{i_1i_2\cdots i_m}$ inductively by setting

$$\overline{f}^{m,j}_{i_1i_2\cdots i_m}=\begin{cases}\displaystyle\sum_{t=1}^{m}(-1)^t\overline{f}^{m-1,j}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m}x_{j+m-1,i_t}, & \text{if } m \text{ is even},\\ \displaystyle\sum_{t=1}^{m}(-1)^{t+1}\overline{f}^{m-1,j}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m}x_{j+m-1,i_t}, & \text{if } m \text{ is odd},\end{cases}\tag{$*$}$$

where $\overline{f}^{-1,j}=0=\overline{f}^{n+1,j}_{i_1i_2\cdots i_{n+1}}$, $j=0,1,\dots,n-m$. Denote by $F^{(m)}=\{\overline{f}^{m,j}_{i_1i_2\cdots i_m}\mid n\geqslant i_1>i_2>\cdots>i_m\geqslant 0, j=0,1,\dots,n-m\}$. Clearly, $|F^{(m)}|=(n+1-m)\binom{n+1}{m}$.

Lemma 2.1. *For all* $1\leqslant m\leqslant n$, $j=0,1,\dots,n-m$, *we have*

$$\overline{f}^{m,j}_{i_1i_2\cdots i_m}=\sum_{t=1}^{m}(-1)^{t+1}x_{ji_t}\overline{f}^{m-1,j+1}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m}.$$

Proof. We prove this by induction on m. It is clear that the lemma holds for $m=1,2$. We assume that the lemma is true for $m-1$. If m is even,

$$\begin{aligned}\overline{f}^{m,j}_{i_1i_2\cdots i_m}&=\sum_{t=1}^{m}(-1)^t\overline{f}^{m-1,j}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m}x_{j+m-1,i_t}\\&=\sum_{t=1}^{m}(-1)^t\Bigg[\sum_{k=1}^{t-1}(-1)^{k+1}x_{ji_k}\overline{f}^{m-2,j+1}_{i_1\cdots i_{k-1}i_{k+1}\cdots i_{t-1}i_{t+1}\cdots i_m}\\&\quad+\sum_{k=t+1}^{m}(-1)^k x_{ji_k}\overline{f}^{m-2,j+1}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_{k-1}i_{k+1}\cdots i_m}\Bigg]x_{j+m-1,i_t}\\&=\sum_{k=1}^{m}\Bigg[\sum_{t=k+1}^{m}(-1)^{t+k+1}x_{ji_k}\overline{f}^{m-2,j+1}_{i_1\cdots i_{k-1}i_{k+1}\cdots i_{t-1}i_{t+1}\cdots i_m}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{t=1}^{k-1}(-1)^{t+k}x_{ji_k}\overline{f}^{m-2,j+1}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_{k-1}i_{k+1}\cdots i_m}\Bigg]x_{j+m-1,i_t}\\
&=\sum_{k=1}^{m}(-1)^{k+1}x_{ji_k}\Bigg[\sum_{t=k+1}^{m}(-1)^t\overline{f}^{m-2,j+1}_{i_1\cdots i_{k-1}i_{k+1}\cdots i_{t-1}i_{t+1}\cdots i_m}x_{j+m-1,i_t}\\
&\quad+\sum_{t=1}^{k-1}(-1)^{t+1}\overline{f}^{m-2,j+1}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_{k-1}i_{k+1}\cdots i_m}x_{j+m-1,i_t}\Bigg]\\
&=\sum_{k=1}^{m}(-1)^{k+1}x_{ji_k}\overline{f}^{m-1,j+1}_{i_1\cdots i_{k-1}i_{k+1}\cdots i_m}
\end{aligned}
$$

as desired. If m is odd, the argument is similar. The proof is completed. □

Recall that a non-zero element x in kQ is called *uniform* if there exist vertices u and v in Q such that $x = uxv$. Note that for any $f \in F^{(m)}$, it is clear that f is uniform. We usually denote by $o(f)$ and $t(f)$ the common origin and terminus of all the paths occurring in f.

Denote $\otimes := \otimes_k$. Let $P_m := \coprod_{\overline{f}\in F^{(m)}} \Lambda o(\overline{f}) \otimes t(\overline{f})\Lambda, m \geqslant 0$, and define $\delta_m : P_m \longrightarrow P_{m-1}$ by

$$
\begin{aligned}
&\delta_m(o(\overline{f}^{m,j}_{i_1i_2\cdots i_m})\otimes t(\overline{f}^{m,j}_{i_1i_2\cdots i_m}))\\
&\quad=\sum_{t=1}^{m}[(-1)^{t+1}x_{ji_t}\otimes t(\overline{f}^{m-1,j+1}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m})+(-1)^t o(\overline{f}^{m-1,j}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m})\otimes x_{j+m-1,i_t}].
\end{aligned}
$$

Theorem 2.2. *Let Λ be the Beilinson algebra of exterior algebra. Then the complex $(\mathbb{P}_\bullet, \delta_\bullet)$:*

$$0 \longrightarrow P_n \xrightarrow{\delta_n} \cdots \to P_{m+1} \xrightarrow{\delta_{m+1}} P_m \xrightarrow{\delta_m} \cdots \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} \Lambda \to 0$$

is a minimal projective bimodule resolution of Λ, where δ_0 is the multiplication map.

Proof. Let $X = \{x_{ti} \mid t = 0, 1, \dots, n-1, i = 0, 1, \dots, n\}$ be the set of arrows in Q, and R be the set of generators of I as above. Since Λ is a Koszul algebra, by [8, Sect. 9], it suffices to show that $F^{(m)}$ is a k-basis of the k-vector space $K_m := \bigcap_{p+q=m-2} X^pRX^q$ for $m \geqslant 2$.

We first show that all the $\overline{f}^{m,j}_{i_1i_2\cdots i_m}$ belong to K_m inductively. It is trivial for $m = 2$. Assume that the assertion holds for $m-1$ and we prove it for m. By induction hypothesis and the formula $(*)$, $\overline{f}^{m,j}_{i_1i_2\cdots i_m} \in RX^{m-2} \cap K_{m-1}X$. And induction hypothesis and Lemma 2.1 show that $\overline{f}^{m,j}_{i_1i_2\cdots i_m} \in X^{m-2}R \cap XK_{m-1}$. The assertion follows from the fact that $K_m = RX^{m-2} \cap X^{m-2}R \cap XK_{m-1} \cap K_{m-1}X$.

Next, $F^{(m)}$ is k-linearly independent since they have distinct supports. Also, the quadratic duality $\Lambda^! = kQ/I^\perp$ of Λ is isomorphic to the Yoneda algebra $E(\Lambda)$ of Λ, where $I^\perp$ is the ideal of kQ generated by $R^\perp = \{x_{ti}x_{t+1,j} + x_{tj}x_{t+1,i} \mid t = 0, 1, \dots, n-1; i, j = 0, 1, \dots, n,\ i \neq j\}$. So the Betti number of the minimal projective bimodule resolution of Λ over Λ^e is $\dim_k K_m = (n+1-m)\binom{n+1}{m}$. Hence $F^{(m)}$ is a k-basis of K_m.

Finally, by [8, Sect. 9] and [19], if m is even,

$$
\begin{aligned}
&\delta_m(o(\overline{f}^{m,j}_{i_1i_2\cdots i_m})\otimes t(\overline{f}^{m,j}_{i_1i_2\cdots i_m}))\\
&\quad=\sum_{t=1}^{m}[(-1)^{t+1}x_{ji_t}\otimes t(\overline{f}^{m-1,j+1}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m})+(-1)^m(-1)^t o(\overline{f}^{m-1,j}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m})\otimes x_{j+m-1,i_t}]\\
&\quad=\sum_{t=1}^{m}[(-1)^{t+1}x_{ji_t}\otimes t(\overline{f}^{m-1,j+1}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m})+(-1)^t o(\overline{f}^{m-1,j}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m})\otimes x_{j+m-1,i_t}],
\end{aligned}
$$

and if m is odd,

$$\delta_m(o(\overline{f}^{m,j}_{i_1i_2\cdots i_m})\otimes t(\overline{f}^{m,j}_{i_1i_2\cdots i_m}))$$

$$= \sum_{t=1}^{m}[(-1)^{t+1}x_{ji_t} \otimes t(\overline{f}^{m-1,j+1}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m}) + (-1)^m(-1)^{t+1}o(\overline{f}^{m-1,j}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m}) \otimes x_{j+m-1,i_t}]$$
$$= \sum_{t=1}^{m}[(-1)^{t+1}x_{ji_t} \otimes t(\overline{f}^{m-1,j+1}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m}) + (-1)^t o(\overline{f}^{m-1,j}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m}) \otimes x_{j+m-1,i_t}].$$

So the maps $\delta_\bullet$ are obtained. □

Since $\Lambda/\mathbf{r} \cong k^{n+1}$ is separable, the theorem above immediately implies the following corollary:

Corollary 2.3. *Let* Λ *be the Beilinson algebra of exterior algebra. Then* $\mathrm{gl.dim}\Lambda = n$.

3 Hochschild cohomology groups

In this section we calculate k-dimensions of Hochschild cohomology groups of Λ. Let X and Y be sets of some uniform elements in kQ. Then one defines a set of parallel paths $X//Y = \{(p,q) \mid o(p) = o(q) \text{ and } t(p) = t(q)\}$ and that $k(X//Y)$ is a k-vector space having the set $X//Y$ as a basis.

For the sake of convenience, we consider another resolution $(\mathbb{Q}_\bullet, \sigma_\bullet)$ which is isomorphic to the minimal resolution $(\mathbb{P}_\bullet, \delta_\bullet)$.

Let $\Gamma^{(m,j)} = \{f^{m,j}_{i_1\cdots i_m} = x_{ji_1}x_{j+1,i_2}\cdots x_{j+m-1,i_m} \mid n \geqslant i_1 > i_2 > \cdots > i_m \geqslant 0\}$, and $\Gamma^{(m)} = \bigcup_{j=0}^{n-m}\Gamma^{(m,j)}$. Then $\Gamma = \bigcup_{m\geqslant 0}\Gamma^{(m)}$ is a k-basis of $\Lambda^!$, the quadratic duality of Λ. It is clear that $|\Gamma^{(m)}| = (n+1-m)\binom{n+1}{m}$, and

$$(-1)^{t-1}x_{ji_t}f^{m-1,j+1}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m} = f^{m,j}_{i_1\cdots i_m} = (-1)^{m-t}f^{m-1,j}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m}x_{j+m-1,i_t}.$$

Denote $Q_m := \coprod_{f\in\Gamma^{(m)}}\Lambda o(f) \otimes t(f)\Lambda$, and define $\sigma_0 : Q_0 \to \Lambda_n$ to be the multiplication map and $\sigma_m : Q_m \longrightarrow Q_{m-1}$ for $m > 0$ as

$$\sigma_m(o(f^{m,j}_{i_1i_2\cdots i_m}) \otimes t(f^{m,j}_{i_1i_2\cdots i_m}))$$
$$= \sum_{t=1}^{m}[(-1)^{t-1}x_{ji_t} \otimes t(f^{m-1,j+1}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m}) + (-1)^m(-1)^{m-t}o(f^{m-1,j}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m}) \otimes x_{j+m-1,i_t}]$$
$$= \sum_{t=1}^{m}[(-1)^{t-1}x_{ji_t} \otimes t(f^{m-1,j+1}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m}) + (-1)^t o(f^{m-1,j}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m}) \otimes x_{j+m-1,i_t}].$$

It is easy to check that $(\mathbb{Q}_\bullet, \sigma_\bullet)$ is isomorphic to the resolution $(\mathbb{P}_\bullet, \delta_\bullet)$, and thus we obtain another minimal projective resolution $(\mathbb{Q}_\bullet, \sigma_\bullet)$ of Λ over Λ^e.

Applying the functor $\mathrm{Hom}_{\Lambda^e}(-,\Lambda)$ to the minimal projective bimodule resolution $(\mathbb{Q}_\bullet, \sigma_\bullet)$, we have $\mathrm{Hom}_{\Lambda^e}((\mathbb{Q}_\bullet, \sigma_\bullet), \Lambda) = (\mathbb{Q}^*_\bullet, \sigma^*_\bullet)$, where $Q^*_m = \mathrm{Hom}_{\Lambda^e}(Q_m, \Lambda)$ and $\sigma^*_m(\eta) = \eta\sigma_m$ for any $\eta \in Q^*_{m-1}$.

We first fix some notations. Given a decreasing sequence $i_1i_2\cdots i_m$, we define a position function of i_k relative to the sequence $i_1i_2\cdots i_m$ by

$$\mu(i_k) = \mu_{i_1i_2\cdots i_m}(i_k) = k.$$

And we denote

$$\Omega_{i_1i_2\cdots i_m} = \{0,1,\ldots,n\} \setminus \{i_1,i_2,\ldots,i_m\}.$$

For any $s \in \Omega_{i_1i_2\cdots i_m}$, if $i_k > s > i_{k+1}$, we denote by $i_1i_2\cdots i_m\hat{s}$ the decreasing sequence $i_1\cdots i_k s i_{k+1}\cdots i_m$. Clearly, $\mu_{i_1i_2\cdots i_m\hat{s}}(s) = k+1$.

We now can describe the complex $(\mathbb{Q}^*_\bullet, \sigma^*_\bullet)$ in terms of parallel paths.

Lemma 3.1. *The complex* $(\mathbb{Q}^*_\bullet, \sigma^*_\bullet)$ *is isomorphic to complex* $(M^\bullet, \tau^\bullet)$, *where* $M^m := k(\mathcal{B}_m//\Gamma^{(m)})$ *and for any* $j = 0, 1, \dots, n-m$, $(b, f^{m,j}_{i_1 i_2\cdots i_m}) \in \mathcal{B}_m//\Gamma^{(m)}$, $\tau^\bullet$ *is given by*

$$\tau^{m+1}(b, f^{m,j}_{i_1 i_2\cdots i_m}) = \sum_{s\in\Omega_{i_1 i_2\cdots i_m}} [(-1)^{\mu(s)-1}(x_{j-1,s}b, f^{m+1,j-1}_{i_1 i_2\cdots i_m \hat{s}}) + (-1)^{\mu(s)}(bx_{j+m,s}, f^{m+1,j}_{i_1 i_2\cdots i_m \hat{s}})].$$

Proof. Clearly, $Q^*_m = \mathrm{Hom}_{\Lambda^e}(Q_m, \Lambda) = \mathrm{Hom}_{\Lambda^e}(\coprod_{f\in\Gamma^{(m)}} \Lambda^e(o(f)\otimes_k t(f)), \Lambda) \cong \coprod_{f\in\Gamma^{(m)}}(o(f)\otimes_k t(f)\Lambda) \cong \coprod_{f\in\Gamma^{(m)}}(o(f)\Lambda t(f)) \cong k(\mathcal{B}_m//\Gamma^{(m)})$. The k-vector spaces isomorphism ϕ from $k(\mathcal{B}_m//\Gamma^{(m)})$ to Q^*_m is given by $\phi(b,f) = \eta_{(b,f)}$, where

$$\eta_{(b,f)}(o(g)\otimes t(g)) = \begin{cases} b, & \text{if } g = f,\\ 0, & \text{otherwise,}\end{cases}$$

for any $(b,f) \in \mathcal{B}_m//\Gamma^{(m)}$.

It suffices to check that $\sigma^*_{m+1}\phi = \phi\tau^{m+1}$. For any $(b, f^{m,j}_{i_1 i_2\cdots i_m}) \in k(\mathcal{B}_m//\Gamma^m)$ and $\sum_{f\in\Gamma^{(m+1)}} \alpha_f o(f)\otimes t(f)\beta_f \in Q_{m+1}$, where $\alpha_f, \beta_f \in \Lambda$, we have

$$\begin{aligned}
&(\sigma^*_{m+1}\phi(b, f^{m,j}_{i_1 i_2\cdots i_m}))\Bigg(\sum_{f\in\Gamma^{(m+1)}} \alpha_f o(f)\otimes t(f)\beta_f\Bigg)\\
&\quad= \eta_{(b, f^{m,j}_{i_1 i_2\cdots i_m})}\sigma_{m+1}\Bigg(\sum_{f\in\Gamma^{(m+1)}} \alpha_f o(f)\otimes t(f)\beta_f\Bigg)\\
&\quad= \sum_{f^{m+1,j'}_{l_1 l_2\cdots l_{m+1}}\in\Gamma^{(m+1)}} \alpha_{f^{m+1,j'}_{l_1 l_2\cdots l_{m+1}}}[\eta_{(b, f^{m,j}_{i_1 i_2\cdots i_m})}\sigma_{m+1}(o(f^{m+1,j'}_{l_1 l_2\cdots l_{m+1}})\otimes t(f^{m+1,j'}_{l_1 l_2\cdots l_{m+1}}))]\beta_{f^{m+1,j'}_{l_1 l_2\cdots l_{m+1}}}\\
&\quad= \sum_{k\in\Omega_{i_1 i_2\cdots i_m}} \alpha_{f^{m+1,j-1}_{i_1 i_2\cdots i_m \hat{k}}}\eta_{(b, f^{m,j}_{i_1 i_2\cdots i_m})}((-1)^{\mu(k)-1}x_{j-1,k}\otimes t(f^{m,j}_{i_1 i_2\cdots i_m}))\beta_{f^{m+1,j-1}_{i_1 i_2\cdots i_m \hat{k}}}\\
&\qquad+ \sum_{k'\in\Omega_{i_1 i_2\cdots i_m}} \alpha_{f^{m+1,j}_{i_1 i_2\cdots i_m \hat{k'}}}\eta_{(b, f^{m,j}_{i_1 i_2\cdots i_m})}((-1)^{\mu(k')}o(f^{m,j}_{i_1 i_2\cdots i_m})\otimes x_{j+m,k'})\beta_{f^{m+1,j}_{i_1 i_2\cdots i_m \hat{k'}}}\\
&\quad= \sum_{k\in\Omega_{i_1 i_2\cdots i_m}} \alpha_{f^{m+1,j-1}_{i_1 i_2\cdots i_m \hat{k}}}(-1)^{\mu(k)-1}x_{j-1,k}b\beta_{f^{m+1,j-1}_{i_1 i_2\cdots i_m \hat{k}}}\\
&\qquad+ \sum_{k'\in\Omega_{i_1 i_2\cdots i_m}} \alpha_{f^{m+1,j}_{i_1 i_2\cdots i_m \hat{k'}}}(-1)^{\mu(k)}bx_{j+m,k}\beta_{f^{m+1,j}_{i_1 i_2\cdots i_m \hat{k'}}},
\end{aligned}$$

and

$$\begin{aligned}
&(\phi\tau^{m+1}(b, f^{m,j}_{i_1 i_2\cdots i_m}))\Bigg(\sum_{f\in\Gamma^{(m+1)}} \alpha_f o(f)\otimes t(f)\beta_f\Bigg)\\
&\quad= \Bigg[\phi\sum_{s\in\Omega_{i_1 i_2\cdots i_m}}[(-1)^{\mu(s)-1}(x_{j-1,s}b, f^{m+1,j-1}_{i_1 i_2\cdots i_m \hat{s}}) + (-1)^{\mu(s)}(bx_{j+m,s}, f^{m+1,j}_{i_1 i_2\cdots i_m \hat{s}})]\Bigg]\\
&\qquad\Bigg(\sum_{f\in\Gamma^{(m+1)}} \alpha_f o(f)\otimes t(f)\beta_f\Bigg)\\
&\quad= \sum_{s\in\Omega_{i_1 i_2\cdots i_m}}\Bigg[(-1)^{\mu(s)-1}\eta_{(x_{j-1,s}b, f^{m+1,j-1}_{i_1 i_2\cdots i_m \hat{s}})}\Bigg(\sum_{f\in\Gamma^{(m+1)}} \alpha_f o(f)\otimes t(f)\beta_f\Bigg)\\
&\qquad+ (-1)^{\mu(s)}\eta_{(bx_{j+m,s}, f^{m+1,j}_{i_1 i_2\cdots i_m \hat{s}})}\Bigg(\sum_{f\in\Gamma^{(m+1)}} \alpha_f o(f)\otimes t(f)\beta_f\Bigg)\Bigg]\\
&\quad= \sum_{s\in\Omega_{i_1 i_2\cdots i_m}}(-1)^{\mu(s)-1}\eta_{(x_{j-1,s}b, f^{m+1,j-1}_{i_1 i_2\cdots i_m \hat{s}})}\Bigg(\sum_{k\in\Omega_{i_1 i_2\cdots i_m}} \alpha_{f^{m+1,j-1}_{i_1 i_2\cdots i_m \hat{k}}} o(f^{m+1,j-1}_{i_1 i_2\cdots i_m \hat{k}})\otimes t(f^{m+1,j-1}_{i_1 i_2\cdots i_m \hat{k}})\beta_{f^{m+1,j-1}_{i_1 i_2\cdots i_m \hat{k}}}\Bigg)
\end{aligned}$$

$$+(-1)^{\mu(s)}\eta_{(bx_{j+m,s},f^{m+1,j}_{i_1i_2\cdots i_m\hat{s}})}\left(\sum_{k'\in\Omega_{i_1i_2\cdots i_m}}\alpha_{f^{m+1,j}_{i_1i_2\cdots i_m\hat{k'}}}o(f^{m+1,j}_{i_1i_2\cdots i_m\hat{k'}})\otimes t(f^{m+1,j}_{i_1i_2\cdots i_m\hat{k'}})\beta_{f^{m+1,j}_{i_1i_2\cdots i_m\hat{k'}}}\right)$$
$$=\sum_{k\in\Omega_{i_1i_2\cdots i_m}}\alpha_{f^{m+1,j-1}_{i_1i_2\cdots i_m\hat{k}}}(-1)^{\mu(k)-1}x_{j-1,k}b\beta_{f^{m+1,j-1}_{i_1i_2\cdots i_m\hat{k}}}$$
$$+\sum_{k'\in\Omega_{i_1i_2\cdots i_m}}\alpha_{f^{m+1,j}_{i_1i_2\cdots i_m\hat{k'}}}(-1)^{\mu(k)}bx_{j+m,k}\beta_{f^{m+1,j}_{i_1i_2\cdots i_m\hat{k'}}}.$$

Thus $\sigma^*_{m+1}\phi=\phi\tau^{m+1}$ as desired. □

To calculate the dimensions of Hochschild cohomology groups of Λ, note that $\mathrm{HH}^m(\Lambda)=\mathrm{Ker}\tau^{m+1}/\mathrm{Im}\tau^m$ by definition, we have

$$\dim_k\mathrm{HH}^m(\Lambda)=\dim_k\mathrm{Ker}\tau^{m+1}-\dim_k\mathrm{Im}\tau^m=\dim_k M^m-\dim_k\mathrm{Im}\tau^{m+1}-\dim_k\mathrm{Im}\tau^m.$$

Since $M^m=k(\mathcal{B}_m//\Gamma^{(m)})$, $\dim_k M^m=(n+1-m)\binom{n+m}{m}\binom{n+1}{m}$. Hence it suffices to determine $\dim_k\mathrm{Im}\tau^{m+1}$ and $\dim_k\mathrm{Im}\tau^m$.

We say the sequences $r_1\cdots r_m<s_1\cdots s_m$ if there is some $1\leqslant t\leqslant m$ such that $r_i=s_i$ for $1\leqslant i<t$ and $r_t<s_t$, where $r_i,s_i\in\{0,1,\dots,n\}$. Then we define an order on $\mathcal{B}_m$ by setting $b^{m,j_1}_{r_1r_2\cdots r_m}<b^{m,j_2}_{s_1s_2\cdots s_m}$, if $j_1<j_2$, or $j_1=j_2$ but $r_1r_2\cdots r_m<s_1s_2\cdots s_m$. Similarly, we can define an order on $\Gamma^{(m)}$ by setting $f^{m,j_1}_{i_1i_2\cdots i_m}<f^{m,j_2}_{k_1k_2\cdots k_m}$, if $j_1<j_2$, or $j_1=j_2$ but $i_1i_2\cdots i_m<k_1k_2\cdots k_m$. The orders defined as above induce an order on $\mathcal{B}_m//\Gamma^{(m)}$ as follows:

$$(b_1,f_1)<(b_2,f_2),\quad\text{ if } f_1<f_2,\quad\text{ or } f_1=f_2 \text{ but } b_1<b_2.$$

We still denote by τ^{m+1} the matrix of the differential τ^{m+1} under the ordered bases above. Based on the description of τ^{m+1} in Lemma 3.1, a rather lengthly analysis shows that τ^{m+1} has the following form:

$$\tau^{m+1}=\begin{pmatrix}A&&&&&\\-A&A&&&&\\&-A&A&&&\\&&&\ddots&\ddots&\\&&&&-A&A\\&&&&&-A\end{pmatrix}_{(n+1-m)\times(n-m)}$$

where A is an $\binom{n+m}{n}\binom{n+1}{m}\times\binom{n+m+1}{n}\binom{n+1}{m+1}$ matrix. And the rows and columns of A are just corresponding to the basis elements in $\mathcal{B}^0_m//\Gamma^{(m,0)}$ and $\mathcal{B}^0_{m+1}//\Gamma^{(m+1,0)}$ respectively. So, for the sake of convenience, we can use the basis elements in $\mathcal{B}^0_m//\Gamma^{(m,0)}$ and $\mathcal{B}^0_{m+1}//\Gamma^{(m+1,0)}$ as the index sets of rows and columns of A respectively, and write the row index $(r_1r_2\cdots r_m,\ j_1j_2\cdots j_m)$ instead of $(b^{m,0}_{r_1r_2\cdots r_m},f^{m,0}_{j_1j_2\cdots j_m})$, and similarly for column indices. The matrix A is described as follows: given the $(r_1r_2\cdots r_m,j_1j_2\cdots j_m)$-th row of A, the $(r_1\cdots r_m\hat{s},\ j_1\cdots j_m\hat{s})$-th component of the row is $(-1)^{\mu_{j_1j_2\cdots j_m\hat{s}}(s)}$ for $s\in\Omega_{j_1j_2\cdots j_m}$, and other components are 0. Obviously, $\mathrm{rank}\tau^{m+1}=(n-m)\mathrm{rank}A$. The following lemma gives the rank of A.

Lemma 3.2.

$$\mathrm{rank}A=\sum_{i=0}^{m}\binom{n-i+m}{n-i}\binom{n-i}{m-i}.$$

Proof. For any $i=0,1,\dots,m$, we denote $T^m_i=\{(r_1\cdots r_m,n(n-1)\cdots(n-i+1)h_1\cdots h_{m-i})\mid n-i\geqslant r_1\geqslant r_2\geqslant\cdots\geqslant r_m\geqslant 0,n-i-1\geqslant h_1>h_2>\cdots>h_{m-i}\geqslant 0\}$ and $T^m=\bigcup_{i=0}^m T^m_i$. And for any $i=0,1,\dots,m-1$, $S^m_i=\{(n-i)r_1r_2\cdots r_{m-1},n(n-1)\cdots(n-i)h_1h_2\cdots h_{m-i-1})\mid n-i\geqslant r_1\geqslant r_2\geqslant\cdots\geqslant r_{m-1}\geqslant 0,n-i-1\geqslant h_1>h_2>\cdots>h_{m-i-1}\geqslant 0\}$ and $S^m=\bigcup_{i=0}^{m-1}S^m_i$. Here, once and for all, we always view $n(n-1)\cdots(n-i+1)h_1\cdots h_{m-i}$ as $h_1\cdots h_m$ when $i=0$, and as $n(n-1)\cdots(n-m+1)$ when $i=m$. It is easy to see that $T^m\cap S^m=\emptyset$ and $T^m\cup S^m$ is the set of indices of all row vectors of A.

We first claim that the row vectors of A with indices in T^m are linearly independent since the last non-zero components of them are pairwise distinct. To see this, it is enough to note that the last non-zero component of the $(r_1\cdots r_m, n(n-1)\cdots(n-i+1)h_1\cdots h_{m-i})$-th row of A is the $((n-i)r_1\cdots r_m, n(n-1)\cdots(n-i)h_1\cdots h_{m-i})$-th component.

Next, we assert that any row vector of A with index in S^m can be written as a linear combination of some row vectors with indices in T^m. More precisely, for any row vector β with row index $((n-i)r_1r_2\cdots r_{m-1}, n(n-1)\cdots(n-i)h_1h_2\cdots h_{m-i-1})\in S^m$, we will prove that

$$\beta=\sum_{s\in\Omega_{n(n-1)\cdots(n-i)h_1h_2\cdots h_{m-i-1}}}(-1)^{\mu_{h_1\cdots h_{m-i-1}\hat{s}}(s)}\beta_{h_1\cdots h_{m-i-1}\hat{s}}, \tag{3.1}$$

where $\beta_{h_1\cdots h_{m-i-1}\hat{s}}$ denotes the row vector of A with the row index $(r_1\cdots r_{m-1}\hat{s}, n(n-1)\cdots(n-i+1)h_1\cdots h_{m-i-1}\hat{s})\in T^m$.

Firstly, we consider the non-zero components of the vector β. We simply denote $\mu(s)=\mu_{h_1\cdots h_{m-i-1}\hat{s}}(s)$ for any $s\in\Omega_{n(n-1)\cdots(n-i)h_1h_2\cdots h_{m-i-1}}$. Then, by definition of τ^{m+1}, the $((n-i)r_1\cdots r_{m-1}\hat{s}, n(n-1)\cdots(n-i)h_1\cdots h_{m-i-1}\hat{s})$-th component of β is $(-1)^{i+1+\mu(s)}$ for $s\in\Omega_{n(n-1)\cdots(n-i)h_1h_2\cdots h_{m-i-1}}$, and other components are 0.

Secondly, consider the non-zero components of $\beta_{h_1\cdots h_{m-i-1}\hat{s}}$ for any given $s\in\Omega_{n(n-1)\cdots(n-i)h_1h_2\cdots h_{m-i-1}}$. By the description of the matrix A in the paragraph before the lemma, there are two classes of non-zero components: the first class is only the $((n-i)r_1\cdots r_{m-1}\hat{s}, n(n-1)\cdots(n-i)h_1\cdots h_{m-i-1}\hat{s})$-th component of $\beta_{h_1\cdots h_{m-i-1}\hat{s}}$ which is $(-1)^{i+1}$, and the other class is those components corresponding to $(r_1\cdots r_{m-1}\hat{s}\hat{h},\ n(n-1)\cdots(n-i+1)h_1\cdots\ h_{m-i-1}\hat{s}\hat{h})$ for $h\in\Omega_{n(n-1)\cdots(n-i)h_1\cdots h_{m-i-1}\hat{s}}$. In the latter case, the $(r_1\cdots r_{m-1}\hat{s}\hat{h},\ n(n-1)\cdots(n-i+1)h_1\cdots h_{m-i-1}\hat{s}\hat{h})$-th component is $(-1)^{i+\mu(h)}$ whenever $h>s$, and $(-1)^{i+\mu(h)+1}$ whenever $h<s$.

Finally, we will show that the formula (3.1) is true by componentwise. We will prove this by dividing components of β into three classes.

We first consider the $((n-i)r_1\cdots r_{m-1}\hat{s},\ n(n-1)\cdots(n-i)h_1\cdots h_{m-i-1}\hat{s})$-th component of the formula (3.1) with $s\in\Omega_{n(n-1)\cdots(n-i)h_1h_2\cdots h_{m-i-1}}$. Clearly, the component of β is $(-1)^{i+1+\mu(s)}$, and for $h\in\Omega_{n(n-1)\cdots(n-i)h_1h_2\cdots h_{m-i-1}}$, the $((n-i)r_1\cdots r_{m-1}\hat{s},\ n(n-1)\cdots(n-i)h_1\cdots h_{m-i-1}\hat{s})$-th component of $\beta_{h_1\cdots h_{m-i-1}\hat{h}}$ is $(-1)^{i+1}$ if $h=s$, and 0 otherwise. Note that $(-1)^{\mu(s)}\cdot(-1)^{i+1}=(-1)^{i+1+\mu(s)}$, the formula (3.1) is true for the $((n-i)r_1\cdots r_{m-1}\hat{s}, n(n-1)\cdots(n-i)h_1\cdots h_{m-i-1}\hat{s})$-th non-zero component.

Given any $u,v\in\Omega_{n(n-1)\cdots(n-i)h_1\cdots h_{m-i-1}}$ with $u\neq v$, we next show that the formula (3.1) is also true for the $(r_1\cdots r_{m-1}\hat{u}\hat{v}, n(n-1)\cdots(n-i+1)h_1\cdots h_{m-i-1}\hat{u}\hat{v})$-th component. Without loss of generality we may assume $u>v$. By the description of the fourth paragraph of the proof, the component of β is zero. On the other hand, among the vectors $\beta_{h_1\cdots h_{m-i-1}\hat{s}}$ appearing in the right-hand side of the formula (3.1) for $s\in\Omega_{n(n-1)\cdots(n-i)h_1h_2\cdots h_{m-i-1}}$, there are only two vectors, $\beta_{h_1\cdots h_{m-i-1}\hat{u}}$ and $\beta_{h_1\cdots h_{m-i-1}\hat{v}}$, such that the $(r_1\cdots r_{m-1}\hat{u}\hat{v}, n(n-1)\cdots(n-i+1)h_1\cdots h_{m-i-1}\hat{u}\hat{v})$-th components of them are non-zero, which are $(-1)^{i+1+\mu(v)}$ and $(-1)^{i+\mu(u)}$, respectively. Since $(-1)^{\mu(u)}\cdot(-1)^{i+1+\mu(v)}+(-1)^{\mu(v)}\cdot(-1)^{i+\mu(u)}=0$, we have the formula (3.1) holds as desired.

The rest components of β are zero by the description of the fourth paragraph of the proof; and the corresponding components of $\beta_{h_1\cdots h_{m-i-1}\hat{s}}$ appearing in the right-hand side of the formula (3.1) for $s\in\Omega_{n(n-1)\cdots(n-i)h_1h_2\cdots h_{m-i-1}}$ are also zero by the description of the fifth paragraph of the proof. Thus we have finished the proof of the formula (3.1) componentwise.

Hence, $\mathrm{rank}A=|T^m|=\sum_{i=0}^{m}\binom{n-i+m}{n-i}\binom{n-i}{m-i}$. □

Lemma 3.3. *For any $0\leqslant m\leqslant n-1$, we have*

$$\mathrm{rank}\tau^{m+1}=(n-m)\binom{n}{m}\binom{n+m+1}{m}.$$

Proof. Since

$$\begin{aligned}\sum_{i=0}^{m}\binom{n+m-i}{m-i} &= \binom{n+m}{m}+\sum_{i=1}^{m-1}\binom{n+m-i}{m-i}+1\\ &= \binom{n+m}{m}+\sum_{i=1}^{m-1}\left(\binom{n+m-i+1}{m-i}-\binom{n+m-i}{m-i-1}\right)+1\\ &= \binom{n+m}{m}+\sum_{i=0}^{m-2}\binom{n+m-i}{m-1-i}-\sum_{i=1}^{m-1}\binom{n+m-i}{m-i-1}+1\\ &= \binom{n+m}{m}+\binom{n+m}{m-1}+\sum_{i=1}^{m-2}\binom{n+m-i}{m-i-1}-\sum_{i=1}^{m-2}\binom{n+m-i}{m-i-1}\\ &= \binom{n+m}{m}+\binom{n+m}{m-1}=\binom{n+m+1}{m},\end{aligned}$$

we have

$$\begin{aligned}\mathrm{rank}\tau^{m+1} &= (n-m)\mathrm{rank}A=(n-m)\sum_{i=0}^{m}\left(\binom{n-i+m}{n-i}\binom{n-i}{m-i}\right)\\ &= (n-m)\sum_{i=0}^{m}\left(\binom{n}{m}\binom{n+m-i}{m-i}\right)=(n-m)\binom{n}{m}\sum_{i=0}^{m}\binom{n+m-i}{m-i}\\ &= (n-m)\binom{n}{m}\binom{n+m+1}{m}.\end{aligned}$$

□

Now we can calculate the dimensions of all the Hochschild cohomology spaces of the Beilinson algebra Λ of exterior algebra.

Theorem 3.4. *Let Λ be the Beilinson algebra of exterior algebra. Then for any $m \geqslant 0$, we have*

$$\dim_k HH^m(\Lambda)=\binom{n}{m}\binom{n+1+m}{m},$$

where $\binom{n}{m}=0$ if $m>n$.

Proof. It is clear that $\dim_k \mathrm{HH}^m(\Lambda)=0$ if $m>n$. By Lemma 3.3 we have

$$\begin{aligned}\mathrm{rank}\tau^{m+1}+\mathrm{rank}\tau^{m} &= (n-m)\binom{n}{m}\binom{n+m+1}{m}+(n-m+1)\binom{n}{m-1}\binom{n+m}{m-1}\\ &= (n-m)\frac{(n+1-m)(n+1+m)}{(n+1)^2}\binom{n+1}{m}\binom{n+m}{m}\\ &\quad +(n-m+1)\frac{m^2}{(n+1)^2}\binom{n+1}{m}\binom{n+m}{m}\\ &= \left(n-m+\frac{m^2}{(n+1)^2}\right)\binom{n+1}{m}\binom{n+m}{m},\end{aligned}$$

so

$$\begin{aligned}\dim_k \mathrm{HH}^m(\Lambda) &= \dim_k M^m-(\dim_k \mathrm{Im}\tau^{m+1}+\dim_k \mathrm{Im}\tau^{m})\\ &= \dim_k M^m-(\mathrm{rank}\tau^{m+1}+\mathrm{rank}\tau^{m})\\ &= (n+1-m)\binom{n+m}{m}\binom{n+1}{m}-\left(n-m+\frac{m^2}{(n+1)^2}\right)\binom{n+1}{m}\binom{n+m}{m}\\ &= \left[n+1-m-\left(n-m+\frac{m^2}{(n+1)^2}\right)\right]\binom{n+1}{m}\binom{n+m}{m}\\ &= \frac{(n+1)^2-m^2}{(n+1)^2}\binom{n+1}{m}\binom{n+m}{m}=\binom{n}{m}\binom{n+1+m}{m}.\end{aligned}$$

□

4 Hochschild cohomology rings

This section is devoted to the description of Hochschild cohomology rings of the Beilinson algebras Λ. As a consequence, we provide a class of algebras of finite global dimensions whose multiplicative structures of Hochschild cohomology rings are non-trivial.

Let $T = \bigcup_{m=0}^{n} T^m$, where the index set T^m is defined in the proof of Lemma 3.2. We first construct a k-basis of the Hochschild cohomology space $\mathrm{HH}^m(\Lambda)$ for any $0 \leqslant m \leqslant n$. Given a non-negative integer $0 \leqslant i \leqslant m$, and $(r_1 r_2 \cdots r_m, n(n-1)\cdots(n-i+1)h_1 \cdots h_{m-i}) \in T$, we denote

$$\xi^i = \xi^i_{(r_1 r_2 \cdots r_m, n(n-1)\cdots(n-i+1)h_1\cdots h_{m-i})} = \sum_{j=0}^{n-m} (b^{m,j}_{r_1\cdots r_m}, f^{m,j}_{n(n-1)\cdots(n-i+1)h_1\cdots h_{m-i}}).$$

Set $\mathcal{H}^{(m,i)} = \{\xi^i_{(r_1 r_2 \cdots r_m, n(n-1)\cdots(n-i+1)h_1\cdots h_{m-i})} \mid (r_1 r_2 \cdots r_m, n(n-1)\cdots(n-i+1)h_1 \cdots h_{m-i}) \in T^m_i\}$ and $\mathcal{H}^m = \bigcup_{i=0}^{m} \mathcal{H}^{(m,i)}$. Then we have the following lemma:

Lemma 4.1. *$\mathcal{H}^m$ forms a k-basis of the k-vector space $HH^m(\Lambda)$.*

Proof. We first consider the k-basis of $\mathrm{Ker}\tau^{m+1}$. It is straightforward to check that $\tau^{m+1}(\xi) = 0$ for $\xi \in \mathcal{H}^m$. For $j = 0, 1, \ldots, n-m$, we denote

$$\begin{aligned}\zeta &= \zeta^j_{((n-i)\ r_1\cdots r_{m-1}, n(n-1)\cdots(n-i)h_1\cdots h_{m-i-1})} \\ &= (b^{m,j}_{(n-i)\ r_1\cdots r_{m-1}}, f^{m,j}_{n(n-1)\cdots(n-i)h_1\cdots h_{m-i-1}}) \\ &\quad + \sum_s (-1)^{\mu(s)+1}(b^{m,j}_{r_1\cdots r_{m-1}\hat{s}}, f^{m,j}_{n(n-1)\cdots(n-i+1)h_1 h_2\cdots h_{m-i-1}\hat{s}}),\end{aligned}$$

where $s \in \Omega_{n(n-1)\cdots(n-i)h_1\cdots h_{m-i-1}}$, and $\mu(s) = \mu_{h_1 h_2\cdots h_{m-i-1}\hat{s}}(s)$. Set

$$\begin{aligned}K^{(m,j)} = \{&\zeta^j_{((n-i)\ r_1\cdots r_{m-1}, n(n-1)\cdots(n-i)h_1\cdots h_{m-i-1})} \mid \\ &((n-i)\ r_1 \cdots r_{m-1}, n(n-1)\cdots(n-i)h_1\cdots h_{m-i-1}) \in S^m\},\end{aligned}$$

and $K^m = \bigcup_{j=0}^{n-m} K^{(m,j)}$.

Formula (3.1) implies that $\tau^{m+1}(a) = 0$ for $a \in K^m$, so $\mathcal{H}^m \cup K^m \subset \mathrm{Ker}\tau^{m+1}$. It is not difficult to check that $\mathcal{H}^m \cup K^m$ is a linearly independent set, and

$$\begin{aligned}|\mathcal{H}^m| &= \sum_{i=0}^{m}\left(\binom{n-i+m}{n-i}\binom{n-i}{m-i}\right) = \binom{n}{m}\binom{n+1+m}{m}, \\ |K^m| &= (n+1-m)\sum_{i=0}^{m-1}\left(\binom{n-i+m-1}{n-i}\binom{n-i}{m-i-1}\right) \\ &= (n+1-m)\sum_{i=0}^{m-1}\left(\binom{n}{m-1}\binom{n+m-1-i}{m-1-i}\right) \\ &= (n+1-m)\binom{n}{m-1}\binom{n+m}{m-1}.\end{aligned}$$

Also,

$$\begin{aligned}\dim_k \mathrm{Ker}\tau^{m+1} &= \dim_k M^m - \dim_k \mathrm{Im}\tau^{m+1} \\ &= (n+1-m)\binom{n+m}{n}\binom{n+1}{m} - (n-m)\binom{n}{m}\binom{n+m+1}{m} \\ &= (n+1-m)\left(\binom{n+m}{n}\binom{n+1}{m} - \binom{n}{m}\binom{n+1+m}{m}\right) + \binom{n}{m}\binom{n+m+1}{m}\end{aligned}$$

$$
\begin{aligned}
&= (n+1-m)\binom{n}{m-1}\binom{n+m}{m-1} + \binom{n}{m}\binom{n+m+1}{m} \\
&= |\mathcal{H}^m| + |K^m|.
\end{aligned}
$$

Hence, $\mathcal{H}^m \cup K^m$ forms a k-basis of $\mathrm{Ker}\tau^{m+1}$.

We next claim that K^m forms a k-basis of $\mathrm{Im}\tau^m$. Indeed,

$$
\begin{aligned}
&\tau^m\left((-1)^{i+1}\sum_{t=0}^{j}(b^{m-1,t}_{r_1\cdots r_{m-1}}, f^{m-1,t}_{n(n-1)\cdots(n-i+1)h_1\cdots h_{m-i-1}})\right) \\
&\quad= (-1)^{i+1}\sum_{t=0}^{j}\sum_{s\in\Omega}[(-1)^{\mu(s)+i-1}(b^{m,t-1}_{r_1\cdots r_{m-1}\hat{s}}, f^{m,t-1}_{n(n-1)\cdots(n-i+1)h_1\cdots h_{m-i-1}\hat{s}}) \\
&\quad\quad + (-1)^{\mu(s)+i}(b^{m,t}_{r_1\cdots r_{m-1}\hat{s}}, f^{m,t}_{n(n-1)\cdots(n-i+1)\ h_1\cdots h_{m-i-1}\hat{s}})] \\
&\quad= \sum_{s\in\Omega}(-1)^{\mu(s)+1}\left(\sum_{t=0}^{j}[-(b^{m,t-1}_{r_1\cdots r_{m-1}\hat{s}}, f^{m,t-1}_{n(n-1)\cdots(n-i+1)h_1\cdots h_{m-i-1}\hat{s}})\right. \\
&\quad\quad \left.+ (b^{m,t}_{r_1\cdots r_{m-1}\hat{s}}, f^{m,t}_{n(n-1)\cdots(n-i+1)\ h_1\cdots h_{m-i-1}\hat{s}})]\right) \\
&\quad= \sum_{s\in\Omega}(-1)^{\mu(s)+1}(b^{m,j}_{r_1\cdots r_{m-1}\hat{s}}, f^{m,j}_{n(n-1)\cdots(n-i+1)h_1\cdots h_{m-i-1}\hat{s}}) \\
&\quad= \sum_{s\in\Omega_{n(n-1)\cdots(n-i)h_1h_2\cdots h_{m-i-1}}}(-1)^{\mu(s)+1}(b^{m,j}_{r_1\cdots r_{m-1}\hat{s}}, f^{m,j}_{n(n-1)\cdots(n-i+1)h_1\cdots h_{m-i-1}\hat{s}}) \\
&\quad\quad + (b^{m,j}_{(n-i)r_1\cdots r_{m-1}}, f^{m,j}_{n(n-1)\cdots(n-i)h_1\cdots h_{m-i-1}}) \\
&\quad= \zeta^{j}_{((n-i)\ r_1\cdots r_{m-1}, n(n-1)\cdots(n-i)h_1\cdots h_{m-i-1})} \in K^{(m,j)},
\end{aligned}
$$

where $\Omega = \Omega_{n(n-1)\cdots(n-i+1)h_1h_2\cdots h_{m-i-1}}$ and $\mu(s) = \mu_{h_1\cdots h_{m-i-1}\hat{s}}(s)$. So $\mathrm{span}K^m \subseteq \mathrm{Im}\tau^m$, but $\dim_k \mathrm{Im}\tau^m = (n-m+1)\binom{n}{m-1}\binom{n+m}{m-1} = |K^m|$, thus K^m forms a k-basis of $\mathrm{Im}\tau^m$.

Since $\mathrm{HH}^m(\Lambda) = \mathrm{Ker}\tau^{m+1}/\mathrm{Im}\tau^m$, $\mathcal{H}^m$ forms a k-basis of $\mathrm{HH}^m(\Lambda)$. □

In the following we will describe the multiplicative structure of Hochschild cohomology ring of Λ in terms of parallel paths. In [26] it was shown that any projective Λ^e-resolution $\mathbb{X}$ of Λ gives rise to a "cup product", which coincides with the ordinary cup product. Let $\mathbb{X}$ be a projective Λ^e-resolution of Λ. There exists a chain map $\Delta : \mathbb{X} \longrightarrow \mathbb{X} \otimes_\Lambda \mathbb{X}$ lifting the identity, which is unique up to homotopy. Siegel and Witherspoon defined in [26] a cup product of two elements η in $\mathrm{HH}^m(\Lambda)$ and θ in $\mathrm{HH}^s(\Lambda)$ as above using the composition

$$
\mathbb{X} \xrightarrow{\Delta} \mathbb{X} \otimes_\Lambda \mathbb{X} \xrightarrow{\eta\otimes\theta} \Lambda \otimes_\Lambda \Lambda \xrightarrow{\nu} \Lambda,
$$

and note that it is independent of the projective resolution $\mathbb{X}$ of Λ and the chain map Δ. Based on this context Buchweitz et al. gave in [5] a description of multiplicative structure of Hochschild cohomology ring of a Koszul algebra. Now we give an explicit formula of Δ for the minimal projective Λ^e-resolution $(\mathbb{Q}_\bullet, \sigma_\bullet)$ of the Beilinson algebra Λ constructed in Section 3.

Recall that $(\mathbb{Q} \otimes_\Lambda \mathbb{Q}, D)$ is still a projective Λ^e-resolution of Λ which is given by

$$
(Q \otimes_\Lambda Q)_m = \coprod_{i+j=m} Q_i \otimes_\Lambda Q_j
$$

and $D_m : (Q \otimes_\Lambda Q)_m \longrightarrow (Q \otimes_\Lambda Q)_{m-1}$ is given by

$$
D_m = \sum_{i=0}^{m-1}((-1)^i \otimes \sigma_{m-i} + \sigma_{i+1} \otimes 1).
$$

Now we will define a chain map $\Delta : (\mathbb{Q}_\bullet, \sigma_\bullet) \longrightarrow (\mathbb{Q} \otimes_\Lambda \mathbb{Q}, D)$. Note that $Q_m = \coprod_{f\in\Gamma^m} \Lambda o(f) \otimes t(f)\Lambda \cong \coprod_{f\in\Gamma^m} \Lambda \otimes f \otimes \Lambda$ as Λ-bimodules. For simplicity, we will adopt the notation $Q_m = \coprod_{f\in\Gamma^m} \Lambda \otimes f \otimes \Lambda$ for the rest of this section.

Definition 4.2. The map $\Delta : (\mathbb{Q}_\bullet, \sigma_\bullet) \longrightarrow (\mathbb{Q} \otimes_\Lambda \mathbb{Q}, D)$ is defined by

$$\Delta_m(1 \otimes f^{m,j}_{i_1\cdots i_m} \otimes 1) = \sum_{s=0}^{m} \sum_{1\leqslant p_1<\cdots<p_s\leqslant m} (-1)^{\epsilon(p_1p_2\cdots p_m)}(1 \otimes f^{s,j}_{i_{p_1}\cdots i_{p_s}} \otimes 1) \otimes_\Lambda (1 \otimes f^{m-s,j+s}_{i_{p_{s+1}}\cdots i_{p_m}} \otimes 1),$$

where $\epsilon(p_1p_2\cdots p_m)$ denotes the sign of the permutation $\kappa = \left(\begin{smallmatrix} 1 & 2 & \cdots & m \\ p_1 & p_2 & \cdots & p_m \end{smallmatrix}\right)$, i.e.,

$$\epsilon(p_1p_2\cdots p_m) = \begin{cases} 1, & \text{if } \kappa \text{ is even;} \\ -1, & \text{if } \kappa \text{ is odd.} \end{cases}$$

Lemma 4.3. *The map $\Delta : (\mathbb{Q}_\bullet, \sigma_\bullet) \longrightarrow (\mathbb{Q} \otimes_\Lambda \mathbb{Q}, D)$ defined as above is a chain map which lifts the identity.*

Proof. It is sufficient to show that the following diagram

$$\begin{array}{ccc} Q_m & \xrightarrow{\sigma_m} & Q_{m-1} \\ \Delta_m \big\downarrow & & \big\downarrow \Delta_{m-1} \\ (Q \otimes_\Lambda Q)_m & \xrightarrow{D_m} & (Q \otimes_\Lambda Q)_{m-1} \end{array} \tag{4.1}$$

is commutative for all $m \geqslant 1$. By the linearity of Δ and σ, it is enough to prove for any $1\otimes f^{m,j}_{i_1\cdots i_m}\otimes 1 \in Q_m$,

$$D_m\Delta_m(1 \otimes f^{m,j}_{i_1\cdots i_m} \otimes 1) = \Delta_{m-1}\sigma_m(1 \otimes f^{m,j}_{i_1\cdots i_m} \otimes 1).$$

Since

$$(Q \otimes_\Lambda Q)_{m-1} = \coprod_{r=0}^{m-1} Q_r \otimes_\Lambda Q_{m-1-r},$$

we will prove that the r-th component of $D_m\Delta_m(1\otimes f^{m,j}_{i_1\cdots i_m}\otimes 1)$ is equal to that of $\Delta_{m-1}\sigma_m(1\otimes f^{m,j}_{i_1\cdots i_m}\otimes 1)$ for each $0 \leqslant r \leqslant m-1$. By definition,

$$\sigma_m(1 \otimes f^{m,j}_{i_1\cdots i_m} \otimes 1) = \sum_{t=1}^{m}[(-1)^{t-1}x_{ji_t} \otimes f^{m-1,j+1}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m} \otimes 1 + (-1)^t 1 \otimes f^{m-1,j}_{i_1\cdots i_{t-1}i_{t+1}\cdots i_m} \otimes x_{j+m-1,i_t}],$$

we obtain the r-th component of $\Delta_{m-1}\sigma_m(1 \otimes f^{m,j}_{i_1\cdots i_m} \otimes 1)$ is the sum of

$$\sum_{t=1}^{m} \sum_{p_1\cdots p_r} (-1)^{\epsilon(p_1p_2\cdots p_{m-1})}(-1)^{t-1}(x_{ji_t} \otimes f^{r,j+1}_{i_{p_1}\cdots i_{p_r}} \otimes 1) \otimes_\Lambda (1 \otimes f^{m-1-r,j+r+1}_{i_{p_{r+1}}\cdots i_{p_{m-1}}} \otimes 1) \tag{4.2}$$

and

$$\sum_{t=1}^{m} \sum_{p_1\cdots p_r} (-1)^{\epsilon(p_1p_2\cdots p_{m-1})}(-1)^{t}(1 \otimes f^{r,j}_{i_{p_1}\cdots i_{p_r}} \otimes 1) \otimes_\Lambda (1 \otimes f^{m-1-r,j+r}_{i_{p_{r+1}}\cdots i_{p_{m-1}}} \otimes x_{ji_t}), \tag{4.3}$$

where $p_1\cdots p_r$ runs over all the strict increasing sequences $p_1\cdots p_r$ with $p_i \in \{1, 2, \ldots, m\}\backslash\{t\}$. Note that the sequence $p_1p_2\cdots p_{m-1}$ is not necessary to be strict increasing though both $p_1\cdots p_r$ and $p_{r+1}\cdots p_{m-1}$ are.

On the other hand, by the definition of Δ, we have

$$\Delta_m(1 \otimes f^{m,j}_{i_1\cdots i_m} \otimes 1) = \sum_{s=0}^{m} \sum_{1\leqslant p_1<\cdots<p_s\leqslant m} (-1)^{\epsilon(p_1p_2\cdots p_m)}(1 \otimes f^{s,j}_{i_{p_1}\cdots i_{p_s}} \otimes 1) \otimes_\Lambda (1 \otimes f^{m-s,j+s}_{i_{p_{s+1}}\cdots i_{p_m}} \otimes 1).$$

Then the r-component of $D_m\Delta_m(1\otimes f^{m,j}_{i_1\cdots i_m}\otimes 1)$ is the sum of the following four terms:

$$\sum_{1\leqslant q_1<\cdots<q_{r+1}\leqslant m}\sum_{t=1}^{r+1}(-1)^{\epsilon(q_1q_2\cdots q_m)}(-1)^{t-1}(x_{ji_{q_t}}\otimes f^{r,j+1}_{i_{q_1}\cdots i_{q_{t-1}}i_{q_{t+1}}\cdots i_{q_{r+1}}}\otimes 1)\otimes_\Lambda(1\otimes f^{m-1-r,j+r+1}_{i_{q_{r+2}}\cdots i_{q_m}}\otimes 1), \tag{4.4}$$

$$\sum_{1\leqslant q_1<\cdots<q_{r+1}\leqslant m}\sum_{t=1}^{r+1}(-1)^{\epsilon(q_1q_2\cdots q_m)}(-1)^{t}(1\otimes f^{r,j}_{i_{q_1}\cdots i_{q_{t-1}}i_{q_{t+1}}\cdots i_{q_{r+1}}}\otimes x_{j+r,i_{q_t}})\otimes_\Lambda(1\otimes f^{m-1-r,j+r+1}_{i_{q_{r+2}}\cdots i_{q_m}}\otimes 1), \tag{4.5}$$

$$\sum_{1\leqslant q_1<\cdots<q_{r}\leqslant m}\sum_{t=1}^{m-r}(-1)^{\epsilon(q_1q_2\cdots q_m)}(-1)^{r+t-1}(1\otimes f^{r,j}_{i_{q_1}\cdots i_{q_r}}\otimes 1)\otimes_\Lambda(x_{j+r,i_{q_{r+t}}}\otimes f^{m-1-r,j+r+1}_{i_{q_{r+1}}\cdots i_{q_{r+t-1}}i_{q_{r+t+1}}\cdots i_{q_m}}\otimes 1), \tag{4.6}$$

$$\sum_{1\leqslant q_1<\cdots<q_{r}\leqslant m}\sum_{t=1}^{m-r}(-1)^{\epsilon(q_1q_2\cdots q_m)}(-1)^{t}(1\otimes f^{r,j}_{i_{q_1}\cdots i_{q_r}}\otimes 1)\otimes_\Lambda(1\otimes f^{m-1-r,j+r}_{i_{q_{r+1}}\cdots i_{q_{r+t-1}}i_{q_{r+t+1}}\cdots i_{q_m}}\otimes x_{j+m-1,i_{q_{r+t}}}). \tag{4.7}$$

To prove that the diagram (4.1) is commutative, it suffices to prove (4.2) + (4.3) = (4.4) + (4.5) + (4.6) + (4.7). In fact, we will show that (4.2) = (4.4), (4.3) = (4.7) and (4.5) + (4.6) = 0.

In order to prove (4.2) = (4.4), we first show that each summand of (4.2) is a summand of (4.4). Taken a summand $(-1)^{\epsilon(p_1p_2\cdots p_{m-1})}(-1)^{t-1}(x_{ji_t}\otimes f^{r,j+1}_{i_{p_1}\cdots i_{p_r}}\otimes 1)\otimes_\Lambda(1\otimes f^{m-1-r,j+r+1}_{i_{p_{r+1}}\cdots i_{p_{m-1}}}\otimes 1)$ in Formula (4.2), we assume $s=\mu_{i_{p_1}\cdots i_{p_r}}(i_t)$ and write the sequence $q_1q_2\cdots q_m$ instead of $p_1\cdots p_{s-1}tp_s\cdots p_{m-1}$. Thus we have

$$\begin{aligned}
&(-1)^{t-1}(-1)^{\epsilon(p_1\cdots p_{m-1})}(x_{ji_t}\otimes f^{r,j+1}_{i_{p_1}\cdots i_{p_r}}\otimes 1)\otimes_\Lambda(1\otimes f^{m-1-r,j+r+1}_{i_{p_{r+1}}\cdots i_{p_{m-1}}}\otimes 1)\\
&\quad=(-1)^{(t-s)}(-1)^{\epsilon(p_1\cdots p_{m-1})}(-1)^{s-1}(x_{ji_t}\otimes f^{r,j+1}_{i_{p_1}\cdots i_{p_r}}\otimes 1)\otimes_\Lambda(1\otimes f^{m-1-r,j+r+1}_{i_{p_{r+1}}\cdots i_{p_{m-1}}}\otimes 1)\\
&\quad=(-1)^{\epsilon(p_1\cdots p_{s-1}tp_{s+1}\cdots p_{m-1})}(-1)^{s-1}(x_{ji_t}\otimes f^{r,j+1}_{i_{p_1}\cdots i_{p_r}}\otimes 1)\otimes_\Lambda(1\otimes f^{m-1-r,j+r+1}_{i_{p_{r+1}}\cdots i_{p_{m-1}}}\otimes 1)\\
&\quad=(-1)^{\epsilon(q_1q_2\cdots q_m)}(-1)^{s-1}(x_{ji_{q_s}}\otimes f^{r,j+1}_{i_{q_1}\cdots i_{q_{s-1}}i_{q_{s+1}}\cdots i_{q_{r+1}}}\otimes 1)\otimes_\Lambda(1\otimes f^{m-1-r,j+r+1}_{i_{q_{r+2}}\cdots i_{q_m}}\otimes 1),
\end{aligned}$$

which is a summand of the formula (4.4). Conversely, given a summand $(-1)^{\epsilon(q_1q_2\cdots q_m)}(-1)^{t-1}(x_{ji_{q_t}}\otimes f^{r,j+1}_{i_{q_1}\cdots i_{q_{t-1}}i_{q_{t+1}}\cdots i_{q_{r+1}}}\otimes 1)\otimes_\Lambda(1\otimes f^{m-1-r,j+r+1}_{i_{q_{r+2}}\cdots i_{q_m}}\otimes 1)$ of the formula (4.4), we assume $s=q_t$ and write the sequence $p_1\cdots p_{m-1}$ as $q_1\cdots q_{t-1}q_{t+1}\cdots q_m$. Then we have

$$\begin{aligned}
&(-1)^{\epsilon(q_1q_2\cdots q_m)}(-1)^{t-1}(x_{ji_{q_t}}\otimes f^{r,j+1}_{i_{q_1}\cdots i_{q_{t-1}}i_{q_{t+1}}\cdots i_{q_{r+1}}}\otimes 1)\otimes_\Lambda(1\otimes f^{m-1-r,j+r+1}_{i_{q_{r+2}}\cdots i_{q_m}}\otimes 1)\\
&\quad=(-1)^{\epsilon(p_1\cdots p_{t-1}sp_t\cdots p_{m-1})}(-1)^{t-1}(x_{ji_s}\otimes f^{r,j+1}_{i_{p_1}\cdots i_{p_r}}\otimes 1)\otimes_\Lambda(1\otimes f^{m-1-r,j+r+1}_{i_{p_{r+1}}\cdots i_{p_{m-1}}}\otimes 1)\\
&\quad=(-1)^{(t-1)-(s-1)}(-1)^{\epsilon(p_1\cdots p_{m-1})}(-1)^{t-1}(x_{ji_s}\otimes f^{r,j+1}_{i_{p_1}\cdots i_{p_r}}\otimes 1)\otimes_\Lambda(1\otimes f^{m-1-r,j+r+1}_{i_{p_{r+1}}\cdots i_{p_{m-1}}}\otimes 1)\\
&\quad=(-1)^{s-1}(-1)^{\epsilon(p_1\cdots p_{m-1})}(x_{ji_s}\otimes f^{r,j+1}_{i_{p_1}\cdots i_{p_r}}\otimes 1)\otimes_\Lambda(1\otimes f^{m-1-r,j+r+1}_{i_{p_{r+1}}\cdots i_{p_{m-1}}}\otimes 1),
\end{aligned}$$

which is a summand of the formula (4.2). Thus (4.2) = (4.4).

A similar argument yields (4.3) = (4.7). It remains to be show that (4.5) + (4.6) = 0. We do this by showing that any summand in Formula (4.5) appears in Formula (4.6) with the opposite sign and the converse holds as well.

Taken a summand $(-1)^{\epsilon(q_1q_2\cdots q_m)}(-1)^t(1\otimes f^{r,j}_{i_{q_1}\cdots i_{q_{t-1}}i_{q_{t+1}}\cdots i_{q_{r+1}}}\otimes x_{j+r,i_{q_t}})\otimes_\Lambda(1\otimes f^{m-1-r,j+r+1}_{i_{q_{r+2}}\cdots i_{q_m}}\otimes 1)$ in Formula (4.5), we assume $p_{r+s}=q_t$ and write $p_1\cdots p_{r+s-1}\,p_{r+s+1}\cdots p_m$ as $q_1\cdots q_{t-1}q_{t+1}\cdots q_m$, then we have

$$\begin{aligned}&(-1)^{\epsilon(q_1q_2\cdots q_m)}(-1)^t(1\otimes f^{r,j}_{i_{q_1}\cdots i_{q_{t-1}}i_{q_{t+1}}\cdots i_{q_{r+1}}}\otimes x_{j+r,i_{q_t}})\otimes_\Lambda(1\otimes f^{m-1-r,j+r+1}_{i_{q_{r+2}}\cdots i_{q_m}}\otimes 1)\\&\quad=(-1)^{r+s-t}(-1)^{\epsilon(p_1\cdots p_m)}(-1)^t(1\otimes f^{r,j}_{i_{p_1}\cdots\cdots i_{p_r}}\otimes x_{j+r,i_{p_{r+s}}})\otimes_\Lambda(1\otimes f^{m-1-r,j+r+1}_{i_{p_{r+1}}\cdots i_{p_m}}\otimes 1)\\&\quad=-(-1)^{\epsilon(p_1\cdots p_m)}(-1)^{r+s-1}(1\otimes f^{r,j}_{i_{p_1}\cdots\cdots i_{p_r}}\otimes 1)\otimes_\Lambda(x_{j+r,i_{p_{r+s}}}\otimes f^{m-1-r,j+r+1}_{i_{p_{r+1}}\cdots i_{p_m}}\otimes 1),\end{aligned}$$

which is in (4.6) with the opposite sign. Conversely, given a summand $(-1)^{\epsilon(q_1q_2\cdots q_m)}(-1)^{r+t-1}(1\otimes f^{r,j}_{i_{q_1}\cdots i_{q_r}}\otimes 1)\otimes_\Lambda(x_{j+r,i_{q_{r+t}}}\otimes f^{m-1-r,j+r+1}_{i_{q_{r+1}}\cdots i_{q_{r+t-1}}i_{q_{r+t+1}}\cdots i_{q_m}}\otimes 1)$ in the formula (4.6), we assume $s=\mu_{i_{q_1}\cdots i_{q_r}}(i_{q_{r+t}})$ and we set $p_s=q_{r+t}$ and write the sequence $p_1\cdots p_{s-1}p_{s+1}\cdots p_m$ as $q_1\cdots q_{r+t-1}q_{r+t+1}\cdots q_m$, thus we have

$$\begin{aligned}&(-1)^{\epsilon(q_1q_2\cdots q_m)}(-1)^{r+t-1}(1\otimes f^{r,j}_{i_{q_1}\cdots i_{q_r}}\otimes 1)\otimes_\Lambda(x_{j+r,i_{q_{r+t}}}\otimes f^{m-1-r,j+r+1}_{i_{q_{r+1}}\cdots i_{q_{r+t-1}}i_{q_{r+t+1}}\cdots i_{q_m}}\otimes 1)\\&\quad=-(-1)^s(-1)^{\epsilon(p_1\cdots p_m)}(1\otimes f^{r,j}_{i_{p_1}\cdots i_{p_{s-1}}i_{p_{s+1}}\cdots i_{p_{r+1}}}\otimes 1)\otimes_\Lambda(x_{j+r,i_{p_s}}\otimes f^{m-1-r,j+r+1}_{i_{p_{r+2}}\cdots i_{p_m}}\otimes 1)\\&\quad=-(-1)^s(-1)^{\epsilon(p_1\cdots p_m)}(1\otimes f^{r,j}_{i_{p_1}\cdots i_{p_{s-1}}i_{p_{s+1}}\cdots i_{p_{r+1}}}\otimes x_{j+r,i_{p_s}})\otimes_\Lambda(1\otimes f^{m-1-r,j+r+1}_{i_{p_{r+2}}\cdots i_{p_m}}\otimes 1),\end{aligned}$$

which is also a summand of Formula (4.5) with the opposite sign.

The proof is finished. □

The following theorem shows that the cup product in Hochschild cohomology ring of the Beilinson algebra Λ is essentially juxtaposition of parallel paths up to sign, which will provide an explicit description of the multiplicative structure of Hochschild cohomology ring.

Theorem 4.4. *Let Λ be the Beilinson algebra of exterior algebra. Suppose that*

$$\xi=(b^{m,i}_{r_1\cdots r_m},f^{m,i}_{i_1i_2\cdots i_m})\in M^m,\quad \theta=(b^{s,j}_{r'_1\cdots r'_s},f^{s,j}_{i'_1i'_2\cdots i'_s})\in M^s$$

represent two cochains in Q^*_m and Q^*_s. Then the cup product of ξ and θ in Q^*_{m+s} is

$$\xi\sqcup\theta=\begin{cases}(-1)^{\epsilon(i_1i_2\cdots i_mi'_1i'_2\cdots i'_s)}(b^{m,i}_{r_1\cdots r_m}b^{s,i+m}_{r'_1\cdots r'_s},f^{m+s,i}_{i_1\cdots i_m\hat{i'_1}\cdots\hat{i'_s}}),\\ \qquad \textit{if } \{i_1,i_2,\ldots,i_m\}\cap\{i'_1,i'_2,\ldots,i'_s\}=\emptyset \textit{ and } j=i+m;\\ 0,\qquad \textit{otherwise.}\end{cases}$$

Recall that $i_1\cdots i_m\hat{i'_1}\cdots\hat{i'_s}$ stands for the strict decreasing sequence consisting of $i_1,\ldots,i_m$, $i'_1,\ldots,i'_s$.

Proof. For simplicity, we still denote by ξ and θ the elements $\phi(\xi)$ and $\phi(\theta)$ in Q^*_m and Q^*_s respectively, where ϕ is the isomorphism defined in the proof of Lemma 3.1. Note that the cup product of ξ and θ is the composition of the following maps

$$\mathbb{Q}\xrightarrow{\Delta}\mathbb{Q}\otimes_\Lambda\mathbb{Q}\xrightarrow{\xi\otimes\theta}\Lambda\otimes_\Lambda\Lambda\xrightarrow{\nu}\Lambda,$$

then for any element $1\otimes f^{m+s,k}_{j_1\cdots j_{m+s}}\otimes 1\in Q_{m+s}$, we have

$$\begin{aligned}&\xi\sqcup\theta(1\otimes f^{m+s,k}_{j_1\cdots j_{m+s}}\otimes 1)=\nu(\xi\otimes\theta)\Delta(1\otimes f^{m+s,k}_{j_1\cdots j_{m+s}}\otimes 1)\\&\quad=\nu(\xi\otimes\theta)\left[\sum_{l=0}^{m+s}\sum_{1\leqslant p_1\leqslant\cdots\leqslant p_l\leqslant m+s}(-1)^{\epsilon(p_1p_2\cdots p_{m+s})}(1\otimes f^{l,k}_{i_{p_1}\cdots i_{p_l}}\otimes 1)\otimes_\Lambda(1\otimes f^{m+s-l,k+l}_{i_{p_{l+1}}\cdots i_{p_{m+s}}}\otimes 1)\right]\\&\quad=\sum_{p_1,\ldots,p_m\in\{1,\ldots,m+s\}}(-1)^{\epsilon(p_1p_2\cdots p_{m+s})}\xi(1\otimes f^{m,k}_{i_{p_1}\cdots i_{p_m}}\otimes 1)\theta(1\otimes f^{s,k+m}_{i_{p_{m+1}}\cdots i_{p_{m+s}}}\otimes 1)\\&\quad=\begin{cases}(-1)^{\epsilon(i_1i_2\cdots i_mi'_1i'_2\cdots i'_s)}b^{m,i}_{r_1\cdots r_m}b^{s,i+m}_{r'_1\cdots r'_s},\\ \qquad\text{if } \{j_1,\ldots,j_{m+s}\}=\{i_1,i_2,\ldots,i_m,i'_1,i'_2,\ldots,i'_s\}\text{ and } k=i,\quad j=i+m;\\ 0,\qquad\text{otherwise.}\end{cases}\end{aligned}$$

So, under the isomorphism ϕ defined as in the proof of Lemma 3.1,

$$\xi \sqcup \theta = \begin{cases} (-1)^{\epsilon(i_1 i_2 \cdots i_m i'_1 i'_2 \cdots i'_s)}(b^{m,i}_{r_1\cdots r_m} b^{s,i+m}_{r'_1\cdots r'_s}, f^{m+s,i}_{i_1\cdots i_m \hat{i'_1}\cdots \hat{i'_s}}), \\ \qquad \text{if } \{i_1, i_2, \ldots, i_m\} \cap \{i'_1, i'_2, \ldots, i'_s\} = \emptyset \text{ and } j = i+m, \\ 0, \quad \text{otherwise,} \end{cases}$$

as desired. □

Lemma 4.5. *The Hochschild cohomology ring $HH^*(\Lambda)$ is generated by $HH^0(\Lambda)$ and $HH^1(\Lambda)$.*

Proof. By Lemma 4.1, $\mathcal{H}^m = \bigcup_{i=0}^m \mathcal{H}^{(m,i)}$ forms a k-basis of $\mathrm{HH}^m(\Lambda)$, and thus it suffices to show that every element in $\mathcal{H}^m$ can be generated by elements in $\mathrm{HH}^0(\Lambda)$ and $\mathrm{HH}^1(\Lambda)$. Taken any element $\xi^0 = \sum_{j=0}^{n-m}(b^{m,j}_{r_1\cdots r_m}, f^{m,j}_{h_1\cdots h_m})$ in $\mathcal{H}^{(m,0)}$ (here $n \geqslant r_1 \geqslant \cdots \geqslant r_m \geqslant 0, n-1 \geqslant h_1 > \cdots > h_m \geqslant 0$), by Theorem 4.4, we have

$$\begin{aligned} \xi^0 &= \sum_{j=0}^{n-m}(b^{1,j}_{r_1} b^{1,j+1}_{r_2} \cdots b^{1,j+m-1}_{r_m}, f^{1,j}_{h_1} f^{1,j+1}_{h_2} \cdots f^{1,j+m-1}_{h_m}) \\ &= \left(\sum_{j=0}^{n-1}(b^{1,j}_{r_1}, f^{1,j}_{h_1})\right) \sqcup \left(\sum_{j=0}^{n-1}(b^{1,j}_{r_2}, f^{1,j}_{h_2})\right) \sqcup \cdots \sqcup \left(\sum_{j=0}^{n-1}(b^{1,j}_{r_m}, f^{1,j}_{h_m})\right), \end{aligned}$$

where $\sum_{j=0}^{n-1}(b^{1,j}_{r_k}, f^{1,j}_{h_k})(1 \leqslant k \leqslant m)$ lies in $\mathrm{HH}^1(\Lambda)$.

If $1 \leqslant i \leqslant m$, given an element $\xi^i = \sum_{j=0}^{n-m}(b^{m,j}_{r_1\cdots r_m}, f^{m,j}_{n(n-1)\cdots(n-i+1)h_1\cdots h_{m-i}})$ in $\mathcal{H}^{(m,i)}$ (here $n-i \geqslant r_1 \geqslant \cdots \geqslant r_m \geqslant 0, n-i-1 \geqslant h_1 > \cdots > h_m \geqslant 0$), we have

$$\begin{aligned} \xi^i &= \sum_{j=0}^{n-m}(b^{1,j}_{r_1} b^{1,j+1}_{r_2} \cdots b^{1,j+m-1}_{r_m}, f^{1,j}_{n} \cdots f^{1,j+i-1}_{n-i+1} f^{1,j+i}_{h_1} \cdots f^{1,j+m-1}_{h_{m-i}}) \\ &= \left(\sum_{j=0}^{n-1}(b^{1,j}_{r_1}, f^{1,j}_{n})\right) \sqcup \cdots \sqcup \left(\sum_{j=0}^{n-1}(b^{1,j}_{r_i}, f^{1,j}_{n-i+1})\right) \sqcup \left(\sum_{j=0}^{n-1}(b^{1,j}_{r_{i+1}}, f^{1,j}_{h_1})\right) \sqcup \cdots \sqcup \left(\sum_{j=0}^{n-1}(b^{1,j}_{r_m}, f^{1,j}_{h_{m-i}})\right). \end{aligned}$$

Therefore, $\mathcal{H}^{(m,i)}$ is generated by $\mathrm{HH}^1(\Lambda)$. □

Recall from Lemma 4.1 that $\mathcal{H} = \bigcup_{m=0}^n \mathcal{H}^m = \bigcup_{m=0}^n (\bigcup_{i=0}^m \mathcal{H}^{(m,i)})$ forms a k-basis of the k-vector space $\mathrm{HH}^*(\Lambda)$, where $\mathcal{H}^{(m,i)} = \{\sum_{j=0}^{n-m}(b^{m,j}_{r_1\cdots r_m}, f^{m,j}_{n(n-1)\cdots(n-i+1)h_1\cdots h_{m-i}}) \mid (r_1 r_2 \cdots r_m, n(n-1)\cdots(n-i+1)h_1 \cdots h_{m-i}) \in T^m_i\}$, $T^m_i = \{(r_1 \cdots r_m, n(n-1)\cdots(n-i+1)h_1 \cdots h_{m-i}) \mid n-i \geqslant r_1 \geqslant r_2 \geqslant \cdots \geqslant r_m \geqslant 0, n-i-1 \geqslant h_1 > h_2 > \cdots > h_{m-i} \geqslant 0\}$ and $T^m = \bigcup_{i=0}^m T^m_i$ defined as in the proof of Lemma 3.2.

It follows from Lemma 4.5 that, as a k-algebra, $\mathrm{HH}^*(\Lambda)$ is generated by $\mathcal{H}^0 \cup \mathcal{H}^1 = \{\sum_{j=0}^n (e_j, e_j)\} \cup \{\sum_{j=0}^{n-1}(x_{jr}, x_{jh}) \mid 0 \leqslant r \leqslant n, 0 \leqslant h \leqslant n, (r,h) \neq (n,n)\}$. Note that $\sum_{j=0}^n (e_j, e_j)$ is the identity in $\mathrm{HH}^*(\Lambda)$ and it is easy to check that the others satisfy the following relations:

(1) $$\sum_{j=0}^{n-1}(x_{jr_1}, x_{jh_1}) \sqcup \sum_{j=0}^{n-1}(x_{jr_2}, x_{jh_2}) = 0, \quad \text{if } h_1 = h_2;$$

(2) $$\begin{aligned} &\sum_{j=0}^{n-1}(x_{jr_1}, x_{jh_1}) \sqcup \sum_{j=0}^{n-1}(x_{jr_2}, x_{jh_2}) \\ &\quad = \begin{cases} -\sum_{j=0}^{n-1}(x_{jr_1}, x_{jh_2}) \sqcup \sum_{j=0}^{n-1}(x_{jr_2}, x_{jh_1}), & \text{if } (r_1, h_2) \neq (n,n), \quad (r_2, h_1) \neq (n,n); \\ 0, & \text{otherwise;} \end{cases} \end{aligned}$$

(3) $$\sum_{j=0}^{n-1}(x_{jr_1}, x_{jh_1}) \sqcup \sum_{j=0}^{n-1}(x_{jr_2}, x_{jh_2})$$

$$= \begin{cases} \sum_{j=0}^{n-1}(x_{jr_2}, x_{jh_1}) \sqcup \sum_{j=0}^{n-1}(x_{jr_1}, x_{jh_2}), & \text{if } (r_1, h_2) \neq (n, n), \quad (r_2, h_1) \neq (n, n); \\ 0, & \text{otherwise.} \end{cases}$$

Replacing the above $\sum_{j=0}^{n}(e_j, e_j)$ and $\sum_{j=0}^{n-1}(x_{jr}, x_{jh})$ in $\mathcal{H}^0 \cup \mathcal{H}^1$ with 1, u_{rh} respectively, and omitting the symbol $\sqcup$ for simplicity, we have the following relations:

$(1)'$ $u_{r_1h_1}u_{r_2h_2} = 0$, if $h_1 = h_2$;

$(2)'$

$$u_{r_1h_1}u_{r_2h_2} = \begin{cases} -u_{r_1h_2}u_{r_2h_1}, & \text{if } (r_1, h_2) \neq (n, n), \quad (r_2, h_1) \neq (n, n); \\ 0, & \text{otherwise;} \end{cases}$$

$(3)'$

$$u_{r_1h_1}u_{r_2h_2} = \begin{cases} u_{r_2h_1}u_{r_1h_2}, & \text{if } (r_1, h_2) \neq (n, n), \quad (r_2, h_1) \neq (n, n); \\ 0, & \text{otherwise.} \end{cases}$$

Note that since $0 \leqslant h_i \leqslant n$, the product $\prod_{i=0}^{n+1} u_{r_ih_i}$ in $\mathrm{HH}^*(\Lambda)$ must be zero because of the relation $(1)'$. Clearly, for any monomial $u_{s_1p_1}u_{s_2p_2}\cdots u_{s_lp_l}$ of degree l ($l > 0$), using the relations $(1)'$–$(3)'$ one can reduce it to 0 or the form with the decreasing indices, namely, $u_{s_1p_1}u_{s_2p_2}\cdots u_{s_lp_l}$ can be changed to the form of

$$u_{r_1h_1}u_{r_2h_2}\cdots u_{r_lh_l} \text{ with } n \geqslant r_1 \geqslant r_2 \geqslant \cdots \geqslant r_l \geqslant 0, n \geqslant h_1 > h_2 > \cdots > h_l \geqslant 0,$$

which is called the *normal form* of $u_{s_1p_1}u_{s_2p_2}\cdots u_{s_lp_l}$, and denoted by

$$N(u_{s_1p_1}u_{s_2p_2}\cdots u_{s_lp_l}) = u_{r_1h_1}u_{r_2h_2}\cdots u_{r_lh_l}.$$

We add the relation

$(4)'$ $u_{s_1p_1}u_{s_2p_2}\cdots u_{s_lp_l} = 0$ if the pair $(r_1r_2\cdots r_l, h_1h_2\cdots h_l)$ of the subscripts of the normal form $N(u_{s_1p_1}u_{s_2p_2}\cdots u_{s_lp_l})$ does not belong to T.

We now can give a presentation of the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda)$. Let Q' be the quiver with one vertex 1 and $n^2 + 2n$ loops $\{u_{rh}|\ 0 \leqslant r \leqslant n, 0 \leqslant h \leqslant n, (r, h) \neq (n, n)\}$, and I' be the ideal of KQ' generated by the relations $(1)'$–$(4)'$.

Theorem 4.6. *Let Λ be the Beilinson algebra of exterior algebra. Then the Hochschild cohomology ring $HH^*(\Lambda)$ is isomorphic to kQ'/I'.*

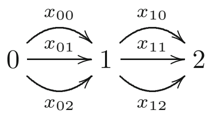
Proof. Denote by Q'_0 and Q'_1 the sets of vertex and arrows of the local quiver Q' defined as above, respectively. Then the maps $\psi_0 : Q'_0 \to \mathrm{HH}^*(\Lambda)$ given by $1 \mapsto \sum_{j=0}^{n}(e_j, e_j)$ and $\psi_1 : Q'_1 \to \mathrm{HH}^*(\Lambda)$ given by $u_{r_1h_1} \mapsto \sum_{j=0}^{n-1}(x_{jr_1}, x_{jh_1}), u_{r_1n} \mapsto \sum_{j=0}^{n-1}(x_{jr_1}, x_{jn})$ can be uniquely extended to an epimorphism of k-algebras $\psi : kQ' \to \mathrm{HH}^*(\Lambda)$.

Let $\bar{I}$ be the ideal of kQ' generated by the relations $(1)'$–$(3)'$. It follows that $\bar{I} \subseteq \mathrm{Ker}\psi$ from the relations (1)–(3) directly, and thus ψ induces an epimorphism $\bar{\psi} : \Gamma_n = kQ'/\bar{I} \to \mathrm{HH}^*(\Lambda)$. For legibility, we do not distinguish a path in KQ' with its image in Γ_n. In particular, Γ_n is a finite dimensional algebra, and $\Gamma_n = \Gamma_n^{(1)} \oplus \Gamma_n^{(2)}$ as vector spaces, where $\Gamma_n^{(1)} = \mathrm{span}\{1, u_{r_1h_1}u_{r_2h_2}\cdots u_{r_lh_l} \mid n \geqslant r_1 \geqslant r_2 \geqslant \cdots \geqslant r_l \geqslant 0, n \geqslant h_1 > h_2 > \cdots > h_l \geqslant 0, (r_1r_2\cdots r_l, h_1h_2\cdots h_l) \in T\}$ and $\Gamma_n^{(2)} = \mathrm{span}\{u_{r_1h_1}u_{r_2h_2}\cdots u_{r_lh_l} \mid n \geqslant r_1 \geqslant r_2 \geqslant \cdots \geqslant r_l \geqslant 0, n \geqslant h_1 > h_2 > \cdots > h_l \geqslant 0, (r_1r_2\cdots r_l, h_1h_2\cdots h_l) \notin T\}$. Let $\tilde{I}$ be an ideal of Γ_n generated by the set $\{u_{r_1h_1}u_{r_2h_2}\cdots u_{r_lh_l} \mid n \geqslant r_1 \geqslant r_2 \geqslant \cdots \geqslant r_l \geqslant 0, n \geqslant h_1 > h_2 > \cdots > h_l \geqslant 0, (r_1r_2\cdots r_l, h_1h_2\cdots h_l) \notin T\}$. We claim that $\bar{\psi}(\tilde{I}) = 0$. Indeed, for any index $(r_1r_2\cdots r_m, n(n-1)\cdots(n-i+1)h_1\cdots h_{m-i}) \in T_i^m$, we have $\bar{\psi}(u_{r_1n}\cdots u_{r_{i+1}h_1}\cdots u_{r_mh_{m-i}}) = \xi^i_{(r_1r_2\cdots r_m, n(n-1)\cdots(n-i+1)h_1\cdots h_{m-i})} \in \mathcal{H}^{(m,i)}$ by the proof of Lemma 4.5. So, as linear maps, the restriction of $\bar{\psi}$ to $\Gamma_n^{(1)}$ is an isomorphism, and zero to $\Gamma_n^{(2)}$. Thus $\tilde{I} \subseteq \mathrm{Ker}\bar{\psi}$ as desired. So $\bar{\psi}$ induces an epimorphism $\tilde{\psi} : \Gamma_n/\tilde{I} \to \mathrm{HH}^*(\Lambda)$.

It is clear that the set $\mathcal{V} = \bigcup_{m=1}^{n}\{u_{r_1h_1}u_{r_2h_2}\cdots u_{r_mh_m} \mid (r_1r_2\cdots r_m, h_1h_2\cdots h_m \in T^m\} \cup \{1\}$ is a k-basis of $\Gamma_n/\tilde{I}$. So $\dim_k \Gamma_n/\tilde{I} = |T| = \dim_k \mathrm{HH}^*(\Lambda)$. Hence $\widetilde{\psi}$ is an isomorphism.

Since $\widetilde{I} \cong I'/\bar{I}$, we have $kQ'/I' \cong (kQ'/\bar{I})/(I'/\bar{I}) \cong \Gamma_n/\widetilde{I} \cong \mathrm{HH}^*(\Lambda)$ as desired. □

Example 4.7. We consider the case $n = 2$. In this case, $\Lambda_n = kQ_2/I_2$, where Q_2 is the quiver with 3 vertices and 6 arrows as follows:



and $I_2 = \langle x_{0i}x_{1j} - x_{0j}x_{1i} \mid 0 \leqslant i, j \leqslant 2\rangle$. From Lemma 4.1, we know that $\mathcal{H} = \mathcal{H}^0 \cup \mathcal{H}^1 \cup \mathcal{H}^2$ forms a k-basis of $\mathrm{HH}^*(\Lambda_2)$, where

$$\mathcal{H}^0 = \left\{ \sum_{j=0}^{2}(e_j, e_j) \right\};$$

$$\mathcal{H}^1 = \left\{ \sum_{j=0}^{1}(x_{jr}, x_{jh}) \mid 2 \geqslant r \geqslant 0, 1 \geqslant h \geqslant 0 \right\} \cup \left\{ \sum_{j=0}^{1}(x_{jr}, x_{j2}) \mid 1 \geqslant r \geqslant 0 \right\};$$

$$\mathcal{H}^2 = \{(b^{2,0}_{r_1r_2}, f^{2,0}_{10}) \mid 2 \geqslant r_1 \geqslant r_2 \geqslant 0\} \cup \{(b^{2,0}_{r_1r_2}, f^{2,0}_{20}) \mid 1 \geqslant r_1 \geqslant r_2 \geqslant 0\} \cup \{(b^{2,0}_{00}, f^{2,0}_{21})\}.$$

Thus $\dim_k \mathrm{HH}^*(\Lambda_2) = 19$. Using Theorem 4.6, we can give a presentation of $\mathrm{HH}^*(\Lambda_2)$. Let kQ'_2 be the path algebra corresponding to the quiver with 1 vertex and 8 loops $\{u_{ij} \mid 2 \geqslant i, j \geqslant 0, (i, j) \neq (2, 2)\}$, and Γ_2 be the quotient of kQ'_2 by the ideal $\bar{I}_2$ generated by the set

$$\{u_{r_1h_1}u_{r_2h_1},\ u_{2h}u_{r2},\ u_{r2}u_{2h},\ u_{r_3h_3}u_{r_4h_4} + u_{r_3h_4}u_{r_4h_3},\ u_{r_3h_3}u_{r_4h_4} - u_{r_4h_3}u_{r_3h_4} \\ \mid 2 \geqslant r_1, r_2, r_3, r_4, h_1 \geqslant 0,\ 1 \geqslant h, r \geqslant 0, (r_i, h_j) \neq (2, 2), i, j \in \{3, 4\}\}.$$

Note that $u_{2h}u_{r2}, u_{r2}u_{2h} \in \bar{I}_2$ by the relation (2′) and (3′). Then $\Gamma_2 = kQ'_2/\bar{I}_2$ has as a k-basis the set $\mathcal{U} = \{1, u_{rh} \mid (r, h) \neq (2, 2)\} \cup \{u_{r_1h_1}u_{r_2h_2} \mid 2 \geqslant r_1 \geqslant r_2 \geqslant 0, 2 \geqslant h_1 > h_2 \geqslant 0, (r_1, h_1) \neq (2, 2)\} \cup \{u_{r_12}u_{r_21}u_{r_30} \mid 1 \geqslant r_1 \geqslant r_2 \geqslant r_3 \geqslant 0\}$. Consider the ideal of Γ_2,

$$\tilde{I}_2 = \langle u_{r_1h_1}u_{r_2h_2}\cdots u_{r_mh_m} \mid (r_1r_2\cdots r_m, h_1h_2\cdots h_m) \notin T^m\rangle \\ = \langle u_{12}u_{r1}, u_{r_12}u_{r_21}u_{r_30} \mid 1 \geqslant r \geqslant 0, 1 \geqslant r_1 \geqslant r_2 \geqslant r_3 \geqslant 0\rangle,$$

and we have an epimorphism $\widetilde{\psi} : \Gamma_2/\tilde{I}_2 \to \mathrm{HH}^*(\Lambda_2)$. Moreover, $\Gamma_2/\tilde{I}_2$ has a k-basis $\mathcal{V} = \{1, u_{rh} \mid (r, h) \neq (2, 2)\} \cup \{u_{r_11}u_{r_20} \mid 2 \geqslant r_1 \geqslant r_2 \geqslant 0\} \cup \{u_{r_12}u_{r_20} \mid 1 \geqslant r_1 \geqslant r_2 \geqslant 0\} \cup \{u_{02}u_{01}\}$. Therefore $\Gamma_2/\tilde{I}_2 \cong \mathrm{HH}^*(\Lambda_2)$ since $\dim_k \Gamma_2/\tilde{I}_2 = 19 = \dim_k \mathrm{HH}^*(\Lambda_2)$.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 10971206 and 11171325) and Important Foundation of Hubei Provincial Department of Education (Grant No. D20101003). The authors would like to thank the three anonymous reviewers for their valuable comments and suggestions to improve the manuscript.

References

1 Assem I, de la Peña J A. The fundamental groups of a triangular algebra. Comm Algebra, 1996, 24: 187–208
2 Auslander M. Representation dimension of Artin algebras. In: Queen Mary College Lecture Notes. London: Queen Mary College, 1971
3 Bautista R, Gabriel P, Roiter A V, et al. Representation-finite algebras and multiplicative bases. Invent Math, 1985, 81: 217–285
4 Beilinson A A. Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebra. Funct Anal Appl, 1978, 12: 214–216
5 Buchweitz R O, Green E L, Snashall N, et al. Multiplicative structures for Koszul algebras. Quart J Math, 2008, 59: 441–454

6 Buchweitz R O, Green E L, Madsen D, et al. Finite Hochschild cohomology without finite global dimension. Math Res Lett, 2005, 12: 805–816

7 Bustamante J C. The cohomology structure of string algebras. J Pure Appl Algebra, 2006, 204: 616–626

8 Butler M C R, King A D. Minimal resolutions of algebras. J Algebra, 1999, 212: 323–362

9 Cartan H, Eilenberg S. Homological Algebra. Princeton: Princeton University Press, 1956

10 Chen X W. Graded self-injective algebras "are" trivial extensions. J Algebra, 2009, 322: 2601–2606

11 Cibils C. Hochschild cohomology algebra of radical square zero algebras. Idun Reiten, Sverre O. Smalo, Oyvind Solberg, eds. In: Algebras and Modules II, CMS Conference Proceedings. Norway: Geiranger, 1998, 24: 93–101

12 Fan J M, Xu Y G. On Hochschild cohomology ring of Fibonacci algebras. Front Math China, 2006, 1: 526–537

13 Happel D. Hochschild cohomology of finite dimensional algebras. Lecture Notes in Math, 1989, 1404: 108–126

14 Hochschild G. On the cohomology groups of an associative algebra. Ann Math, 1945, 46: 58–67

15 Iyama O. Finiteness of representation dimension. Proc Amer Math Soc, 2002, 131: 1011–1014

16 Krause H, Kussin D. Rouquier's theorem on representation dimension. In: Trends in Representation Theory of Algebras and Related Topics. Contemp Math, Amer Math Soc, 2006, 406: 95–103

17 Gerstenhaber M. On the deformation of rings and algebras. Ann of Math, 1964, 79: 59–103

18 Gerstenhaber M. The cohomology structure of an associative ring. Ann of Math, 1963, 78: 267–288

19 Green E L, Hartman G, Marcos E N, et al. Resolutions over Koszul algebras. Archiv der Mathematik, 2005, 85: 118–127

20 Green E L, Huang Rosa Q. Projective resolutions of straightening closed algebras generated by Minors. Adv in Math, 1995, 110: 314–333

21 Green E L, Marcos E N, Snashall N. The Hochschild cohomology ring of a one point extension. Comm Algebra, 2003, 31: 357–379

22 Green E L, Solberg Ø. Hochschild cohomology rings and triangular rings. Happel D, Zhang Y B, eds. In: Proceedings of the Ninth International Conference. Beijing: Beijing Normal University Press, 2002, 192–200

23 Li H, Yao H L. The Hochschild cohomology of the quasi-entwining structure. Sci China Math, 2010, 53: 1103–1110

24 Rickard J. Derived equivalences as derived functors. J London Math Soc, 1991, 43: 37–48

25 Rouquier R. Representation dimension of exterior algebras. Invent Math, 2006, 165: 357–367

26 Siegel S F, Witherspoon S J. The Hochschild cohomology ring of a group algebra. London Math Soc, 1999, 79: 131–157

27 Skowroński A. Simply connected algebras and Hochschild cohomology. Can Math Soc Proc, 1993, 14: 431–447

28 Xi C C. On the representation dimension of finite dimensional algebras. J Algebra, 2000, 226: 332–346

29 Xi C C. Representation dimension and quasi-hereditary algebras. Adv Math, 2002, 168: 280–298

30 Xu Y G, Xiang H L. Hochschild cohomology rings of d-Koszul algebras. J Pure Appl Algebra, 2011, 215: 1–12

31 Xu Y G, Zhang C. Gerstenhaber bracket product of truncated quiver algebras. Sci China Math, 2011, 41: 17–32

32 Zhang P. Hochschild cohomology of truncated basic cycle. Sci China Ser A, 1997, 40: 1272–1278

Information for authors

SCIENCE CHINA Mathematics, a peer review mathematical journal cosponsored by Chinese Academy of Sciences and National Natural Science Foundation of China, and published monthly in both print and electronic forms by Science China Press and Springer, is committed to publishing high-quality, original results in both basic and applied research.

Categories of articles:

Reviews summarize representative results and achievements in a particular topic or an area, comment on the current state of research, and advise on the research directions. The author's own opinion and related discussion are requested.

Articles report on important original results in all areas of mathematics.

Brief reports present short reports in a timely manner of the latest important results.

Authors are recommended to use the online submission services. To submit a manuscript, please visit www.scichina.com, log on at JoMaSy© (Journal Management System), get an account, and follow the instructions to upload the text and image/table files.

Authors should also submit such accompanying materials as a short statement on the research background, area/subarea and significance of the work, a brief introduction to the first and corresponding authors including their mailing address, post code, telephone number, fax number, and email address. Authors may suggest several referees (please supply full names, addresses, phone, fax and email), and/or request the exclusion of specific reviewers.

All submissions will be reviewed by referees selected by the editorial board. The decision of acceptance or rejection of a manuscript is made by the editorial board based on the referees' reports. The entire review process may take 60 to 90 days, and the editorial office will inform the author of the decision as soon as the process is completed.

Authors should guarantee that their submitted manuscript has not been published before, and has not been submitted elsewhere for print or electronic publication consideration.

Submission of a manuscript is taken to imply that all the named authors are aware that they are listed as co-authors, and they have seen and agreed to the submitted version of the paper. No change in the order of listed authors can be made without an agreement signed by all the authors.

Once a manuscript is accepted, the authors should send a copyright transfer form signed by all authors to Science China Press. Authors of one published paper will be presented one sample copy. If offprints and more sample copies are required, please contact the managing editor and pay the extra fee. The full text in Chinese and in English opens freely to the readers in China at www.scichina.com, and the full text in English is available to overseas readers at www. springerlink.com.

Subscription information

ISSN print edition: 1674-7283
ISSN electronic edition: 1869-1862

Subscription rates:
For information on subscription rates please contact:
Customer Service
China: sales@scichina.org
North and South America:
journals-ny@springer.com
Outside North and South America:
subscriptions@springer.com

Orders and inquiries:
China
Science China Press
16 Donghuangchenggen North Street,
Beijing 100717, China
Tel: 86-10-64034559 or 86-10-64034134
Fax: 86-10-64016350
Email: sales@scichina.org
North and South America
Springer New York, Inc.
Journal Fulfillment
P.O. Box 2485
Secaucus, NJ 07096, USA
Tel: 1-800-SPRINGER or 1-201-348-4033
Fax: 1-201-348-4505
Email: journals-ny@springer.com
Outside North and South America
Springer Distribution Center
Customer Service Journals
Haberstr. 7, 69126
Heidelberg, Germany
Tel: 49-6221-345-0, Fax: 49-6221-345-4229
Email: subscriptions@springer.com

Cancellations must be received by September 30 to take effect at the end of the same year.

Changes of address: Allow for six weeks for all changes to become effective. All communications should include both old and new addresses (with postal codes) and should be accompanied by a mailing label from a recent issue. According to § 4 Sect. 3 of the German Postal Services Data Protection Regu lations, if a subscriber's address changes, the German Federal Post Office can inform the publisher of the new address even if the subscriber has not submitted a formal application for mail to be forwarded. Subscribers not in agreement with this procedure may send a written complaint to Customer Service Journals, Karin Tiks, within 14 days of publication of this issue.

Microform editions are available from: ProQuest. Further information available at http://www.il.proquest.com/uni.

Electronic edition:
An electronic version is available at springerlink.com.

Production:
Science China Press
16 Donghuangchenggen North Street, Beijing 100717, China
Tel: 86-10-64034559 or 86-10-64034134
Fax: 86-10-64016350
Printed in the People's Republic of China
Jointly Published by
Science China Press and Springer

SCIENCE CHINA
Mathematics

CONTENTS

Vol. 55 No. 6 June 2012

Articles

math.scichina.com www.springer.com/scp

Indexed by:
SCI-CD
MR
Z Math
MathSciNet

ISSN 1674-7283